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ON TRANSFER THEOREMS FOR FINITE GROUPS

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ABSTRACT. We shall study some transfer theorems of finite groups with respect to a certain commutator subgroup, called "F-commutator" relative to any field F and apply the transfer to the fusion of a group H or to the focal subgroup of H.

1. Introduction

Let H be a subgroup of a finite group G with $|G:H| = \mu$ and A be any abelian group. For a homomorphism $\theta : H \to A$, it is possible to construct a homomorphism $\theta^* : G \to A$ from θ in a canonical way. In fact, θ^* is the transfer map defined by

$$\theta^*(g) = \prod_{i=1}^{\mu} \theta(s_i g \overline{s_i g}^{-1}), \ g \in G$$

where $S = \{s_i\}_{i=1}^{\mu}$ is a set of transversals of H in G, and $\bar{g}$ is the unique element in S such that $g = h\bar{g}$, $h \in H$. The transfer map was originally defined analogous to the determinations of a monomial representation, and offers an effective method of solving the problem of finding the properties by which the group under investigation differs from its commutator subgroup.

Conjugacy in H plays very important role in transfer homomorphism, as in (1). Many transfer theorems assert that conjugacy of G is completely determined by "local" property - specially, by the normalizers of the nonidentity p-subgroups of G.

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In this paper, we shall use the F-conjugacy ([2], [3]) instead of conjugacy in H and then define a notion F-focal subgroup in order to study transfer theorems.

2. Preliminaries

It is known that that the transfer homomorphism is independent of the choices of transversal. Thus by choosing a specific set of conjugate elements of g as a transversal set, the map reduces to

$$\theta^*(g) = \prod_{i=1}^{k} \theta(s_i g^{l_i} s_i^{-1}) \tag{1}$$

where g^{l_i} is the first positive power of g such that $Hs_i g^{l_i} = Hs_i$, and of course $\sum_{i=1}^{k} l_i = \mu$ (refer to [8]). The most important case of transfer arises when θ is a map from H to H/H', where H' is the commutator subgroup of H. In this case, θ^* is referred to the transfer of G into H. Two cases of special interest are when H is central or H is a Sylow subgroup of G.

Let F be a field of char $p \geq 0$, and E be a normal closure of F. For integer n divisible by $\exp(G)$, write $n = n_p n_{p'}$, where $n_p = 1$ if $p = 0$. For each $\sigma \in \mathrm{Gal}\,(E/F) = \mathcal{G}$, let $m(\sigma)$ be a positive integer such that

$$\zeta_{n_{p'}}^{\sigma} = \zeta_{n_{p'}}^{m(\sigma)}, \quad m(\sigma) \equiv 1 \pmod{n_p}, \tag{2}$$

where $\zeta_{n_{p'}} \in E$ is a primitive $n_{p'}$-th root of unity. For any $a, b \in G$ and $\sigma \in \mathcal{G}$, the element $[a,b]_\sigma = a^{-1}b^{-1}a^{m(\sigma)}b$ is an F-commutator of a and b, and the group generated by all F-commutators is the F-commutator subgroup, denoted by $G'(F)$. Moreover, a and b are F-conjugate if $a = x^{-1}b^{m(\sigma)}x$ for some $x \in G$, $\sigma \in \mathcal{G}$. If F is an algebraically closed field or $m(\sigma) = 1$ for all $\sigma \in \mathcal{G}$, then $[a,b]_\sigma = [a,b]$, so that $G'(F) = G'$, the commutator subgroup.

A group G is an abelian F-group if all representations of G in E of degree 1 have values in F. For an abelian group G, G is an abelian F-group if ane only if F contains a primitive root of unity $\zeta_{\exp(G)_{p'}}$, or equivalently $\exp(G) | m(\sigma) - 1$ for each $\sigma \in \mathcal{G}$. Thus, G/N is an abelian

F-group if and only if $G'(F) \subseteq N$; and it follows that $G'(F)$ is the unique smallest normal subgroup N of G whose quotient G/N is an abelian F-group (refer to [2]).

The following lemma is useful for next use.

LEMMA 1. [2].

(a) *Every finite direct product of abelian F-groups is an abelian F-group.*

(b) *Any subgroup and any epimorphic image of an abelian F-group are abelian F-groups.*

(c) *If G/N_1 is an abelian F-group and $N_1 \subseteq N_2$ where N_i is a normal subgroup of G for $i = 1, 2$, then G/N_2 is an abelian F-group.*

3. Abelian F-groups and F-commutator subgroups

We shall first study some properties of abelian F-group which are analogues of properties of abelian groups.

THEOREM 2.

(a) *Any group homomorphism carries an F-commutator to the F-commutator of the image.*

(b) *Let f be a homomorphism from a group G onto another group H. Then, f maps the F-commutator subgroup of G onto that of H. That is, $f(G'(F)) = (f(G))'(F)$.*

(c) *The F-commutator subgroup $G'(F)$ of G is fully invariant; in particular, $G'(F)$ is characteristic.*

(d) *For a normal subgroup N of a group G contained in $G'(F)$, let $\bar{G} = G/N$. If H is a subgroup of G corresponding to the F-commutator subgroup $(\bar{G})'(F)$, i.e., $H/N = (\bar{G})'(F)$, then $H = NG'(F)$.*

(e) *If $G = H_1 \times \cdots \times H_k$, then $G'(F) = H_1'(F) \times \cdots \times H_k'(F)$.*

PROOF. Choose a positive integer n divisible by $|G|$, and then $m(\sigma)$ for n as in (2). For any homomorphism f on G the integers n and $m(\sigma)$ also work for $\mathrm{Im}(f)$, too, since $|\mathrm{Im}(f)|$ divides $|G|$. Thus $[f(x), f(y)]_\sigma = f(x)^{-1}f(y)^{-1}f(x)^{m(\sigma)}f(y) = f([x,y]_\sigma)$. This is (a). Further this proves (b), that is, for any $[y_1, y_2]_\sigma \in (f(G))'(F)$ with

$y_i \in f(G)$, there are $x_i \in G$ such that $f(x_i) = y_i$. Thus $[y_1, y_2]_\sigma = [f(x_1), f(x_2)]_\sigma = f[x_1, x_2]_\sigma \in f(G'(F))$. Conversely, choose any element $f(a) \in f(G'(F))$ where $a = [x_1, x_2]_\sigma \in G'(F)$. Then $f(a) = f([x_1, x_2]_\sigma) = [f(x_1), f(x_2)]_\sigma \in (f(G))'(F)$.

(c) is obvious and for (d), let f be a surjection from G to $G/N = \bar{G}$. Then $G'(F)/N = f(G'(F)) = (f(G))'(F) = (\bar{G})'(F) = H/N$, hence $H = G'(F)$ and $H \subseteq NG'(F)$. Conversely, $N \subseteq H$ and $G'(F) \subseteq H$ implies that $NG'(F) \subseteq H$.

It is enough to prove (e) when $k = 2$. Let $H_1'(F) \times H_2'(F) = D$. Since $H_i'(F) \subset G'(F)$ for $i = 1, 2$, $D \subseteq G'(F)$. Moreover, $G/D \cong H_1/H_1'(F) \times H_2/H_2'(F)$. Since $H_i/H_i'(F)$ is an abelian F-group for $i = 1, 2$, so is G/D by Lemma 1. Hence $G'(F) \subseteq D$.

A finite group G is called F-solvable provided that there is an F-solvable series of G; that is, a subnormal series

$$(e) = G_k \subseteq G_{k-1} \subseteq \cdots \subseteq G_1 \subseteq G_0 = G$$

satisfies that each factor group G_i/G_{i+1} is an abelian F-group. The next theorem contains a known result that G is solvable if and only if any solvable series of G is terminated by (e).

THEOREM 3. *A finite group G is F-solvable if and only if the F-commutator series reaches (e) in a finite number of steps; thus, $G^{(k)} = (e)$ for some $k > 0$. Here $G^{(k)}$ is defined by $G^{(k)} = (G^{(k-1)})'(F)$, $\cdots, G^{(2)} = (G^{(1)})'(F)$, $G^{(1)} = G'(F)$.*

PROOF. Suppose that $G^{(k)} = (e)$ for some k. Consider $(e) = G^{(k)} \subseteq \cdots \subseteq G^{(1)} = G'(F) \subseteq G$. This is a subnormal series of G. For each i, since $G^{(i+1)}$ is an F-commutator subgroup $(G^{(i)})'(F)$ of $G^{(i)}$, $G^{(i)}/G^{(i+1)}$ is an abelian F-group. On the other hand, let $(e) = G_n \subseteq \cdots \subseteq G_1 \subseteq G_0 = G$ be an F-solvable series of G. Then G_i/G_{i+1} is an abelian F-group, so that the F-commutator subgroup $G_i'(F)$ is contained in G_{i+1}. We claim that $G^{(i)} \subseteq G_i$ for all $i = 1, \ldots, n$. Certainly, an abelian F-group G/G_1 implies that $G^{(1)} = G'(F) \subseteq G_1$. Suppose that $G^{(i)} \subseteq G_i$ for some i. Since G_i/G_{i+1} is an abelian F-group, we have $G_i'(F) \subseteq G_{i+1}$, and it follows that $G^{(i+1)} = (G^{(i)})'(F) \subseteq G_i'(F) \subseteq G_{i+1}$. This completes the proof.

The next corollary follows immediately.

COROLLARY 4. *Let G be an F-solvable group. Then any subgroup, as well as any factor group, of G is F-solvable. Conversely, if a normal subgroup H and the factor group G/H are both F-solvable, then G is F-solvable.*

The F-kernel of G is the intersection of the kernel of all homomorphisms of G to F^*, and denoted by G_F. Then G/N is an abelian F-group and p'-group if and only if $G_F \subseteq N$. The p-commutator subgroup $G'(p)$ is the intersection of all normal subgroups of G whose quotient is an abelian p-group, while the p'-commutator subgroup $G'(p')$ is that with resepct to p'.

LEMMA 5. *(refer to [2]) Let G be a group.*

(a) $G_F \cdot G'(p) = G$; $G_F \cap G'(p) = G'(F)$.
(b) $G'(F) \cdot G'(p') = G_F$; $G'(F) \cap G'(p') = G'$.
(c) $G'(F)/G' = (G_F/G')_{p'}$; $G_F/G'(F) = (G/G'(F))_p$.

The above lemma shows that $G'(F)$ and G_F can be expressed in terms of each other, and if $p = 0$ or if $p > 0$ and p does not divide $|G|$ then $G_F = G'(F)$, $G'(p) = G$ and $G'(p') = G'$. Analogous results in Theorem 2 are true for F-kernel G_F of G.

COROLLARY 6. *Let G be a group.*

(a) *$f(G_F) = f(G)_F$ for any homomorphism f on G.*
(b) *Let N be a normal subgroup of G, $\bar{G} = G/N$, and let H be a subgroup of G corresponding to $\bar{G}_F$ so that $H/N = \bar{G}_F$. Then $H = NG_F$.*
(c) *If $G = H \times K$, then $G_F = H_F \times K_F$. This can be extended to a finite direct product group.*

4. Transfer theorems

Though the transfer θ^* of a group G into an abelian group A via $\theta : H \to A$ is constructed in a canonical way, it is in general difficult to calculate $\theta^*(g)$ explicitly and decide whether $g \in \ker(\theta^*)$. The fusion of g in H gives a very useful information about that.

Two elements of H are said to be fused in G if they are conjugate in G. The focal subgroup $\mathrm{Foc}_G(H)$ of H in G due to D.G.Higman [5] is the subgroup generated by the quotients of pairs of elements of H which are fused in G. It was shown that $\mathrm{Foc}_G(H)$ contains H' and is normal in H so that $H/\mathrm{Foc}_G(H)$ is abelian. Further, if K is a subgroup of H containing $\mathrm{Foc}_G(H)$ then K is normal in H and H/K is abelian. Therefore, for $h \in H$, by (1) we have that

$$\theta^* : G \to H/K, \quad \theta^*(h) = h^{\mu}K \text{ where } \mu = |G:H|. \tag{3}$$

Two $x, y \in H$ are said to be F-fused in G, denoted by $x \sim_{F,G} y$, if they are F-conjugate in G. That is, if x and $g^{-1}x^{m(\sigma)}g$ are in H for some $g \in G$ and $\sigma \in \mathcal{G}$, then $g^{-1}x^{m(\sigma)}g$ is F-fused to x.

LEMMA 7. *Let H be a subgroup of a group G. Then for $x, y \in H$, $x \sim_{F,G} y$ if and only if $y \sim_{F,G} x$. Moreover, $\sim$ is an equivalence relation.*

PROOF. $x \sim_{F,G} y$ if and only if $y = g^{-1}x^{m(\sigma)}g$ for some $g \in G$ and $\sigma \in \mathcal{G}$. Then $x^{m(\sigma)} = gyg^{-1}$, and $x = x^{m(\sigma)m(\sigma^{-1})} = gy^{m(\sigma^{-1})}g^{-1}$ so that $y \sim_{F,G} x$, and vice versa. Here, we used the fact that $m(\sigma)m(\sigma^{-1}) = m(\sigma\sigma^{-1}) = m(1) = 1$.

We now define the F-focal subgroup $\mathrm{Foc}_{F,G}(H)$ of H by the subgroup generated by quotients of pairs of elements of H which are F-fused in G, that is,

$$\begin{aligned}\mathrm{Foc}_{F,G}(H) &= \langle h^{-1}k \mid h, k \in H, h \sim_{F,G} k\rangle \\ &= \langle h^{-1}(h^g)^{m(\sigma)} \mid h \in H,\ (h^g)^{m(\sigma)} \in H,\ g \in G, \sigma \in \mathcal{G}\rangle.\end{aligned}$$

Certainly, $\mathrm{Foc}_G(H)$ is contained in $\mathrm{Foc}_{F,G}(H)$.

Suppose we take a positive integer n divisible by $\exp(G)$ and then $m(\sigma)$ as in (2). Then for any subgroup H of G and any factor group G/N, both $\exp(H)$ and $\exp(G/N)$ divide $\exp(G)$, so that the n and $m(\sigma)$ work for H as well as G/N.

THEOREM 8. *Let H be a subgroup of G. Then the F-focal subgroup $Foc_F(H)$ of H contains $H'(F)$ and is a normal in H, so that $H/Foc_F(H)$ is an abelian F-group. Further, if K is a subgroup of H containing $Foc_F(H)$ then K is normal in H and H/K is an abelian F-group.*

PROOF. Since $[u,v]_\sigma = u^{-1}(u^v)^{m(\sigma)}$ for any $u, v \in H$ and $\sigma \in \mathcal{G}$, it is clear that $H'(F) \subset \mathrm{Foc}_F(H)$. And for any element h in H, $u^{-1}h^{-1}(h^g)^{m(\sigma)}u = u^{-1}h^{-1}uu^{-1}g^{-1}uu^{-1}h^{m(\sigma)}uu^{-1}gu = (h^u)^{-1}(g^u)^{-1}(h^u)^{m(\sigma)}g^u =\in \mathrm{Foc}_F(H)$, because $(h^u) \in H$ and $((h^u)^{g^u})^{m(\sigma)} = u^{-1}(h^g)^{m(\sigma)}u \in H$. Therefore, $\mathrm{Foc}_F(H)$ is normal in H containing $H'(F)$ and $H/\mathrm{Foc}_F(H)$ is an abelian F-group.

Suppose that a subgroup K of H contains $\mathrm{Foc}_F(H)$. Then K is normal in H and H/K is abelian, since $H' \subset \mathrm{Foc}_G(H) \subset \mathrm{Foc}_{F,G}(H) \subset K \subset H$. Furthermore since $H'(F) \subset \mathrm{Foc}_{F,G}(H) \subset K \subset H$, the fact $H/H'(F)$ is an abelian F-group implies that H/K is an abelian F-group too, by Lemma 1.

Let K be the same group as in Theorem 8, and let $\theta^* : G \to H/K$ be the transfer of $\theta : H \to H/K$. Then $\theta^*(h) = h^\mu K$ for $h \in H$ by (3). Further this map also sends $h^{m(\sigma)}$ to $h^\mu K$ for any $\sigma \in \mathcal{G}$. Indeed, since $s_i^{-1}(h^{l_i})^{m(\sigma)}s_i$ is F-fused to h^{l_i}, we have that $(h^{l_i})^{-1}s_i^{-1}(h^{l_i})^{m(\sigma)}s_i \in \mathrm{Foc}_{F,G}(H) \subset K$, and $(s_i^{-1}(h^{l_i})^{m(\sigma)}s_i)K = h^{l_i}K$. Hence, for each $\sigma \in \mathcal{G}$, we have

$$\theta^*(h^{m(\sigma)}) = \prod_{i=1}^{k}(s_i^{-1}(h^{m(\sigma)})^{l_i}s_i)K = \prod_{i=1}^{k} h^{l_i}K = h^\mu K. \tag{4}$$

Note that, when G is an abelian F-group, $\exp(G)$ divides $m(\sigma)-1$ so that $g^{m(\sigma)} = g$ for each $g \in G$, $\sigma \in \mathcal{G}$ and $m(\sigma)$. In the case of Theorem 8, since H/K is an abelian F-group, $(hK)^{m(\sigma)} = h^{m(\sigma)}K = hK$ for any $h \in H$. This fact together with (3) shows (4) directly.

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