Comm. Korean Math. Soc. **11** (1996), No. 4, pp. 903–916

MINIMUM PERMANENTS ON CERTAIN FACES OF Ω_n

SI-JU KIM AND JAE-BONG SHIM

ABSTRACT. In this paper we investigate the minimum permanents and minimizing matrices on the faces $\Omega(D)$ of Ω_n for two fully indecomposable (0,1) matrices D which are slight changes of both a convertible matrix and the matrix with zero trace.

1. Introduction

An $n \times n$ matrix with nonnegative entries is called a *doubly stochastic* matrix if all of its row sums and column sums are equal to 1. The set of all n-square doubly stochastic matrices is denoted by Ω_n.

Let $D = [d_{ij}]$ be an n-square (0,1) matrix, and let

$$\Omega(D) = \{X = [x_{ij}] \in \Omega_n | x_{ij} = 0 \text{ whenever } d_{ij} = 0\}.$$

Then $\Omega(D)$ is a face of the polytope Ω_n for D with positive permanent. Since it is compact, there exists a matrix $A \in \Omega(D)$ such that $perA \leq perX$ for all $X \in \Omega(D)$. Such a matrix A is called a *minimizing* matrix of $\Omega(D)$. In 1981, Falikman and Egorycêv[2] proved the van der Waerden conjecture: if $A \in \Omega_n$, then

$$perA \geq perJ_n = \frac{n!}{n^n}$$

where J_n is n-square matrix all of whose entries equal $\frac{1}{n}$. After the resolution of the conjecture, there has been interested in determining minimizing matrices and minimum permanents on faces of Ω_n [3,4,5,6,7,8].

Received April 11, 1996. Revised August 30, 1996.

1991 AMS Subject Classification: 53C25.

Key words and phrases: faces, permanents.

This work was partially supported by the Grant of TGRC-KOSEF and Andong University.

Recall that an n-square nonnegative matrix is said to be *fully indecomposable* if it contains no $k \times (n-k)$ zero submatrix for $k = 1, \cdots, n-1$. Brualdi[3] defined an n-square $(0,1)$ matrix D to be *cohesive* if there is a matrix Z in the interior of $\Omega(D)$ for which $perZ = min\{perX \mid X \in \Omega(D)\}$. The barycenter $\mathbf{b}(D)$ of $\Omega(D)$ is given by $\mathbf{b}(D) = \frac{1}{perD} \sum_{P \leq D} P$, where the summation extends over the set of all permutation matrices P with $P \leq D$, and $perD$ is their number. An n-square (0,1)-matrix D said to be *barycentric* if $per\mathbf{b}(D) = min\{perX \mid X \in \Omega(D)\}$.

In this paper we investigate the minimum permanents and minimizing matrices on the faces $\Omega(D)$ of Ω_n for two fully indecomposable (0,1) matrices D which are slight changes of both a convertible matrix and the matrix with zero trace.

Let I_n denote the identity matrix of order n and let $J_{k,p}(and\ O_{k,p})$ be the $k \times p$ matrix all of whose entries are equal to $1(and\ 0)$ respectively.

2. Minimum Permanent of $\Omega(E_{k,p})$

We shall rewrite the following well-known results [5] as Lemmas before we state our first result.

LEMMA 1. *If $D = [d_{ij}]$ be a n-square fully indecomposable $(0,1)$ matrix, and $A = [a_{ij}]$ be a minimizing matrix on $\Omega(D)$, then A is fully indecomposable.*

LEMMA 2. *Let $A = [a_{ij}]$ be a minimizing matrix on $\Omega(D)$. Then for (i,j) such that $d_{ij} = 1$,*

$$perA(i \mid j) = perA \quad if \quad a_{ij} > 0$$

$$perA(i \mid j) \geq perA \quad if \quad a_{ij} = 0.$$

LEMMA 3. *If A is a minimizing matrix on $\Omega(D)$, $i_1, \cdots, i_t$ rows(columns) have the same Z pattern, then the matrix obtained from A by replacing each of these rows (columns) by the average of the t rows(columns) is also minimizing in $\Omega(D)$.*

Consider the $(k+p+1)$-square $(0,1)$ matrix $E_{k,p}$;

$$E_{k,p} = \left(\begin{array}{c|ccc|ccc} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & & & & & & \\ \vdots & & J_{k,k} & & & J_{k,p} & \\ 0 & & & & & & \\ \hline 1 & & & & & & \\ \vdots & & O_{p,k} & & & I_p & \\ 1 & & & & & & \end{array}\right)$$

Notice that if the submatrix $E_{k,p}[2,3,\cdots,k+1|2,3,\cdots,k+1]$ of $E_{k,p}$ is replaced by I_k, then the new matrix $E^*_{k,p}$ is convertible for some p, k. That is, for some (1,−1) matrix H, $per(E^*_{k,p}) = det(E^*_{k,p} \circ H)$ where $\circ$ means the Hadamard(entrywise) product.

Now we determine the minimum permanents on $\Omega(E_{k,p})$.

THEOREM 2.1. *For $k \geq 2$,*

(1) $E_{k,p}$ is cohesive for p=1,2, and the minimum permanent of the face $\Omega(E_{k,p})$ is

$$k!a^k(\frac{p+ka-1}{p})^p,$$

where $a = \frac{2k^2-k+p-\sqrt{(4p+1)k^2-2pk+p^2}}{2k^2(k-1)}$.

(2) $E_{k,p}$ is not cohesive for $p \geq 3$, and the minimum permanent of $\Omega(E_{k,p})$ is

$$(k-1)!(\frac{k-1}{k^2})^{k-1}\frac{(p-1)^{p-1}}{p^p}.$$

PROOF. By Lemma 3, we may assume that a minimizing matrix of

$\Omega(E_{k,p})$ is of the form

(1)

$$A_{k,p} = \left(\begin{array}{c|ccc|ccc} z & b & \cdots & b & 0 & \cdots & 0 \\ \hline 0 & a & \cdots & a & x_1 & \cdots & x_p \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & a & \cdots & a & x_1 & \cdots & x_p \\ \hline y_1 & & & & c_1 & & 0 \\ \vdots & & O_{p,k} & & & \ddots & \\ y_p & & & & 0 & & c_p \end{array}\right),$$

where $z + kb = z + \sum_{i=1}^{p} = 1$, $b + ka = ka + \sum_{i=1}^{p} = 1$, $kx_i + c_i = 1$, $y_i + c_i = 1$, for $i = 1, \cdots, p$.

By Lemma 1 and Lemma 2, $per A_{k,p} = per A_{k,p}(i + k + 1 \mid 1)$ for $i = 1, \cdots, p$.

Thus

$$\begin{aligned} kbx_1c_2\cdots c_pa^kk! &= kbx_2c_1c_3\cdots c_pa^kk! \\ &= \cdots \\ &\vdots \\ &= kbx_pc_1\cdots c_{p-1}a^kk! \end{aligned}$$

From this equation and (1), we have

$$x_1 = x_2 = \cdots = x_p$$

$$c_1 = c_2 = \cdots = c_p$$

$$y_1 = y_2 = \cdots = y_p.$$

Hence

$$z = 1 - k + k^2a, \quad b = 1 - ka, \quad x_i = \frac{1 - ka}{p},$$

$$c_i = \frac{p - k + k^2a}{p}, \quad y_i = \frac{k - k^2a}{p} \quad for\ \ i = 1, \cdots, p.$$

Since we should have $b > 0$ and $z \geq 0$ by Lemma 1,

$$\frac{k-1}{k^2} \leq a < \frac{1}{k}.$$

Thus

$$\begin{aligned} per A_{k,p} &= (1-k+k^2a) per A_{k,p}(1 \mid 1) + k(1-ka) per A_{k,p}(1 \mid 2) \\ &= \frac{k!}{p^p} a^{k-1}(p-k+k^2a)^{p-1}\{k^4(1-k)a^3 \\ &\quad + k^2(3k^2-2k+p+1)a^2 - (3k^3+pk+k-p-k^2)a + k^2\}. \end{aligned}$$

Let

(2)

$$\begin{aligned} f(a) =& \frac{k!}{p^p} a^{k-1}(k^2x+p-k)^{p-1}\{k^4(1-k)a^3 + k^2(3k^2-2k+p+1)a^2 \\ & -(3k^3+pk+k-p-k^2)a + k^2\}. \end{aligned}$$

Then $per A_{k,p} = f(a)$, where $\frac{k-1}{k^2} \leq a < \frac{1}{k}$.
For $p = 1$,

$$\begin{aligned} f(a) &= (1-k+k^2a)^2 k! a^k + k^2(1-ka)^3 k! a^{k-1} \\ &= -k! a^{k-1}\{k^4(k-1)a^3 - k^2(3k^2-2k+2)a^2 \\ &\quad + (3k^3-k^2+2k-1)a - k^2\} \end{aligned}$$

and

$$f'(a) = -k \cdot k! a^{k-2}\{k(k+2)a-(k-1)\}\{k^2(k-1)a-(2k^2-k+1)a+k\}.$$

Thus $f(a) = per A_{k,1}$ has the minimum value at $a = \frac{2k^2-k+1-\sqrt{5k^2-2k+1}}{2k^2(k-1)}$ under the condition $\frac{k-1}{k^2} \leq a < \frac{1}{k}$.

Similarly, for $p = 2$, $f(a) = per A_{k,2}$ has the minimum value at $a = \frac{2k^2-k+2-\sqrt{9k^2-4k+4}}{2k^2(k-1)}$ under the condition $\frac{k-1}{k^2} \leq a < \frac{1}{k}$.

Hence for $p = 1$ or 2, $f(a) = per A_{k,p}$ has the minimum value at

$$a = \frac{2k^2 - k + p - \sqrt{(4p+1)k^2 - 2pk + p^2}}{2k^2(k-1)},$$

and

$$per A_{k,p} = per A_{k,p}(1 \mid 1) = k! a^k (\frac{p + ka - 1}{p})^p.$$

Thus the corresponding entry in $A_{k,p}$ to each nonzero entry in $E_{k,p}$ is nonzero.

Hence $E_{k,p}$ is cohesive.

For $p \geq 3$, differentiating $f(a)$, we have

$$f'(a) = -k\frac{k!}{p^p} a^{k-2}(k^2 a + p - k)^{p-2}\{k^2(k-1)a^2 + (k - p - 2k^2)a + k\}$$
$$\times \{k^3(p + k + 1)a^2 - 2k(k^2 - p)a + (k-1)(k-p)\}.$$

Now we put

$$p(a) = k^2 a + p - k,$$
$$q(a) = k^2(k-1)a^2 + (k - p - 2k^2)a + k,$$
$$r(a) = k^3(p + k + 1)a^2 - 2k(k^2 - p)a + (k-1)(k-p).$$

Then the roots of $r(a) = 0$ are

$$\begin{cases} a_1 &= \frac{k^2 - p - \sqrt{(p^2 - p + 1)k^2 - (p^2 + p)k + p^2}}{k^2(p+k+1)} \\ a_3 &= \frac{k^2 - p + \sqrt{(p^2 - p + 1)k^2 - (p^2 + p)k + p^2}}{k^2(p+k+1)}, \end{cases}$$

the root of $p(a) = 0$ is $a_2 = \frac{k-p}{k^2}$, and the roots of $q(a) = 0$ are

$$\begin{cases} a_4 &= \frac{2k^2 - k + p - \sqrt{(4p+1)k^2 - 2pk + p^2}}{2k^2(k-1)} \\ a_5 &= \frac{2k^2 - k + p + \sqrt{(4p+1)k^2 - 2pk + p^2}}{2k^2(k-1)}. \end{cases}$$

Notice that a_1, a_2, a_3, a_4 and a_5 are real numbers. It is easy to show that a_3 is the largest real number among a_1, a_2 and a_3. Hence we compare a_3 with $\frac{k-1}{k^2}$.

$$\begin{aligned}
&\frac{k-1}{k^2} - \frac{k^2 - p + \sqrt{(p^2-p+1)k^2 - (p^2+p)k + p^2}}{k^2(p+k+1)} \\
&= \frac{1}{k^2(p+k+1)}\{(pk-1) - \sqrt{(p^2-p+1)k^2 - (p^2+p)k + p^2}\}.
\end{aligned}$$

Let

$$\begin{aligned}
g_1(k) &= (pk-1)^2 - \{(p^2-p+1)k^2 - (p^2+p)k + p^2\} \\
&= (p-1)\{k^2 + pk - (p+1)\}.
\end{aligned}$$

Then $g_1(2) = (p-1)(p+3) > 0$, and $g_1'(k) = (p-1)(2k+p) > 0$ for $k \geq 2$, $p \geq 3$. Thus we have

$$a_3 < \frac{k-1}{k^2}.$$

Now we compare a_4 with $\frac{k-1}{k^2}$;

$$\begin{aligned}
&\frac{k-1}{k^2} - \frac{2k^2 - k + p - \sqrt{(4p+1)k^2 - 2pk + p^2}}{2k^2(k-1)} \\
&= \frac{1}{2k^2(k-1)}[\sqrt{(4p+1)k^2 - 2pk + p^2} - \{3k + (p-2)\}].
\end{aligned}$$

Now let

$$\begin{aligned}
g_2(k) &= \{(4p+1)k^2 - 2pk + p^2\} - \{3k + (p-2)\}^2 \\
&= (4p-8)k^2 - (8p-12)k + 4(p-1) \\
&= 4\{(p-2)k^2 - (2p-3)k + (p-1)\}.
\end{aligned}$$

Then $g_2(2) = 4(p-3) \geq 0$ and $g_2'(k) = 8(p-2)k - 4(2p-3) > 0$ for $k \geq 2$, $p \geq 3$. Hence

$$a_4 \leq \frac{k-1}{k^2} < \frac{1}{k} < a_5.$$

Therefore $f(a) = perA_{k,p}$ (condition $\frac{k-1}{k^2} \le a < \frac{1}{k}$) has the minimum value at $a = \frac{k-1}{k^2}$ and

$$\begin{aligned} perA_{k,p} &= perA_{k,p}(1 \mid 2) \\ &= (k-1)!(\frac{k-1}{k^2})^{k-1}\frac{(p-1)^{p-1}}{p^p}. \end{aligned}$$

Since $a = \frac{k-1}{k^2}$, $z = 0$ and hence $E_{k,p}$ is not cohesive. □

THEOREM 2.2. *$E_{1,p}$ is cohesive for any natural number p, and the minimum permanent of $\Omega(E_{1,p})$ is*

$$\frac{p^p}{(p+1)^{p+1}}.$$

PROOF. Without loss of generality, we may asume that a minimizing matrix of $\Omega(E_{1,p})$ is the form of

$$A_{1,p} = \left(\begin{array}{c|c|cccc} a & 1-a & 0 & \cdots & & 0 \\ \hline 0 & a & \frac{1-a}{p} & \cdots & & \frac{1-a}{p} \\ \hline \frac{1-a}{p} & 0 & \frac{p+a-1}{p} & & & 0 \\ \vdots & \vdots & & & \ddots & \\ \frac{1-a}{p} & 0 & 0 & & & \frac{p+a-1}{p} \end{array}\right),$$

where $0 < a < 1$. Thus

$$\begin{aligned} perA &= a^2(\frac{p+a-1}{p})^p + (1-a)^2\frac{1-a}{p}(\frac{p+a-1}{p})^{p-1} \\ &= \frac{1}{p^p}(a+p-1)^{p-1}\{(p+a-1)a^2 + (1-a)^3\} \\ &= \frac{1}{p^p}(a+p-1)^{p-1}\{(p+2)a^2 - 3a + 1\}. \end{aligned}$$

Let

$$f(a) = (a+p-1)^{p-1}\{(p+2)a^2 - 3a + 1\}.$$

Then

$$f'(a) = (a+p-1)^{p-2}\{(p^2+3p+2)a^2 + (2p^2-p-4)a - 2p+2\}.$$

Hence the minimum permanent is attained at $a = \frac{1}{p+1}$ and

$$per A_{1,p} = f(a) = \frac{p^p}{(p+1)^{p+1}}. \quad \square$$

3. Minimum Permanent of $\Omega(Z_n)$

Let R_n denote $n \times n\,(0,1)$ matrix with zero trace and off-diagonal entries which are equal to 1, and let E_{ij} be $n \times n$ matrix with 1 in (i,j) position and zeros elsewhere.

Henryk Minc[6] determined the minimum permanent of $\Omega(C_n)$, where $C_n = R_n + E_{n,n}$, under a plausible assumption that there exists a minimizing matrix in $\Omega(C_n)$ of the form

$$X_n(a) = \begin{pmatrix} 0 & c & c & c & \cdots & c & a \\ c & 0 & c & c & \cdots & c & a \\ c & c & 0 & c & \cdots & c & a \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ c & c & \cdots & c & 0 & c & a \\ c & c & c & \cdots & c & 0 & a \\ a & a & a & \cdots & a & a & b \end{pmatrix}.$$

We consider $Z_n = C_n - E_{1,n} - E_{n,1}$, and make a plausible assumption in $\Omega(Z_n)$. Let

$$(3) \qquad Z_n(a) = \begin{pmatrix} 0 & \frac{1}{n-2} & \frac{1}{n-2} & \cdots & \frac{1}{n-2} & 0 \\ \frac{1}{n-2} & 0 & c & \cdots & c & b \\ \frac{1}{n-2} & c & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c & b \\ \frac{1}{n-2} & c & \vdots & c & 0 & b \\ 0 & b & \cdots & b & b & a \end{pmatrix},$$

where $a + (n-2)b = 1, \quad b + (n-3)c = \frac{n-3}{n-2}$.

THEOREM 3.1. *If a matrix of the form(3) is minimizing in $\Omega(Z_n)$, $n \geq 4$, then the minimum occurs only for*

$$a = \frac{(n-3)^2 perD_{n-2} - (n-2)(n-4)perC_{n-2}}{(n-3)^2 perD_{n-2} + (n-2)perC_{n-2}},$$
$$b = \frac{(n-3)perC_{n-2}}{(n-3)^2 perD_{n-2} + (n-2)perC_{n-2}}$$
$$c = \frac{(n-3)^2 perD_{n-2}}{(n-2)\{(n-3)^2 perD_{n-2} + (n-2)perC_{n-2}\}}.$$

Moreover, min $\{perS \mid S \in \Omega(Z_n)\} =$

$$\frac{perC_{n-2}}{n-2}\left(\frac{(n-3)^2 perD_{n-2}}{(n-2)\{(n-3)^2 perD_{n-2} + (n-2)perC_{n-2}\}}\right)^{n-3},$$

where $D_n = R_n + E_{n-1,n-1} + E_{n,n}$.

PROOF. Since Z_n is fully indecomposable, $b \neq 0$ and $c \neq 0$ by Lemma 1. First we will prove $a \neq 0$. If not, then $b = \frac{1}{n-2}$ and $c = \frac{n-4}{(n-2)(n-3)}$. Hence

$$perZ_n(0) = \frac{n-3}{(n-2)^{n-1}}\left(\frac{n-4}{n-3}\right)^{n-4} perD_{n-2}$$

and

$$perZ_n(0)(n \mid n) = \frac{1}{n-2}\left\{\frac{n-4}{(n-2)(n-3)}\right\}^{n-3} perC_{n-2}.$$

Since

$$\begin{aligned} perD_n &= perC_n + perC_{n-1} \\ &= perR_n + 2perR_{n-1} + perR_{n-2}, \end{aligned}$$

$$\begin{aligned} &perZ_n(0) - perZ_n(0)(n \mid n) \\ &= C(n)\left\{\frac{(n-3)^2 perD_{n-2} - (n-2)(n-4)perC_{n-2}}{(n-2)(n-3)}\right\} > 0 \end{aligned}$$

for $n \geq 4$, where $C(n) = (\frac{1}{n-2})^{n-2} \cdot (\frac{n-4}{n-3})^{n-4}$.

Therefore

$$perZ_n(0)(n \mid n) < perZ_n(0).$$

This is contradictory to lemma 2. Hence $a \neq 0$.

Now, from Lemma 2, we have $perZ_n(a)(n \mid n) = perZ_n(a)(2 \mid n)$ and hence

$$(4) \qquad \frac{1}{n-2}c^{n-3}perC_{n-2} = \frac{n-3}{(n-2)^2}bc^{n-4}perD_{n-2}.$$

From (3) and (4), we have

$$b = \frac{perC_{n-2}}{(n-3)perD_{n-2} + \frac{n-2}{n-3}perC_{n-2}}$$
$$= \frac{(n-3)perC_{n-2}}{(n-3)^2perD_{n-2} + (n-2)perC_{n-2}},$$

$$a = 1-(n-2)b = \frac{(n-3)perD_{n-2} - \frac{(n-2)(n-4)}{(n-3)}perC_{n-2}}{(n-3)perD_{n-2} + \frac{n-2}{n-3}perC_{n-2}}$$
$$= \frac{(n-3)^2perD_{n-2} - (n-2)(n-4)perC_{n-2}}{(n-3)^2perD_{n-2} + (n-2)perC_{n-2}}$$

and

$$c = \frac{1}{n-2} - \frac{perC_{n-2}}{(n-3)^2perD_{n-2} + (n-2)perC_{n-2}}$$
$$= \frac{(n-3)^2perD_{n-2}}{(n-2)\{(n-3)^2perD_{n-2} + (n-2)perC_{n-2}\}}.$$

Consequently the minimum permanent in $\Omega(Z_n)$, under our assumption, is

$$perZ_n(a) = perZ_n(a)(n \mid n) = \frac{1}{n-2}c^{n-3}perC_{n-2}$$
$$= \frac{perC_{n-2}}{n-2}\left(\frac{(n-3)^2perD_{n-2}}{(n-2)\{(n-3)^2perD_{n-2} + (n-2)perC_{n-2}\}}\right)^{n-3}.$$

THEOREM 3.2. *Z_n is not barycentric for $n \geq 4$.*

PROOF. The barycenter of $\Omega(Z_n)$ is

$$\mathbf{b}(Z_n) = \frac{1}{p}\begin{pmatrix} 0 & k & k & \cdots & k & 0 \\ k & 0 & c & \cdots & c & b \\ k & c & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c & \vdots \\ k & c & \cdots & c & 0 & b \\ 0 & b & \cdots & \cdots & b & a \end{pmatrix},$$

where $a = perR_{n-1}$, $b = (n-3)perD_{n-2}$, $c = perC_{n-2} + (n-4)perD_{n-2}$,
$k = perC_{n-2} + (n-3)perD_{n-2}$, $p = (n-2)\{perC_{n-2} + (n-3)perD_{n-2}\}$.

Then

$$\begin{aligned} per\mathbf{b}(Z_n) &= \frac{a}{p}per\mathbf{b}(Z_n)(n \mid n) + (n-2)\frac{b}{p}per\mathbf{b}(Z_n)(2 \mid n) \\ &= \frac{1}{n-2}\frac{a}{p}\left(\frac{c}{p}\right)^{n-3}perC_{n-2} + \frac{n-3}{n-2}\left(\frac{b}{p}\right)^2\left(\frac{c}{p}\right)^{n-4}perD_{n-2}. \end{aligned}$$

Suppose that $\mathbf{b}(Z_n)$ is a minimizing matrix. Then we have

$$per\mathbf{b}(Z_n) = per\mathbf{b}(Z_n)(2 \mid n) = per\mathbf{b}(Z_n)(n \mid n)$$

by Lemma 2. Thus

$$\frac{n-3}{n-2} \cdot \frac{1}{n-2} \cdot \frac{b}{p}\left(\frac{c}{p}\right)^{n-4} = \frac{1}{n-2}\left(\frac{c}{p}\right)^{n-3}perC_{n-2}.$$

Then

$$(n-3)^2(perD_{n-2})^2 - (n-2)perC_{n-2}\{perC_{n-2} + (n-4)perD_{n-2}\} = 0.$$

Hence

$$\text{(5)} \qquad \frac{(n-2)perC_{n-2}}{(n-3)perD_{n-2}} \cdot \frac{\{perC_{n-2} + (n-4)perD_{n-2}\}}{(n-3)perD_{n-2}} = 1.$$

Let

$$f(n) = \frac{(n-2)perC_{n-2}}{(n-3)perD_{n-2}}$$
$$= 1 + \frac{(n-2)perC_{n-2} - (n-3)perD_{n-2}}{(n-3)perD_{n-2}} = 1 + \alpha,$$
$$g(n) = \frac{perC_{n-2} + (n-4)perD_{n-2}}{(n-3)perD_{n-2}}$$
$$= 1 - \frac{perD_{n-2} - perC_{n-2}}{(n-3)perD_{n-2}} = 1 - \beta$$

where α, β are positive real numbers.

Compare α with $\frac{\beta}{1-\beta}$. Since

$$\alpha = \frac{(n-2)perC_{n-2} - (n-3)perD_{n-2}}{(n-3)perD_{n-2}}, \tag{6}$$

$$\frac{\beta}{1-\beta} = \frac{perD_{n-2} - perC_{n-2}}{(n-4)perD_{n-2} + perC_{n-2}} \tag{7}$$

and $perD_n = perC_n + perC_{n-1}$, $perR_n = (n-1)perC_{n-1}$,

(Denominator of right hand of (6))–(Denominator of right hand of (7))

$$=(n-3)perD_{n-2} - \{(n-4)perD_{n-2} + perC_{n-2}\}$$
$$>(n-3)perD_{n-2} - \{(n-4)perD_{n-2} + perD_{n-2}\}$$
$$=0$$

and

(Numerator of right hand of (6))–(Numerator of right hand of (7))

$$=(n-2)perC_{n-2} - (n-3)perD_{n-2} - perD_{n-2} + perC_{n-2}$$
$$=(n-1)perC_{n-2} - (n-2)perD_{n-2}$$
$$=perC_{n-2} - (n-2)perC_{n-3}$$
$$=perR_{n-2} + perR_{n-3} - perR_{n-2} - perC_{n-3}$$
$$=perR_{n-3} - (perR_{n-3} + perR_{n-4})$$
$$= - perR_{n-4}$$
$$<0.$$

Hence $\alpha < \frac{\beta}{1-\beta}$. Since $(1+\alpha)(1-\beta) = 1\,(\alpha, \beta;\ positive)$ if and only if $\alpha = \frac{\beta}{1-\beta}$, $(1+\alpha)(1-\beta) \neq 1$,which is contradicts to (5). Therefore $per\mathbf{b}(Z_n)(2 \mid n) > per\mathbf{b}(Z_n)(n \mid n)$.

Thus $\mathbf{b}(Z_n)$ is not minimizing matrix on $\Omega(Z_n)$. That is , Z_n is not barycentric for $n \geq 4$.

References

1. H. Minc, *Permanents. Encyclopedia of Mathematics and its Applications*, Addison-Wesley **6** (1978).
2. ———, *Thoery of Permanents* 1978 – 1981, Linear and Multilinear Algebra Vol.12 (1983).
3. R. A. Brualdi, *An interesting face of the polytope of doubly stochastic matrices*, Linear and Multilinear Algebra Vol.17 (1985).
4. S. Song, *Minimum permanents on certain faces of matrices containing an identity submatrix*, Linear Algebra Appl. (1988).
5. H. Minc, *Minimum permanents of doubly stochastic matrices with prescribed zoro entries*, Linear and Multilinear Algebra (1984).
6. ———, *On a Conjecture of R.A. Brualdi*, Linear Algebra and its Applications 94 (1987).
7. S. Song, *Minimum Permanents on Certain Doubly Stochastic Matrices*, Linear Algebra and Its Applications 161 (1992).
8. H. Minc, *Minimum permanents of doubly stochastic matrices with prescribed zero entries on the main diagonal*, Linear Algebra and its Applications 201 (1994).

Department of Mathematic Education
Andong University
Andong 760-749, Korea