Comm. Korean Math. Soc. **11** (1996), No. 4, pp. 983–996

ON THE THREE OPERATOR SPACE STRUCTURES OF HILBERT SPACES

DONG-YUN SHIN

ABSTRACT. In this paper, we show that $\|\xi\|_r = \|\sum_{i\in I} x_i x_i^*\|^{\frac{1}{2}}$, $\|\xi\|_c = \|\sum_{i\in I} x_i^* x_i\|^{\frac{1}{2}}$ for $\xi = \sum_{i\in I} x_i e_i$ in $M_n(H)$, that subspaces as Hilbert spaces are subspaces as column and row Hilbert spaces, and that the standard dual of column (resp., row) Hilbert spaces is the row (resp., column) Hilbert spaces differently from [1, 6]. We define operator Hilbert spaces differently from [10], show that our definition of operator Hilbert spaces is the same as that in [10], show that subspaces as Hilbert spaces are subspaces as operator Hilbert spaces, and for a Hilbert space H we give a matrix norm which is not an operator space norm on H.

1. Introduction

The theory of operator spaces and their completely bounded maps has provided a powerful tool for studying operator algebras. D. P. Blecher and V. I. Paulsen [3] and E. Effros and Z. J. Ruan [5] replace bounded operators by completely bounded operators, isometries by complete isometries, and Banach spaces by operator spaces. E. Effros and Z. J. Ruan [6, 7], D. P. Blecher [1] and G.Pisier [10] study Hilbert spaces as operator spaces. E. Effros and Z. J. Ruan [6, 7], D. P. Blecher [1] study the column and the row Hilbert spaces and G.Pisier [10] studies the operator Hilbert spaces.

In section 2, we study column and row Hilbert spaces. We show that $\|\xi\|_r = \|\sum_{i\in I} x_i x_i^*\|^{\frac{1}{2}}$, $\|\xi\|_c = \|\sum_{i\in I} x_i^* x_i\|^{\frac{1}{2}}$ for $\xi = \sum_{i\in I} x_i e_i$ in $M_n(H)$, that subspaces as Hilbert spaces are subspaces as column and row Hilbert spaces, and that the standard dual of column (resp., row)

Received April 11, 1996. Revised July 15, 1996.

1991 AMS Subject Classification: 46L05.

Key words and phrases: Column, Row, and Operator Hilbert spaces.

The Present Studies were Supported in part by the Basic Science Research Institute Program, Ministry of Education, 1995, Project No. 1420.

Hilbert spaces is the row (resp., column) Hilbert spaces differently from [1, 6].

In section 3, we study operator Hilbert spaces. We define operator Hilbert spaces differently from [10], show that our definition of operator Hilbert spaces is the same as that in [10], and show that subspaces as Hilbert spaces are subspaces as operator Hilbert spaces. Finally, for a Hilbert space H we give a matrix norm which is not an operator space norm on H.

The author would like to express his gratitude to Professor A. M. Sinclair for many valuable discussions and suggestions. This work was done while the author was staying at The University of Edinburgh.

2. Column and Row Operator Space Structures of Hilbert Spaces

Let E be a vector space over the complex field C, let $M_n(E)$ denote the vector space of $n \times n$ matrices with entries from E, let M_n denote the set of all $n \times n$ complex matrices with C^*-norm, and let $\{e_{ij}\}$ denote the standard matrix units for M_n, that is, e_{ij} is 1 in the (i, j)-entry and 0 elsewhere.

For $x = [x_{ij}] \in M_m(E)$, $y = [y_{ij}] \in M_n(E)$, $\alpha = [\alpha_{ij}]$, $\beta = [\beta_{ij}] \in M_m$, we write

$$x \oplus y = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in M_{m+n}(E),$$

$$\alpha x = [z_{ij}], x\beta = [w_{ij}] \in M_m(E),$$

where $z_{ij} = \sum_{p=1}^{m} \alpha_{ip} x_{pj}$ and $w_{ij} = \sum_{p=1}^{m} \beta_{pj} x_{ip}$. Here we use the symbol 0 for a rectangular matrix of zero element over E.

If there is a norm $\|\cdot\|_n$ on $M_n(E)$ for each positive integer n, the family of the norms $\{\|\cdot\|_n\}$ is called a matrix norm on E. E is called a space with a matrix norm. If there no danger of confusion, we set $\|\cdot\| = \|\cdot\|_n$.

A space E with a matrix norm is called a matrix normed space if for $\alpha, \beta \in M_n$, $x \in M_n(E)$,

$$\|\alpha x \beta\|_n \leq \|\alpha\| \, \|x\|_n \|\beta\|.$$

A matrix normed space E is an operator space if it satisfies

$$\|x \oplus y\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}.$$

If H is a Hilbert space, we set $H^n = \overbrace{H \oplus \cdots \oplus H}^{n \text{ times}}$. Let $B(H, K)$ denote the set of all bounded linear operators from a Hilbert space H to a Hilbert space K and $B(H) = B(H, H)$. We may identify $B(H^n)$ with $M_n(B(H))$. Then $B(H)$ is an operator space. Since a C^*-algebra $\mathcal{A}$ can be embedded in $B(H)$, $\mathcal{A}$ is also an operator space with the canonical matrix norm.

Suppose that E and F are matrix normed spaces and $\phi : E \to F$ is a linear map. We define the map $\phi_n : M_n(E) \to M_n(F)$ by $\phi_n([x_{ij}]) = [\phi(x_{ij})]$ for $[x_{ij}] \in M_n(E)$. We write $\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in N\}$, where $\|\phi\| = \sup\{\|\phi(x)\| : x \in E, \|x\| = 1\}$. We call ϕ completely bounded if $\|\phi\|_{cb} < \infty$. We call ϕ a complete isometry if for every positive integer n, $\phi_n : M_n(E) \to M_n(F)$ is an isometry. Let $CB(E, F)$ denote the set of all completely bounded linear maps from E to F and $CB(E) = CB(E, E)$. Two matrix normed spaces are completely isometrically isomorphic if there is a complete isometry of the first space onto the second.

Given Hilbert space H, we may identify $M_n(H)$ with $B(C^n, H^n)$ for every positive integer n. Then the Hilbert space H with this matrix norm is an operator space which is called a column Hilbert space and is indicated by H_c and the norm on $M_n(H_c)$ is indicated by $\|\cdot\|_c$. Secondly, we may identify $M_n(H)$ with $B(H^n, C^n)$ for $n \in N$. This gives an operator space structure on H, which is called a row Hilbert space and is indicated by H_r and the norm on $M_n(H_r)$ is indicated by $\|\cdot\|_r$.

DEFINITION 2.1. Let I be a set, let $x_i \in M_n$ for $i \in I$, and let $S_J = \sum_{i \in J} x_i \in M_n$ for any finite subset J of I. A series $\sum_{i \in I} x_i$ is said to converge if there is a matrix $a \in M_n$ with the following property : For any $\epsilon > 0$ there is a finite subset J of I such that $J \subseteq J_1 \subseteq I$ implies that $\|\sum_{i \in J_1} x_i - a\| < \epsilon$, where $\|\cdot\|$ is the C^*-norm.

In this case we also say that $\sum_{i \in I} x_i$ converges to a and we write $\sum_{i \in I} x_i = a$.

We may consider x_{ij} as a column (resp., a row) vector for $[x_{ij}] \in M_n(C^k)$. Then we may consider $[x_{ij}]$ as a $nk \times n$ (resp., a $n \times kn$)

matrix. We denote this $nk \times n$ (resp., a $n \times kn$) matrix by α (resp., β). The next proposition shows why we call H_c (resp., H_r) a column (resp., a row) Hilbert space.

PROPOSITION 2.2. *For $[x_{ij}] \in M_n(C^k)$, we have $\|[x_{ij}]\|_c = \|\alpha\|$ and $\|[x_{ij}]\|_r = \|\beta\|$.*

PROOF. We may consider $[x_{ij}] \in M_n(C^k)$ a linear transformation from C^n to C^{kn} (resp., a linear transformation from C^{kn} to C^n). Then α (resp., β) is the matrix of $[x_{ij}]$ relative to the standard bases. Hence $\|[x_{ij}]\|_c = \|\alpha\|$ and $\|[x_{ij}]\|_r = \|\beta\|$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis for a Hilbert space H, let $[\xi_{ij}]$ be in $M_n(H)$, let $\xi_{ij} = \sum_{l\in I} x_{ij}^l e_l$ be H, and let $x_l = [x_{ij}^l]$ be in M_n. We formally write $[\xi_{ij}] = \sum_{i\in I} x_i e_i$. For a $n \times n$ matrix x, let x^t denote the transpose matrix of x and let $\bar{x}$ denote the transpose matrix of x^*. We can easily show that $\sum_{i\in I} x_i$ converges for $\xi = \sum_{i\in I} x_i e_i \in M_n(H)$. We show that $\|\xi\|_r = \|\sum_{i\in I} x_i x_i^*\|^{\frac{1}{2}}$ and $\|\xi\|_c = \|\sum_{i\in I} x_i^* x_i\|^{\frac{1}{2}}$ for $\xi = \sum_{i\in I} x_i e_i \in M_n(H)$ and that they are independent to orthonormal bases.

LEMMA 2.3. *Let $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ be orthonormal bases for a Hilbert space H, let $\xi = [\xi_{ij}]$ be in $M_n(H)$ and let $\xi = \sum_{i\in I} x_i e_i = \sum_{i\in I} y_i f_i$ for $x_i, y_i \in M_n$. Then the following hold :*

(1) $\sum_{i\in I} x_i^* x_i = \sum_{i\in I} y_i^* y_i$,
(2) $\sum_{i\in I} x_i x_i^* = \sum_{i\in I} y_i y_i^*$,
(3) $\sum_{i\in I} x_i \otimes \bar{x}_i = \sum_{i\in I} y_i \otimes \bar{y}_i$,
(4) $\sum_{i\in I} x_i \otimes x_i^* = \sum_{i\in I} y_i \otimes y_i^*$.

PROOF. Let $e_i = \sum_{l\in I} \lambda_{il} f_l$. Then $f_i = \sum_{l\in I} \bar{\lambda}_{li} e_l$ and $\sum_{l\in I} \lambda_{il} \bar{\lambda}_{kl} = \delta_{ik}$. Since $\sum_{i\in I} x_i e_i = \sum_{i\in I}(x_i \sum_{l\in I} \lambda_{il} f_l) = \sum_{l\in I}(\sum_{i\in I} \lambda_{il} x_i) f_l$, $y_l = \sum_{i\in I} \lambda_{il} x_i$. Hence $\sum_{l\in I} y_l^* y_l = \sum_{l,k,i\in I} \bar{\lambda}_{il} \lambda_{kl} x_i^* x_k = \sum_{i\in I} x_i^* x_i$ and similarly, the others hold.

THEOREM 2.4. *Let $\{e_i\}_{i\in I}$ be a fixed orthonormal basis for a Hilbert space H and let $\xi = \sum_{i\in I} x_i e_i$ be in $M_n(H)$. Then $\|\xi\|_r = \|\sum_{i\in I} x_i x_i^*\|^{\frac{1}{2}}$ and $\|\xi\|_c = \|\sum_{i\in I} x_i^* x_i\|^{\frac{1}{2}}$.*

PROOF. First, we identify $M_n(H)$ with $B(H^n, C^n)$. Let $y = \sum_{i\in I} y_i e_i$ be in H^n and let $\xi = \sum_{i\in I} x_i e_i$ be in $M_n(H)$. Then $\xi(y) = \sum_{i\in I} x_i y_i$ and $\|y\| = (\sum_{i\in I} \|y_i\|^2)^{\frac{1}{2}}$. Hence

$$\begin{aligned}\|\xi\|_r &= \sup\{\|\xi(y)\| : y \in H^n, \|y\| = 1\}\\ &= \sup\{\|\sum_{i\in I} x_i y_i\| : \sum_{i\in I} \|y_i\|^2 = 1\}\\ &= \|\sum_{i\in I} x_i x_i^*\|^{\frac{1}{2}}.\end{aligned}$$

Secondly, we identify $M_n(H)$ with $B(C^n, H^n)$. Let λ be in C^n and $\xi = \sum_{i\in I} x_i e_i$ be in $M_n(H)$. Then $\xi(\lambda)= \sum_{i\in I} x_i \lambda e_i$. Hence

$$\begin{aligned}\|\xi\|_c &= \sup\{\|\xi(\lambda)\| : \lambda \in C^n, \|\lambda\| = 1\}\\ &= \sup\{\|\sum_{i\in I} x_i \lambda e_i\| : \lambda \in C^n, \|\lambda\| = 1\}\\ &= \|\sum_{i\in I} x_i^* x_i\|^{\frac{1}{2}}.\end{aligned}$$

Let $X \otimes_\wedge Y$ be the projective tensor product of two operator spaces X and Y, and X_1 (resp.,Y_1) be a subspace of X (resp., Y). Then $X_1 \otimes_\wedge Y_1$ is a subspace of the normed space $X \otimes_\wedge Y$, but it is not a subspace of the operator space $X \otimes_\wedge Y$. But, the next theorem shows that if H is a subspace of a Hilbert space K, H_r is a subspace of the row Hilbert space K_r, and H_c is a subspace of the column Hilbert space K_c.

THEOREM 2.5. *Let H and K be Hilbert spaces and $\phi : H \longrightarrow K$ be isometric. Then $\phi_r : H_r \longrightarrow K_r$ defined by $\phi_r(\xi) = \phi(\xi)$ and $\phi_c : H_c \longrightarrow K_c$ defined by $\phi_c(\xi) = \phi(\xi)$ are completely isometric.*

PROOF. Since ϕ is isometric, it preserves inner product, $i, e, (\phi(\xi), \phi(\eta)) = (\xi, \eta)$ for $\xi, \eta \in H$. Hence if $\{e_i\}_{i\in I}$ is an orthonormal basis for H, then $\{\phi(e_i)\}_{i\in I}$ is an orthonormal set in K. Since $\phi_{r_n}(\xi)= \sum_{i\in I} x_i \phi(e_i)= \phi_{c_n}(\xi)$ for $\xi \in \sum_{i\in I} x_i e_i \in M_n(H)$, ϕ_r and ϕ_c are completely isometric by Theorem 2.4.

Let $a = [a_{ij}]$ be a $n \times n$ matrix, let b and c be $m \times m$ matrices, let I_n be the $n \times n$ identity matrix, and let $a \otimes b = [a_{ij} b]$ be a $mn \times mn$ matrix. Then $a \otimes b = (a \otimes I_m)(I_n \otimes b)$, $(a \otimes b)^* = a^* \otimes b^*$, $a \otimes b + a \otimes c = a \otimes (b+c)$, $b \otimes a + c \otimes a = (b+c) \otimes a$ and $\| \sum_{i=1}^n a_i \otimes b_i \| = \| \sum_{i=1}^n b_i \otimes a_i \|$ for $a_i \in M_n$, $b_i \in M_m$.

It was proved in [1,6] that the standard dual space of a row Hilbert space H_r (resp., a column Hilbert space H_c) is the column Hilbert space H_c (resp., row Hilbert space H_r). But to give a new Proof we need the following Lemmas.

LEMMA 2.6. *Let k, m, n be positive integers and let $a_i \in M_n$, $b_i \in M_m$ for $1 \le i \le k$. Then the following hold :*

(1) $\| \sum_{i=1}^k a_i \otimes b_i \| \le \| \sum_{i=1}^k a_i^* a_i \|^{\frac{1}{2}} \| \sum_{i=1}^k b_i b_i^* \|^{\frac{1}{2}}$,
(2) $\| \sum_{i=1}^k a_i \otimes b_i \| \le \| \sum_{i=1}^k a_i a_i^* \|^{\frac{1}{2}} \| \sum_{i=1}^k b_i^* b_i \|^{\frac{1}{2}}$,
(3) *if* $k \le m$, $\| \sum_{i=1}^k a_i a_i^* \|^{\frac{1}{2}} = \sup\{ \| \sum_{i=1}^k c_i \otimes a_i \| : c_i \in M_m, \| \sum_{i=1}^k c_i^* c_i \| = 1 \}$,
(4) *if* $k \le m$, $\| \sum_{i=1}^k a_i^* a_i \|^{\frac{1}{2}} = \sup\{ \| \sum_{i=1}^k c_i \otimes a_i \| : c_i \in M_m, \| \sum_{i=1}^k c_i c_i^* \| = 1 \}$.

PROOF. Let $\mathcal{A}$ be a C^*-algebra and $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{A}$. Then it is well known that $\| \sum_{i=1}^k x_i y_i \| \le \| \sum_{i=1}^k x_i x_i^* \|^{\frac{1}{2}} \| \sum_{i=1}^k y_i^* y_i \|^{\frac{1}{2}}$. Since $a \otimes b = (a \otimes I_m)(I_n \otimes b)$, $a \otimes b + a \otimes c = a \otimes (b + c)$, $b \otimes a + c \otimes a = (b+c) \otimes a$ and $\| \sum_{i=1}^k a_i \otimes b_i \| = \| \sum_{i=1}^k b_i \otimes a_i \|$ for $a_i \in M_n$, $b_i \in M_m$, $\| \sum_{i=1}^k a_i \otimes b_i \| = \| \sum_{i=1}^k (a_i \otimes I_m)(I_n \otimes b_i) \| \le \| \sum_{i=1}^k a_i^* a_i \|^{\frac{1}{2}} \| \sum_{i=1}^k b_i b_i^* \|^{\frac{1}{2}}$ and $\| \sum_{i=1}^n a_i \otimes b_i \| \le \| \sum_{i=1}^k a_i a_i^* \|^{\frac{1}{2}} \| \sum_{i=1}^k b_i^* b_i \|^{\frac{1}{2}}$. Hence (1) and (2) hold.

Put $c_i = e_{1i}$. Then $\| \sum_{i=1}^k c_i^* c_i \| = 1$ and $\| \sum_{i=1}^k c_i \otimes a_i \| = \| \sum_{i=1}^k a_i a_i^* \|$. Put $c_i = e_{i1}$. Then $\| \sum_{i=1}^k c_i c_i^* \| = 1$ and $\| \sum_{i=1}^k c_i \otimes a_i \| = \| \sum_{i=1}^k a_i^* a_i \|$. Hence (3) and (4) hold.

Note that we identify $M_n(H_c)$ (resp., $M_n(H_r)$) with $B(C^n, H^n)$ (resp., $B(H^n, C^n)$. For $\xi \in H$, we define $\omega_\xi : H \to C$ and $i_\xi : C \to H$ by

$$\omega_\xi(\eta) = (\eta, \xi), \quad i_\xi(\lambda) = \lambda\xi.$$

Then $H_r = \{\omega_\xi : \xi \in H\}$, $H_c = \{i_\xi : \xi \in H\}$ and

$$\lambda\omega_\xi + \omega_\eta = \omega_{\bar{\lambda}\xi+\eta}, \quad \lambda i_\xi + i_\eta = i_{\lambda\xi+\eta}$$

for $\lambda \in C$, ξ, $\eta \in H$. We define $p_\xi : H_c \to C$ and $q_\xi : H_r \to C$ by

$$p_\xi(i_\eta) = (\eta, \xi), \quad q_\xi(\omega_\eta) = (\xi, \eta).$$

For $\xi = [\xi_{st}] \in M_n(H)$, we set $\omega_\xi = [\omega_{\xi_{st}}]$, $i_\xi = [i_{\xi_{st}}]$, $p_\xi = [p_{\xi_{st}}]$, and $q_\xi = [q_{\xi_{st}}]$. Let H_c^*(resp., H_c^*) be the standard dual of H_c(resp., H_r), i, e, we identify $M_n(H_c^*)$(resp., $M_n(H_r^*)$ with $CB(H_c, M_n)$ (resp., $CB(H_r, M_n)$. Hence for $p_\xi \in M_n(H_c^*)$, the norm of p_ξ in the operator space H_c^* is the completely bounded norm of the map $p_\xi : H_c \to M_n$ and it is $\|p_{\xi_n}\|$, and for $q_\xi \in M_n(H_r^*)$, the norm of q_ξ in the operator space H_r^* is the completely bounded norm of the map $q\xi : H_r \to M_n$ and it is $\|q_{\xi_n}\|$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis for a Hilbert space H, let $[\xi_{kl}]$ be in $M_n(H)$, let $\xi_{kl} = \sum_{i\in I} x_{kl}^i e_i$ be H, and let $x_i = [x_{kl}^i]$ be in M_n. We formally write

$$[\xi_{kl}] = \sum_{i\in I} x_i e_i.$$

Let $\zeta = \sum_{i\in I} z_i e_i$ be in H, let $\xi = [\xi_{kl}] = \sum_{i\in I} x_i e_i$ be in $M_n(H)$, let $\eta = [\eta_{kl}] = \sum_{i\in I} y_i e_i$ be in $M_m(H)$, let $\xi_{kl} = \sum_{i\in I} x_{kl}^i e_i$ be in H, and let $\eta_{kl} = \sum_{i\in I} y_{kl}^i e_i$ be in H. Then

$$\begin{aligned} p_\xi(i_\zeta) &= [(\zeta, \xi_{kl})] = [\sum_{i\in I} z_i \bar{x}_{kl}^i] \\ &= \sum_{i\in I} z_i \bar{x}_i, \\ p_{\xi_m}(i_\eta) &= [p_\xi(i_{\eta_{kl}})] = \left[[\sum_{i\in I} y_{kl}^i \bar{x}_i] \right] \\ &= \sum_{i\in I} y_i \otimes \bar{x}_i, \\ q_\xi(\omega_\zeta) &= [(\xi_{kl}, \zeta)] \\ &= [\sum_{i\in I} \bar{z}_i x_i], \\ q_{\xi_m}(\omega_\eta) &= \sum_{i\in I} \bar{y}_i \otimes x_i. \end{aligned}$$

LEMMA 2.7. *Let k and m be positive integers with $k \leq m$, let H be a k-dimensional Hilbert space, and let $\xi \in M_n(H)$. Then $\|p_{\xi_m}\|= \|p_\xi\|_{cb}= \|\xi\|_r$ and $\|q_{\xi_m}\|= \|q_\xi\|_{cb}= \|\xi\|_c$.*

PROOF. Let $\{e_i\}_{i=1}^k$ be an orthonormal basis for H, $\xi= x_1e_1 + \cdots + x_ke_k$ with $x_i \in M_n$, and $\eta= y_1e_1 + \cdots + y_ke_k \in M_m(H)$. Then $p_{\xi_m}(i_\eta)= \sum_{i=1}^k y_i \otimes \bar{x}_i$, $q_{\xi_m}(\omega_\eta)= \sum_{i=1}^k \bar{y}_i \otimes x_i$, $\|i_\eta\|= \|\sum_{i=1}^k y_i^* y_i\|^{\frac{1}{2}}$, and $\|\omega_\eta\|= \|\sum_{i=1}^k y_i y_i^*\|^{\frac{1}{2}}$. Obviously, $\|x\|= \|\bar{x}\|$ for $x \in M_m$. By Lemma 2.6, if $k \leq m$, then $\|p_{\xi_m}\|= \|\sum_{l\in I} x_l x_l^*\|^{\frac{1}{2}}$ and $\|q_{\xi_m}\|_c= \|\sum_{l\in I} x_l^* x_l\|^{\frac{1}{2}}$. Hence by Theorem 2.4, $\|p_{\xi_m}\| = \|p_\xi\|_{cb}= \|\xi\|_r$ and $\|q_{\xi_m}\|= \|q_\xi\|_{cb}= \|\xi\|_c$.

THEOREM 2.8. *Let K be a Hilbert space and let $\xi \in M_n(K)$. Then $\|p_\xi\|_{cb}= \|\xi\|_r$ and $\|q_\xi\|_{cb}= \|\xi\|_c$.*

PROOF. Let $\xi = [\xi_{ij}]$ be in $M_n(K)$ and $H =$ span $\{\xi_{ij} : 1 \leq i, j \leq n\}$. Then H is finite dimensional. Let p_ξ^H (resp., q_ξ^H) be the restriction of p_ξ (resp., q_ξ) to H. By Lemma 2.7, $\|p_\xi^H\|_{cb}= \|\xi\|_r$ and $\|q_\xi^H\|_{cb}= \|\xi\|_c$.

For $\eta \in M_m(K)$, we can decompose $\eta = \eta_1 + \eta_2$ with $\eta_1 \in M_m(H)$, $\eta_2 \in M_m(H^\perp)$. Then $\|i_{\xi_1}\| \leq \|i_\xi\|$, $\|\omega_{\xi_1}\| \leq \|\omega_\xi\|$, $p_{\xi_m}(i_\eta)= p_{\xi_m}^H(i_{\eta_1})$ and $q_{\xi_m}(\omega_\eta)= q_{\xi_m}^H(\omega_{\eta_1})$. Hence $\|p_\xi\|_{cb}= \|\xi\|_r$ and $\|q_\xi\|_{cb}= \|\xi\|_c$.

We define $\Phi : K_c^* \to K_r$ by $\Phi(p_\xi)= \omega_\xi$ and define $\Psi : K_r^* \to K_c$ by $\Psi(q_\xi)= i_\xi$. Then Φ and Ψ are linear. The following was proved in [1,6]. But we give here another Proof.

COROLLARY 2.9. *Φ and Ψ are complete isometries, i, e, we can identify K_c^* (resp., K_r^*) with K_r (resp., K_c).*

PROOF. By definition, we have $\|\omega_\xi\| = \|\xi\|_r$, $\|i_\xi\| = \|\xi\|_c$, $\Phi(p_\xi)= \omega_\xi$ and $\Psi(p_\xi)= i_\xi$ for $\xi \in M_n(H)$. Hence Φ and Ψ are complete isometries by Theorem 2.8.

COROLLARY 2.10. *Let k, m, n be positive integers with $\min\{k, n\} \leq m$, let $a_i \in M_n$, let $\alpha_m= \sup\{\|\sum_{i=1}^k c_i \otimes a_i\| : c_i \in M_m, \|\sum_{i=1}^k c_i^* c_i\| = 1\}$, and let $\beta_m= \sup\{\|\sum_{i=1}^k c_i \otimes a_i\| : c_i \in M_m, \|\sum_{i=1}^k c_i c_i^*\| = 1\}$. Then $\alpha_m= \|\sum_{i=1}^n a_i a_i^*\|^{\frac{1}{2}}$ and $\beta_m= \|\sum_{i=1}^n a_i^* a_i\|^{\frac{1}{2}}$.*

PROOF. Let $\{e_i\}_{i=1}^k$ be the standard basis for C^n, $\xi = a_1e_1+\cdots+a_ke_k$ with $a_i \in M_n$, and $\eta = b_1e_1 + \cdots + b_ke_k \in M_m(C^n)$. Then $p_{\xi_m}(\eta) = \sum_{i=1}^k a_i \otimes b_i$, $q_{\xi_m}(\eta) = \sum_{i=1}^k \bar{a}_i \otimes b_i$, $\|p_\xi\|_{cb} = \|p_{\xi_n}\|$, and $\|q_\xi\|_{cb} = \|q_{\xi_n}\|$. Hence by Theorem 2.8, $\alpha_n = \|\xi\|_r$ and $\beta_n = \|\xi\|_c$, i, e, $\alpha_n = \|\sum_{i=1}^k a_ia_i^*\|^{\frac{1}{2}}$ and $\beta_n = \|\sum_{i=1}^k a_i^*a_i\|^{\frac{1}{2}}$. By Lemma 2.6(1), $\alpha_l \leq = \|\sum_{i=1}^k a_ia_i^*\|^{\frac{1}{2}}$ and $\beta_l \leq \|\sum_{i=1}^k a_i^*a_i\|^{\frac{1}{2}}$. for any positive integer l. Hence if $n \leq m$, $\alpha_m = \alpha_n$ and $\beta_m = \beta_n$. By Lemma 2.6(2) and 2.6(3), if $k \leq m$, $\alpha_m = \alpha_k$ and $\beta_m = \beta_k$. Therefore if $\min\{k,n\} \leq m$, then $\alpha_m = \|\sum_{i=1}^n a_ia_i^*\|^{\frac{1}{2}}$ and $\beta_m = \|\sum_{i=1}^n a_i^*a_i\|^{\frac{1}{2}}$.

REMARK. $\alpha_1 = \beta_1$, but if $2 \leq k$ and $2 \leq n$, there always exist matrices $b_i \in M_n$ for $1 \leq i \leq k$ such that $\alpha_n \neq \beta_n$.

3. Operator Hilbert Space Structure of Hilbert Spaces

G. Pisier [10] showed that a Hilbert space H is isometrically embeded in $B(K)$ for some Hilbert space K, gave H the operator space structure induced by $B(K)$, and defined this operator space structure on H by operator Hilbert space.

In this section, we define $\|\xi\|_o = \|\sum_{i\in I} x_i \otimes \bar{x}_i\|^{\frac{1}{2}}$ for $x = \sum_{i\in I} x_ie_i$ in $M_n(H)$. We show that H is an operator space with this matrix norm, and show that this operator space is the same as the operator Hilbert space which is defined in [10].

LEMMA 3.1. *Let $\{e_i\}_{i\in I}$ be a fixed orthonormal basis for a Hilbert space H, let $\xi = \sum_{i\in I} x_ie_i$ be in $M_n(H)$, and let $\eta = \sum_{i\in I} y_ie_i$ be in $M_m(H)$. Then the following hold :*

(1) $\|\sum_{i\in I} x_i \otimes \bar{y}_i\| \leq \|\sum_{i\in I} x_i \otimes \bar{x}_i\|^{\frac{1}{2}}\|\sum_{i\in I} y_i \otimes \bar{y}_i\|^{\frac{1}{2}}$,

(2) *if* $n \leq m$, $\|\sum_{i\in I} x_i \otimes \bar{x}_i\|^{\frac{1}{2}} = \sup\{\|\sum_{i\in I} x_i \otimes \bar{y}_i\| : \|\sum_{i\in I} y_i \otimes \bar{y}_i\| = 1\}$.

PROOF. Let J be a finite subset of I. Then $\|\sum_{i\in J} x_i \otimes \bar{y}_i\| \leq \|\sum_{i\in J} x_i \otimes \bar{x}_i\|^{\frac{1}{2}}\|\sum_{i\in J} y_i \otimes \bar{y}_i\|^{\frac{1}{2}}$ by [8, Lemma 2.4.]. Hence (1) holds.

Assume $n \leq m$. Put $\alpha = \|\sum_{i\in I} x_i \otimes \bar{x}_i\|^{\frac{1}{2}}$ and $\alpha y_i = x_i \oplus 0$. Then $\|\sum_{i\in I} x_i \otimes \bar{x}_i\|^{\frac{1}{2}} = \|\sum_{i\in I} x_i \otimes \bar{y}_i\|$. Hence (2) holds.

THEOREM 3.2. *Let $\{e_i\}_{i\in I}$ be an orthonormal basis for a Hilbert space H and let $\|x\|_o = \|\sum_{i\in I} x_i \otimes \bar{x}_i\|^{\frac{1}{2}}$ for $x = \sum_{i\in I} x_i e_i \in M_n(H)$. Then H is an operator space with this matrix norm $\|\cdot\|_o$.*

PROOF. Let $\xi = \sum_{i\in I} x_i e_i$, $\eta = \sum_{i\in I} y_i e_i$ be in $M_n(H)$ Then $\xi + \eta = \sum_{i\in I}(x_i + y_i)e_i$ and

$$\begin{aligned}\|\xi + \eta\|_o^2 &= \|\sum_{i\in I}(x_i + y_i)\otimes \overline{x_i + y_i}\| \\ &\leq \|\xi\|_o^2 + \|\eta\|_o^2 + \|\sum_{i\in I} x_i \otimes \bar{y}_i\| + \|\sum_{i\in I} y_i \otimes \bar{x}_i\| \\ &\leq \|\xi\|_o^2 + \|\eta\|_o^2 + 2\|\xi\|_o\|\eta\|_o.\end{aligned}$$

Hence $\|\cdot\|_o$ is a norm on $M_n(H)$.

Let a, b be in M_n, and $\xi = \sum_{i\in I} x_i e_i$ be in $M_n(H)$. Then $a\xi b = \sum_{i\in I} a x_i b e_i$, $(ax_ib)\otimes(\overline{ax_ib}) = (a\otimes\bar{a})(x_i\otimes\bar{x}_i)(b\otimes\bar{b})$ and $\|a\otimes\bar{a}\| \leq \|a\|^2$.

$$\begin{aligned}\|a\xi b\|_o &= \|\sum_{i\in I}(ax_ib)\otimes(\overline{ax_ib})\|^{\frac{1}{2}} \\ &\leq \|a\|\|b\|\|\xi\|_o.\end{aligned}$$

Note that $(a\oplus c)\otimes(b\oplus d) = (a\otimes b)\oplus(a\otimes d)\oplus(c\otimes b)\oplus(c\otimes d)$ for $a,b \in M_n$ and $c, d \in M_m$. Let $\xi = \sum_{i\in I} x_i e_i$ be in $M_n(H)$ and $\eta = \sum_{i\in I} y_i e_i$ be in $M_m(H)$. Then by Lemma 3.1(1),

$$\begin{aligned}\|\xi\oplus\eta\|_o^2 &= \|\sum_{i\in I}(x_i\oplus y_i)\otimes\overline{x_i\oplus y_i}\| \\ &= \|\sum_{i\in I} x_i\otimes\bar{x}_i \oplus \sum_{i\in I} x_i\otimes\bar{y}_i \oplus \sum_{i\in I} y_i\otimes\bar{x}_i \oplus \sum_{i\in I} y_i\otimes\bar{y}_i\| \\ &= \max\{\|\sum_{i\in I} x_i\otimes\bar{x}_i\|, \|\sum_{i\in I} x_i\otimes\bar{y}_i\|, \|\sum_{i\in I} y_i\otimes\bar{x}_i\|, \|\sum_{i\in I} y_i\otimes\bar{y}_i\|\} \\ &= \max\{\|\xi\|_o^2, \|\eta\|_o^2\|\}\end{aligned}$$

Hence $\|\xi\oplus\eta\|_o = \max\{\|\xi\|_o, \|\eta\|_o\}$. Therefore H is an operator space with this matrix norm.

If H is a subspace of a Hilbert space K, H_r is a subspace of the row Hilbert space K_r, and H_c is a subspace of the column Hilbert space K_c. The next theorem shows that if H is a subspace of a Hilbert space K, H_o is a subspace of the operator Hilbert space K_o.

THEOREM 3.3. *Let H and K be Hilbert spaces and $\phi : H \longrightarrow K$ be isometric. Then $\phi_o : H_o \longrightarrow K_o$ defined by $\phi_o(\xi) = \phi(\xi)$ is completely isometric.*

PROOF. Similar to Theorem 2.5.

Let H be a Hilbert space, let ξ be in H, and let $\omega_\xi : H \to C$ be defined by $\omega_\xi(\eta) = (\eta, \xi)$. Then the standard dual space of H_o, H_o^* is $\{\omega_\xi : \xi \in H\}$ and $M_n(H_o^*) = CB(H_o, M_n)$. We set $\omega_\xi = [\omega_{\xi_{ij}}] \in M_n(H_o^*)$ for $\xi = [\xi_{ij}] \in M_n(H_o)$. Now, we prove that H_o^* is completely isometrically isomorphic to H_o. Then this operator space is the same as the operator Hilbert space which is defined in [10] by [10, Theorem 1.1.].

THEOREMY 3.4. *Let $\Phi : H_o \to H_o^*$ be defined by $\Phi(p_\xi) = \omega_\xi$. Then Φ is a complete isometry, i.e, we can identify H_o^* with H_o.*

PROOF. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for H, let $\zeta = \sum_{i\in I} z_i e_i$ be in H, let $\xi = [\xi_{kl}] = \sum_{i\in I} x_i e_i$ be in $M_n(H)$, and let $\eta = [\eta_{kl}] = \sum_{i\in I} y_i e_i$ be in $M_m(H)$. Then $\omega_\xi(i_\zeta) = \sum_{i\in I} z_i \bar{x}_i$, $\omega_{\xi_m}(i_\eta) = \sum_{i\in I} y_i \otimes \bar{x}_i$. Hence if $n \le m$, then by Lemma 3.1,

$$\begin{aligned}
\|\omega_{\xi_m}\| &= \sup\{\|\omega_{\xi_m}(y)\| : y \in M_m(H), \|y\|_o = 1\} \\
&= \sup\{\|\sum_{i\in I} y_i \otimes \bar{x}_i\| : \|\sum_{i\in I} y_i \otimes \bar{y}_i\| = 1\} \\
&= \sup\{\|\sum_{i\in I} x_i \otimes \bar{y}_i\| : \|\sum_{i\in I} y_i \otimes \bar{y}_i\| = 1\} \\
&= \|\sum_{i\in I} x_i \otimes \bar{x}_i\|.
\end{aligned}$$

Therefore $\|\omega_\xi\|_{cb} = \|\xi\|$, i.e, Φ is a complete isometry.

REMARK. Let $\{e_i\}_{i\in I}$ be an orthonormal base for a Hilbert space H, let $\xi = \sum_{i\in I} x_i e_i$ be in $M_n(H)$, and let $\|\xi\| = \|\sum_{i\in I} x_i \otimes x_i^*\|^{\frac{1}{2}}$. Then by Lemma 2.3, it is independent to orthonormal bases. But, it is not a norm on $M_n(H)$. For example, we put $H = C^2$, $x_1 = y_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $x_2 = y_2^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $e_1 = (1,0)$, $e_2 = (0,1)$, $\xi = x_1 e_1 + x_2 e_2$, $\eta = y_1 e_1 + y_2 e_2$. Then $\|\xi\| = \|\eta\| = 1$, $\|\xi + \eta\| = \sqrt{2+\sqrt{5}}$ and $\|x_1 \otimes y_1^* + x_2 \otimes y_2^*\| = \sqrt{2}$. Thus $\|\cdot\|$ is not a norm on $M_2(C^2)$ and does not satisfy $\|\sum_{i\in I} x_i \otimes y_i^*\| \le \|\sum_{i\in I} x_i \otimes x_i^*\|^{\frac{1}{2}} \|\sum_{i\in I} y_i \otimes y_i^*\|^{\frac{1}{2}}$.

Simarly, $\|\xi\| = \|\sum_{i\in I} x_i \otimes x_i\|^{\frac{1}{2}}$ and $\|\xi\| = \|\sum_{i\in I} x_i \otimes x_i^t\|^{\frac{1}{2}}$ are not norms on $M_2(C^2)$.

Finally we construct a matrix norm on H which is not an operator space norm on H.

LEMMA 3.5. *Let $\mathcal{A}$ be a C^*-algebra, $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathcal{A}$, $P = \sum_{i=1}^n x_i^* x_i + x_i x_i^*$, $Q = \sum_{i=1}^n y_i^* y_i + y_i y_i^*$, and $R = \sum_{i=1}^n x_i^* y_i + x_i y_i^*$. Then $\|R\|^2 \le \|P\|\|Q\|$.*

PROOF. We may assume that $\mathcal{A}$ is a subalgebra of $B(H)$ for some Hilbert space H.

Note that $\sum_{i=1}^n [x_i, y_i]^*[x_i, y_i] + [x_i^*, y_i^*]^*[x_i^*, y_i^*] = \begin{bmatrix} P & R \\ R^* & Q \end{bmatrix}$.

Let ξ, η be unit vectors of H such that $(R\eta, \xi)$ is a real number. Then any real number t,

$$0 \le \left(\begin{bmatrix} P & R \\ R^* & Q \end{bmatrix} \begin{pmatrix} t\xi \\ \eta \end{pmatrix}, \begin{pmatrix} t\xi \\ \eta \end{pmatrix} \right) = t^2(P\eta, \eta) + 2t(R\xi, \eta) + (Q\eta, \eta).$$

Hence $(R\eta, \xi)^2 \le (P\xi, \xi)(Q\eta, \eta) \le \|P\|\|Q\|$, *i, e,* $(R\eta, \xi)^2 \le \|P\|\|Q\|$. Thus $\|R\|^2 \le \|P\|\|Q\|$.

PROPOSITION 3.6. *Let $\{e_i\}_{i\in I}$ be an orthonormal base for a Hilbert space H, let $\xi = \sum_{i\in I} x_i e_i$ be in $M_n(H)$, and let $\|\xi\| = \frac{1}{2}\|\sum_{i\in I} x_i x_i^* + x_i^* x_i\|^{\frac{1}{2}}$. Then it is a matrix norm on H, but is not an operator space norm on H.*

PROOF. Let $\xi = \sum_{i\in I} x_i e_i$ and $\eta = \sum_{i\in I} y_i e_i \in M_n(H)$. Then by Lemma 3.5,

$$\begin{aligned}\|\xi+\eta\|^2 &= \frac{1}{2}\|\sum_{i\in I}(x_i+y_i)(x_i+y_i)^* + (x_i+y_i)^*(x_i+y_i)\| \\ &\le \|\xi\|^2 + \|\eta\|^2 + \frac{1}{2}\|\sum_{i\in I} y_i^* x_i + y_i x_i^*\| + \frac{1}{2}\|\sum_{i\in I} x_i y_i^* + x_i^* y_i\| \\ &\le \|\xi\|^2 + \|\eta\|^2 + 2\|\xi\|\|\eta\|.\end{aligned}$$

Hence $\|\cdot\|$ is a norm on $M_n(H)$. But, we put $\xi = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} e_1$, $\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\|\alpha\xi\|^2 = \frac{5+2\sqrt{5}}{2}$ and $\|\xi\|^2 = \frac{5+\sqrt{5}}{2}$. Thus $\|\cdot\|$ is not an operator space norm on H.

References

1. D. P. Blecher, *Tensor product of operator spaces II*. Can. J. Math. **44** (1992), 75–90.
2. ———, *The standard dual of operator spaces*, Pacific J. Math. **153** (1992), 15–30.
3. D. P. Blecher and V. I. Paulsen, *Tensor product of operator spaces*, J. Funct. Anal. **99** (1991), 262–291.
4. E. Christensen and A. M. Sinclair, *A survey of completely bounded operators*, Bull. London Math. Soc. **21** (1989), 417–448.
5. E. Effros and Z. J. Ruan, *A new approach to operator spaces*, Can. Math. Bull. **34** (1991), 329–337.
6. ———, *Self duality for the Haagerup tensor product and Hilbert space factorizations*, J. Funct. Anal. **100** (1991), 257–284.
7. ———, *Recent development in operator spaces*, preprint.
8. U. Haagerup, *Injectivity and decomposition of completely bounded maps*, Lecture Notes in Math. **1132** (1983), Springer-Verlag, Berlin. 170–222.
9. V. I. Paulsen, *Completely Bounded Maps and Dilations*, Pitman Research Notes in Math. 146, Longman, London, 1986.
10. G. Pisier, *The operator Hilbert space OH, complex interpolation and tensor norms*, preprint.
11. Z. J. Ruan, *Subspaces of C^*-algebras*, J. Funct. Anal. **76** (1988), 217–230.

Department of Mathematics
Seoul City University
90 Jeonnong-Dong, Dongdaemun-Ku
Seoul 130-743, Korea