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ON A STABILITY THEOREM FOR HYPEREXACT OPERATORS

YONG BIN CHOI AND CHOON KYUNG CHUNG

ABSTRACT. In this paper we study the index stability theorem for a bounded linear operator with closed range and extend the Kato's decomposition theorem for an absence of the index.

Suppose X and Y are normed spaces, write $BL(X,Y)$ for the set of all bounded linear operators from X to Y. We recall that if $k > 0$ and if $||x|| \leq k\,||Tx||$ for each $x \in X$ then we call $T \in BL(X,Y)$ *bounded below*, if $y \in \{Tx : ||x|| \leq k\,||y||\}$ for each $y \in Y$ then we call T *open*. The operator $T \in BL(X,Y)$ will be called *relatively open* ([2],[8]) if its *truncation*

$$T^{\vee} : X \to T(X)$$

is open. Thus bounded below is just relatively open one-one, open is the same as relatively open onto. If X and Y are complete then ([1],[5])

$$\text{(0.1)} \qquad T \text{ is relatively open} \iff T \text{ has a closed range} \iff T^{\dagger} \text{ has a closed range},$$

where $T^{\dagger}$ is the adjoint operator of T. When X and Y are the same space then we can introduce ([2],[3]) the *hyperrange* and the *hyperkernel* of $T \in BL(X,X)$:

$$T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^{n}(X)$$

and

$$T^{-\infty}(0) = \bigcup_{n=1}^{\infty} T^{-n}(0) :$$

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it is clear that both subspaces are invariant under any operator $S \in BL(X,X)$ which commutes with T, although for bounded operators on Banach spaces neither need be closed. If S commutes with T, so that also $ST^{\infty}(X) \subseteq T^{\infty}(X)$, we shall write

$$S^{\wedge} : T^{\infty}(X) \to T^{\infty}(X) \tag{0.2}$$

for the operator induced by S. If in particular S is invertible and commutes with T then

$$(T-S)^{-1}(0) \subseteq T^{\infty}(X), \tag{0.3}$$

so that the null space of $T-S$ is the same as the null space of $(T-S)^{\wedge}$. An operator $T \in BL(X,X)$ is called *hyperexact* ([3],[4]) if

$$T^{-1}(0) \subseteq T^{\infty}(X).$$

Also we have

$$T^{-1}(0) \subseteq T^{\infty}(X) \Longleftrightarrow T^{-\infty}(0) \subseteq T(X). \tag{0.4}$$

Our first observation was noticed by Goldberg ([1], Theorem V.1.2), which relies on Borsuk's antipodal lemma. But our argument avoids Borsuk's lemma; this was very nearly established by Harte ([2], (6.10.2.9)):

1. LEMMA. *Let $S, T \in BL(X,Y)$. If T is relatively open and if S has a sufficiently small norm then*

$$dim\,(T-S)^{-1}(0) \leq dim\,T^{-1}(0) \tag{1.1}$$

PROOF. If $\dim T^{-1}(0) = \infty$, this is evident. Suppose $\dim T^{-1}(0) < \infty$. Then we can find a closed subspace W of X for which

$$T^{-1}(0) \bigoplus W = X.$$

By [5] Theorem 4, the restriction of T to the subspace W, T_W is relatively open and hence bounded below. Therefore there is $k > 0$ for which

$$x \in W \Longrightarrow ||Tx|| \geq k||x||.$$

Thus it follows that

$$x \in W \Longrightarrow ||(T-S)x|| \geq ||Tx|| - ||S|| ||x|| \geq (k - ||S||)||x||,$$

which gives that if $||S|| < k$ then the restriction of $T-S$ to W is bounded below and hence one-one:

$$(T-S)^{-1}(0) \bigcap W = \{0\}.$$

Thus we have that $\dim (T-S)^{-1}(0) \leq \dim X/W = \dim T^{-1}(0)$. •

For brevity, we shall write

$$\alpha(T) = \dim T^{-1}(0) \quad \text{and} \quad \beta(T) = \dim Y/\mathrm{cl}\, T(X).$$

Thus $\alpha(T)$ and $\beta(T)$ will be either a non-negative integer or ∞.

We are ready for a stability theorem :

2. THEOREM. *Let X be a Banach space and let $T \in BL(X, X)$ have a closed range. If $S \in BL(X, X)$ is invertible, commutes with T and has a sufficiently small norm then*

(2.1) T *is hyperexact* $\Longrightarrow \alpha(T-S) = \alpha(T)$ *and* $\beta(T-S) = \beta(T)$.

Further, if $T^{-1}(0) \cap T^{\infty}(X)$ is finite dimensional and if $|\lambda|$ is sufficiently small then

(2.2) $\alpha(T-\lambda) = \alpha(T) \Longrightarrow T$ *is hyperexact.*

PROOF. By (1.1), we have that $\alpha(T-S) \leq \alpha(T)$. For the reverse inequality, suppose that $x \in T^{-1}(0) \subseteq T^{\infty}(X)$. Thus there exists a sequence $\{z_n\}$ in X such that

$$x = Tz_1 = T^2 z_2 = T^3 z_3 = \cdots$$

By using (0.4) and (0.1), we can find a constant $k > 0$ and a sequence (x_n) in X with $x_1 = x$ for which

(2.3) $x_n = Tx_{n+1}$ and $k||x_{n+1}|| \leq ||x_n||$ for each $n = 1, 2, \cdots$

Since X is complete, the series

$$y_x = \sum_{k=1}^{\infty} S^{k-1} x_k$$

is convergent in X because if $||S|| < k$ then by (2.3),

$$||\sum_{k=1}^{m} S^{k-1} x_k|| \leq ||x_1|| + \frac{||S||}{k}||x_1|| + \cdots + \frac{||S||^{m-1}}{k^{m-1}}||x_1||$$

$$\leq \frac{||x||}{1-\delta} \quad \text{with } \delta = \frac{||S||}{k} < 1.$$

Thus $Ty_x = Sy_x$, which says that $y_x \in (T-S)^{-1}(0)$ whenever $x \in T^{-1}(0)$; therefore we have that $\alpha(T) \leq \alpha(T-S)$.

For the argument for β, observe, by (0.1) and (0.4), that

$$T^{-1}(0) \subseteq T^{\infty}(X) \Longrightarrow T^{-1}(0)^{\perp} \supseteq T^{n}(X)^{\perp} \ (n = 1, 2, \cdots)$$

$$\Longrightarrow T^{\dagger}(X^{\dagger}) \supseteq (T^{\dagger n})^{-1}(0) \ (n = 1, 2, \cdots) \Longrightarrow T^{\dagger -1}(0) \subseteq T^{\dagger\infty}(X^{\dagger}),$$

where $K^{\perp}$ denotes the annihilator of K.

Now applying the equality for α to $T^{\dagger}$ gives the equality for β. Towards (2.2), we claim that

$$\begin{aligned} \dim T^{\wedge -1}(0) \leq \dim T^{-1}(0) &= \dim (T-\lambda)^{-1}(0) \\ &= \dim (T-\lambda)^{\wedge -1}(0) \leq \dim T^{\wedge -1}(0) : \end{aligned}$$

indeed the first inequality is evident, the second equality is the assumption, the third equality is (0.3), and the last inequality is (1.1). Thus $\dim T^{-1}(0) = \dim T^{\wedge -1}(0)$. Therefore if $T^{-1}(0) \cap T^{\infty}(X)$ is finite dimensional, we can conclude that $T^{-1}(0) \subseteq T^{\infty}(X)$.

If, in (2.2), the assumption "dim $\left(T^{-1}(0) \cap T^{\infty}(X)\right) < \infty$ "is dropped , (2.2) may fail: for example, take $X = \ell_2$ and consider the operator

$$T(x_1, x_2, x_3, x_4, \cdots) = (0, 0, x_5, x_7, x_9, x_{11}, \cdots).$$

Then T has a closed range and $\alpha(T-\lambda) = \alpha(T) = \infty$ for sufficiently small λ; but T is not hyperexact.

If $T \in BL(X, X)$ is semi-Fredholm then, by the punctured neighborhood theorem ([1],[2],[5]) , there is $\epsilon > 0$ for which $\alpha(T-\lambda)$ and $\beta(T-\lambda)$ are both constant for $0 < |\lambda| < \epsilon$. Thus we can define ([9]) the *jump*, $j(T)$, of a semi-Fredholm operator T:

$$j(T) = \alpha(T) - \alpha(T-\lambda) \quad \text{for } 0 < |\lambda| < \epsilon \quad \text{if } T \text{ is upper semi-Fredholm}$$

and

$$j(T) = \beta(T) - \beta(T-\lambda) \quad \text{for } 0 < |\lambda| < \epsilon \quad \text{if } T \text{ is lower semi-Fredholm.}$$

From Theorem 2, we can see that if T is semi-Fredholm then ([10])

$$j(T) = 0 \Longleftrightarrow T \text{ is hyperexact.} \tag{2.4}$$

Then *Kato's decomposition theorem* ([7],[10]) says that if $T \in BL(X, X)$ is semi-Fredholm, T can be decomposed as:

$$T = T_1 \oplus T_2, \tag{2.5}$$

where T_1 is nilpotent and T_2 is hyperexact.

We can now have (2.5) for an absence of the index. For this we need ([6]):

3. LEMMA. *If $T \in BL(X, Y)$ and $S \in BL(Y, Z)$ for Banach spaces X, Y and Z then*

$$S(Y) \text{ and } S^{-1}(0) + T(X) \text{ are both closed} \Longrightarrow ST(X) \text{ is closed} \tag{3.1}$$

and

$$S^{-1}(0) + T(X) \text{ is closed and } S^{-1}(0) \cap T(X) \text{ is finite dimensional} \Longrightarrow T(X) \text{ is closed.} \tag{3.2}$$

For the sake of completeness, we sketch a proof.

The implication (3.1) is a lemma of Kato ([7] Lemma 331). When the intersection $S^{-1}(0) \cap T(X)$ is $\{0\}$ then (3.2) is a simple application of the open mapping theorem ([2] Theorem 4.8.2); consider the operator $W : Z = S^{-1}(0) \times X/T^{-1}(0) \to Y$ defined by setting

$$W(y, x + T^{-1}(0)) = y + Tx \text{ for each } y \in Y, x \in X.$$

Evidently W is well-defined, bounded, and onto with finite dimensional null space: there is therefore $W' : Y \to Z$, also bounded and linear, for which $W = WW'W$. Now observe that

$$E = TW' : Y \to Y \text{ satisfies } E = E^2 \text{ and } T(X) = E(Y) = (I - E)^{-1}(0) \text{ is closed .}$$

We meet a decomposition theorem:

4. THEOREM. *Let $T \in BL(X, X)$. If $T^{-1}(0) \cap T(X)$ is finite dimensional and if there is $k > 0$ for which $T^{-k}(0) + T(X)$ is complemented then T can be decomposed as $T = T_1 \oplus T_2$, where T_1 is nilpotent and T_2 is hyperexact.*

PROOF. Suppose $T^{-1}(0) \cap T(X)$ is finite dimensional and $T^{-k}(0) + T(X)$ is complemented. Remembering the isomorphism ([2])

$$(ST)^{-1}(0)/T^{-1}(0) \cong T(X) \cap S^{-1}(0),$$

we have

$$T^{-k}(0)/T^{-1}(0) \cong T^{-(k-1)}(0) \cap T(X) \cong \bigoplus_{i=1}^{k-1} T^{-1}(0) \cap T^i(X),$$

where each summand is finite dimensional. Thus

$$\left(T^{-k}(0) + T(X)\right) / \left(T^{-1}(0) + T(X)\right) \quad \text{is finite dimensional}$$

and hence $T^{-1}(0) + T(X)$ is complemented. Thus we can find closed subspaces L, W and Z of X for which

$$X = \overbrace{L \oplus T^{-1}(0) \cap T(X)}^{T^{-1}(0)} \oplus W \oplus Z \quad \text{and} \quad T^{-1}(0) \cap T(X) \oplus W = T(X).$$

Write

$$M = T^{-1}(0) \cap T(X) \oplus W \oplus Z.$$

Then $T(M) \subseteq M$ and

$$T_L = 0, \quad T_M(M) = T(X) \quad \text{and} \quad T_M^{-1}(0) = T^{-1}(0) \cap T(X).$$

Since, by (3.2) and our assumption, T_M has a closed range with finite dimensional null space, T_M is upper semi-Fredholm. By Kato's decomposition theorem, T_M can be decomposed as

$$T_M = T'_M \oplus T''_M,$$

where T'_M is nilpotent and T''_M is hyperexact. Now putting $T_1 = T_L \oplus T'_M$ and $T_2 = T''_M$ gives the result. •

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Department of Mathematics Education
Kwan Dong University
Kangnung 210-701, Korea