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NUMERICAL SOLUTION OF LINEAR ELASTICITY BY PRECONDITIONING CUBIC SPLINE COLLOCATION

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ABSTRACT. Numerical approximations to the linear elasticity are traditionally based on the finite element method. In this paper we propose a new formulation based on the cubic spline collocation method for linear elastic problem on the unit square. We present several numerical results for the eigenvalues of the matrix represented by cubic collocation method and preconditioner matrix which is preconditioned by FEM and FDM. Finally we present the numerical solution for some example equation.

1. Introduction

Let us denote by $\Omega = [a,b] \times [c,d]$ the volume of the body and $\partial\Omega$ its boundary. Let us consider a linear elastic problem defined on Ω

$$A\mathbf{u} := -\mu\Delta\mathbf{u} - (\lambda+\mu)\text{grad div }\mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega \tag{1.1a}$$

with boundary conditions

$$\mathbf{u} = \Phi \quad \text{on} \quad \Gamma_{Di} \tag{1.1b}$$

and

$$(B\mathbf{u})_i := \sum_{j=1}^{2} \sigma_{ij}(\mathbf{u})n_j = \Psi_i \quad \text{on} \quad \Gamma_{Ne}, \quad i = 1,2, \tag{1.1c}$$

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where $\partial\Omega = \Gamma_{Di} \cup \Gamma_{Ne}$ and $\mathbf{n} = (n^1, n^2)$ is the outward unit normal to $\partial\Omega$. The problem of interest here is the determination of the displacement field $\mathbf{u} = (u^1, u^2)$ produced in the body which is assumed to be in equilibrium and in static conditions by the quasi static application of known distributions of external actions. Neglecting thermal effects, they consist of distributed loads $\Psi = (\Psi^1, \Psi^2)$ over Γ_{Ne}, of given displacements $\Phi = (\Phi^1, \Phi^2)$ over Γ_{Di}, of body forces $\mathbf{f} = (f^1, f^2)$. The stress field $\sigma(\mathbf{u}) = \sigma_{ij}(\mathbf{u})$ is determined from the displacement field $\mathbf{u} = (u^1, u^2)$. Two parameters used for the isotropic linear elastic materials are the constants λ and μ

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{1+\nu},$$

where E is Young's modulus and ν is the Poisson ratio of the material. For the sake of simplicity we consider only homogeneous Dirichlet boundary condition, i.e., $\partial\Omega = \Gamma_{Di}$ (i.e, $\Gamma_{Ne} = \emptyset$) and $\Phi = 0$ and let $\lambda = \mu = 1$.

For the numerical approximation for this problem, we propose the C^1 interpolatory cubic spline collocation method[KP3]. This method is to find such a cubic spline solution $\mathbf{u}(x, y)$ which satisfies

$$A\mathbf{u}(\xi_i, \xi_j) = \mathbf{f}(\xi_i, \xi_j) \tag{1.2}$$

and boundary conditions, where the collocation points $\{(\xi_i, \xi_j)\}_{i,j=1}^{2N}$ are chosen as the local Legendre-Gauss[=:LG] points(see [BS]).

Further, we propose the preconditioning matrix to the finite element stiffness matrix or finite difference matrix corresponding to the operator

$$L\mathbf{u} := -\Delta\mathbf{u} + \mathbf{u} \tag{1.3}$$

with a homogeneous Dirichlet boundary condition.

One of the main results is to get a uniform bound for the eigenvalues of the preconditioned matrix. Such an estimate is important for the successful application of the iteration method, such as Bi-CGSTAB, GMRES, etc. (see [H], [SS])

Since Orszag suggested a finite difference preconditioner for the Chebyshev spectral method for the Poisson equation [O], the two dimensional theoretical proof of uniform bounds of eigenvalues (or singular values) for the preconditioning Chebyshev or Legendre spectral collocation method by a particular finite element method or finite difference method were recently proved (see [KP1],[KP2] and [PR]). For the elliptic equation, finite element preconditioning and finite difference preconditioning of the cubic spline collocation method was investigated in [KP3] and [KKL]. In [CQZ], the authors suggests a finite element preconditioner for a spectral collocation method of elasticity. In this paper, using the ideas appeared in [KP3] and [KKL], we propose a finite element preconditioning for the linear elastic problem using the cubic splines and a finite difference preconditioning. Without analytic proof, we present the numerical results for the boundedness of the eigenvalues of preconditioned matrix.

In Section 2 we collect some preliminary ideas, notations, etc. Section 3 deals with C^1 cubic interpolatory collocation method and eigenvalues of collocation matrix. In Section 4 we present the results for the preconditioned matrix. Finally we present an approximate solution of an example in Section 5.

2. Preliminaries

In this section we introduce some notations, definitions, and basic facts to be used in the sequel.

Le $I = [a, b]$ be an interval. Let $N > 1$ be an integer and set $h := (b-a)/N$. The "knots" are the points $t_k := a + kh$, $k = 0, 1, \cdots, N$, and $I_k := (t_{k-1}, t_k)$, $k = 1, \cdots, N$, is the k^{th} subinterval. Let

(2.1)
$$S_{h,3}[a,b] := \{u \in C^1[a,b] : u|_{I_k} \text{ is a cubic polynomial, } u(a) = u(b) = 0\}.$$

In this work we are concerned with the Hermite cubic spline which is given by the translation and dilation of the functions $v(t)$, $s(t)$ given by

$$v(t) := (1 + 2|t|)(1 - |t|)^2, \quad -1 \le t \le 1, \tag{2.2a}$$

and

$$s(t) := t(1 - |t|)^2, \quad -1 \le t \le 1. \tag{2.2b}$$

Then the basis for $S_{h,3}[a,b]$ is given by $\{v_j(t), s_j(t)\}$, where

$$\text{(2.3a)}\quad v_j(t) = v\left(\frac{t-t_j}{h}\right), t_{j-1} \le t \le t_{j+1}, \quad j = 1, \dots, N-1,$$

$$\text{(2.3b)}\quad s_j(t) = hs\left(\frac{t-t_j}{h}\right), t_{j-1} \le t \le t_{j+1}, \quad j = 1, \dots, N-1,$$

$$\text{(2.3c)}\quad s_0(t) = hs\left(\frac{t-t_0}{h}\right), t_0 \le t \le t_1,$$

and

$$\text{(2.3}d\text{)}\quad s_N(t) = hs\left(\frac{t-t_N}{h}\right), t_{N-1} \le t \le t_N.$$

For the convenience, let us denote by $\{\psi_j\}$ the basis for $S_{h,3}[a,b]$ in the order of $s_0, v_1, s_1, \dots, v_{N-1}, s_{N-1}, s_N$.

Let $\{\xi_i\}_{i=1}^{2N}$ be the local LG points, that is to say, ξ_{2i-1} and ξ_{2i} on each subintervals $I_i = [x_{i-1}, x_i]$ are given by

$$\text{(2.4}a\text{)}\quad \xi_{2i-1} = t_{i-1} + hc_1, \quad \xi_{2i} = t_{i-1} + hc_2, \qquad i = 1, \cdots, N,$$

where

$$\text{(2.4}b\text{)}\quad c_1 := \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), \quad c_2 := \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right),$$

and

$$\text{(2.4}c\text{)}\quad \xi_0 = 0, \qquad \xi_{2N+1} = 1.$$

The two dimensional space $S_{h^2,3}[\Omega]$ is given as the tensor product of the appropriate one-dimensional spaces. That is,

$$\text{(2.5)}\quad S_{h^2,3}[\Omega] := S_{h,3}[a,b] \otimes S_{h,3}[c,d],$$

which has the basis functions $\{\Psi_\mu\}_{\mu=1}^{4N^2}$,

$$\text{(2.6)}\quad \Psi_\mu(x,y) := \psi_i(x)\psi_j(y), \quad \mu = i + 2(j-1)N,$$

where $i, j = 1, \ldots, 2N$, $\{\psi_i(x)\}$ is the basis for $S_{h,3}[a,b]$ and $\{\psi_j(y)\}$ is the basis for $S_{h,3}[c,d]$.

The usual norm and inner product notations are used. For example, If $U = (u_k)$ and $V = (v_k)$ are K-tuples of complex numbers, then the inner product is defined by

$$(U,V) := \sum_{k=1}^{K} u_k \bar{v}_k. \tag{2.7}$$

If $u(x,y), v(x,y)$ are functions defined on Ω, then

$$(u,v),\ \|u\|_{L_2},\ \|u\|_s$$

denote the usual L_2 inner product, L_2 norm and H^s norm respectively. In addition, we define the discrete inner product $\langle u,v\rangle_N$ and the discrete norm $\|u\|_N$, for u and v in $S_{h^2,3}[\Omega]$,

$$\langle u,v\rangle_N := \sum_{i,j=1}^{2N} (u\bar{v})(\xi_i,\eta_j) w_i^1 w_j^2, \qquad \|u\|_N^2 := \langle u,u\rangle_N, \tag{2.8}$$

where $w_i^1 = \frac{b-a}{2}$ and $w_j^2 = \frac{d-c}{2}$.

We define the discrete inner product $\langle \mathbf{u},\mathbf{v}\rangle_N$ and the discrete norm $\|\mathbf{u}\|_N$ for $\mathbf{u}$ and $\mathbf{v}$ in $[S_{h^2,3}] \times [S_{h^2,3}]$,

$$\langle \mathbf{u},\mathbf{v}\rangle_N := \sum_{j=1}^{2} \langle u^j, v^j\rangle_N, \qquad \|\mathbf{u}\|_N^2 := \langle \mathbf{u},\mathbf{u}\rangle_N. \tag{2.9}$$

3. Cubic spline collocation method

In this section we consider a linear elastic problem defined on the rectangular body Ω which is given by

$$A\mathbf{u} := -\Delta\mathbf{u} - 2\text{grad div }\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \tag{3.1a}$$

with a homogeneous Dirichlet boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{3.1b}$$

We denote by

$$C := \{P_\mu := (\xi_k, \eta_l) \ : \ 1 \le k, l \le 2N, \quad \mu = k + 2N(l-1)\} \tag{3.2}$$

the set of the local LG points in Ω, where $\{\xi_k\}_{k=1}^{2N}$ is the local LG points in $[a,b]$, and $\{\eta_l\}_{l=1}^{2N}$ is the local LG points in $[c,d]$ as defined in (2.2) along x-axis and y-axis, respectively.

Let A_N be the cubic collocation operator such that

$$A_N : [S_{h^2,3}]^2 \longrightarrow [S_{h^2,3}]^2. \tag{3.3a}$$

The cubic spline collocation discretization in the space $[S_{h^2,3}]^2$ is determined by the bilinear form

$$a_N(\mathbf{u}, \mathbf{v}) = \langle A_N \mathbf{u}, \mathbf{v} \rangle_N, \tag{3.3b}$$

where

$$\langle A_N \mathbf{u}, \mathbf{v} \rangle_N = \langle -\Delta \mathbf{u} - 2\text{grad div } \mathbf{u}, \mathbf{v} \rangle_N, \quad \mathbf{u}, \mathbf{v} \in [S_{h^2,3}]^2. \tag{3.3c}$$

For any $\mathbf{u}, \mathbf{v} \in [S_{h^2,3}]^2$, let

$$\mathbf{u}(x,y) = \sum_{i,j=1}^{2N} \mathbf{u}_{ij} \Psi_{ij}(x,y), \qquad \mathbf{v}(x,y) = \sum_{i,j=1}^{2N} \mathbf{v}_{ij} \Psi_{ij}(x,y), \tag{3.3d}$$

where $\{\Psi_{ij}\}$ is the bases of $S_{h^2,3}$, and let $U = (\mathbf{u}_1, \cdots, \mathbf{u}_{4N^2})^t$ and $V = (\mathbf{v}_1, \cdots, \mathbf{v}_{4N^2})^t$ be complex vectors. Then

$$a_N(\mathbf{u}, \mathbf{v}) = \frac{h^2}{4} V^* \tilde{A}_N U, \quad (\text{where } \tilde{A}_N := \hat{E}_N^t W_N \hat{A}_N), \tag{3.3e}$$

where

(3.3f)

$$\hat{A}_N(\mu,\nu) := A_N \Psi_\nu(P_\mu), \ \ \hat{E}_N(\mu,\nu) := \Psi_\nu(P_\mu) \text{ and } W_N(\mu,\nu) := \text{diag}(\frac{h^2}{4}),$$

and the matrix $\hat{E}_N$ satisfies

$$\hat{E}_N U = (\mathbf{u}(P_1), \mathbf{u}(P_2), \dots, \mathbf{u}(P_{4N^2}))^t.$$

Then we can define the cubic spline collocation approximation to (3.1), which leads as a generalized Galerkin method.

Find $\mathbf{u} \in [S_{h^2,3}]^2$ such that

$$a_N(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_N, \qquad \forall \mathbf{v} \in [S_{h^2,3}]^2. \tag{3.4}$$

If we choose as $\mathbf{v}$ in (3.4) the cubic spline basis $\Psi_\mu(x, y)$, $\mu = 1, \dots, 4N^2$, then it can be easily seen that (3.4) is equivalent to the collocation problem.

Find $\mathbf{u} \in [S_{h^2,3}]^2$ such that

$$A\mathbf{u}(P_\mu) = \mathbf{f}(P_\mu), \qquad \mu = 1, \dots, 4N^2, \tag{3.5}$$

which leads to the linear system

$$\hat{E}_N^t W_N \hat{A}_N U = \hat{E}_N^t W_N F, \tag{3.6}$$

where $F = (\mathbf{f}(\xi_1), \dots, \mathbf{f}(\xi_{4N^2}))^t$.

Table 1 shows the result of the eigenvalues of matrix $\tilde{A}_N$, the size of M by M, where $M = 2 * 4N^2$. We investigate the total number of eigenvalues of the matrix, the number of the real eigenvalues and the number of the eigenvalues which has real positive part, and find the minimum, maximum values and the ratio of them. for N = 2, 4, 6, 8, 10 and 12.

N	M	real #	pos. #	min value	max value	max/min
2	32	32	32	0.004358	8.524043	1956.1014
4	128	114	128	0.000288	13.413777	46501.6082
6	288	250	288	0.000057	14.759145	258046.1448
8	512	448	512	0.000018	15.282123	844187.6927
10	800	704	800	0.000007	15.534331	2094952.3974
12	1152	1018	1152	0.000004	15.674186	4383137.2181

TABLE 1. The eigenvalues of $\tilde{A}_N$

4. Preconditioning method

In the previous section we get the collocation scheme for the equation (3.1), and we have known from the numerical results that the eigenvalues of the collocation matrix $\tilde{A}_N$ have positive real part. Moreover, for large N, the condition number $\mathcal{K}$ of $\tilde{A}_N$ (the ratio between the maximum and minimum absolute value of the eigenvalues of $\tilde{A}_N$), i.e.,

$$(4.1a) \qquad \mathcal{K}(\tilde{A}_N) := \frac{\max|\lambda(\tilde{A}_N)|}{\min|\lambda(\tilde{A}_N)|}$$

behaves like

$$(4.1b) \qquad \mathcal{K}(\tilde{A}_N) \cong CN^4.$$

where C is a constant independent of N.

In view of (4.1), preconditioning techniques are in order, as large systems become increasingly ill-conditioned.

We propose a very effective preconditioner for (3.1) based on the C^1 interpolatory cubic finite element discretization of the basic operator

$$(4.2) \qquad L\mathbf{u} := -\Delta\mathbf{u} + \mathbf{u}, \quad \text{on } \Omega$$

with a homogeneous Dirichlet boundary condition with nodes given precisely by the LG points. Hence the preconditioner A_p is the finite element stiffness matrix $\hat{\beta}_N$ associated with (4.2) which has been introduced in [KP1, HP3]. The another preconditioner A_p is the finite difference discretization $\hat{L}_N$ of the operator (4.2) with same nodes, has been introduced in [KP2]. The precise construction is reported in [KKL].

The preconditioned collocation matrix A_{cp} leads therefore:

$$(4.3) \qquad A_{cp} := A_p^{-1}\tilde{A}_N = \hat{\beta}_N^{-1}\tilde{A}_N \text{ (or } \hat{L}_N^{-1}\tilde{A}_N)$$

Therefore it leads the following from (3.6)

$$(4.4a) \qquad A_{cp}U := \hat{\beta}_N^{-1}\hat{E}_N^t W_N \hat{A}_N U = \hat{\beta}_N^{-1}\hat{E}_N^t W_N F =: F'$$

or

$$A_{cp}U := \hat{L}_N^{-1}\hat{E}_N^t W_N \hat{A}_N U = \hat{L}_N^{-1}\hat{E}_N^t W_N F =: F'' \tag{4.4b}$$

A theoritical analysis of the eigenvalues of the preconditioned matrix A_{cp} is not available at the moment. However, numerical results in Table 2 and Table 3 show that the eigenvalues of preconditioned matrix have positive real part, moreover the condition number $\mathcal{K}(A_{cp})$ is uniformly bounded with respect to N.

The above preconditioning method can be applied to any iterative method, for example, Bi-CGSTAB method(see [H]).

N	M	real #	positive #	min value	max value	max/min
2	32	32	32	1.002555	6.481481	6.464966
4	128	102	128	0.986362	6.481481	6.571096
6	288	232	288	0.985454	6.481481	6.577153
8	512	378	512	0.985631	6.481481	6.575971
10	800	704	800	0.985443	6.481481	6.577226
12	1152	892	1152	0.985281	6.481481	6.578304
14	1568	1214	1568	0.985144	6.481481	6.579223
18	2592	2052	2592	0.985121	6.481481	6.579374

TABLE 2. The eigenvalues of preconditioned matrix by FEM

N	M	real #	positive #	min value	max value	max/min
2	32	22	32	0.277785	1.355247	4.878770
4	128	90	128	0.064072	0.386122	6.026356
6	288	198	288	0.028079	0.177242	6.312351
8	512	390	512	0.015719	0.100985	6.424464
10	800	636	800	0.010038	0.065045	6.479816
12	1152	816	1152	0.006963	0.045336	6.511187
14	1568	1104	1568	0.005112	0.033384	6.530683
18	2592	1902	2592	0.003090	0.020248	6.552660

TABLE 3. The eigenvalues of preconditioned matrix by FDM

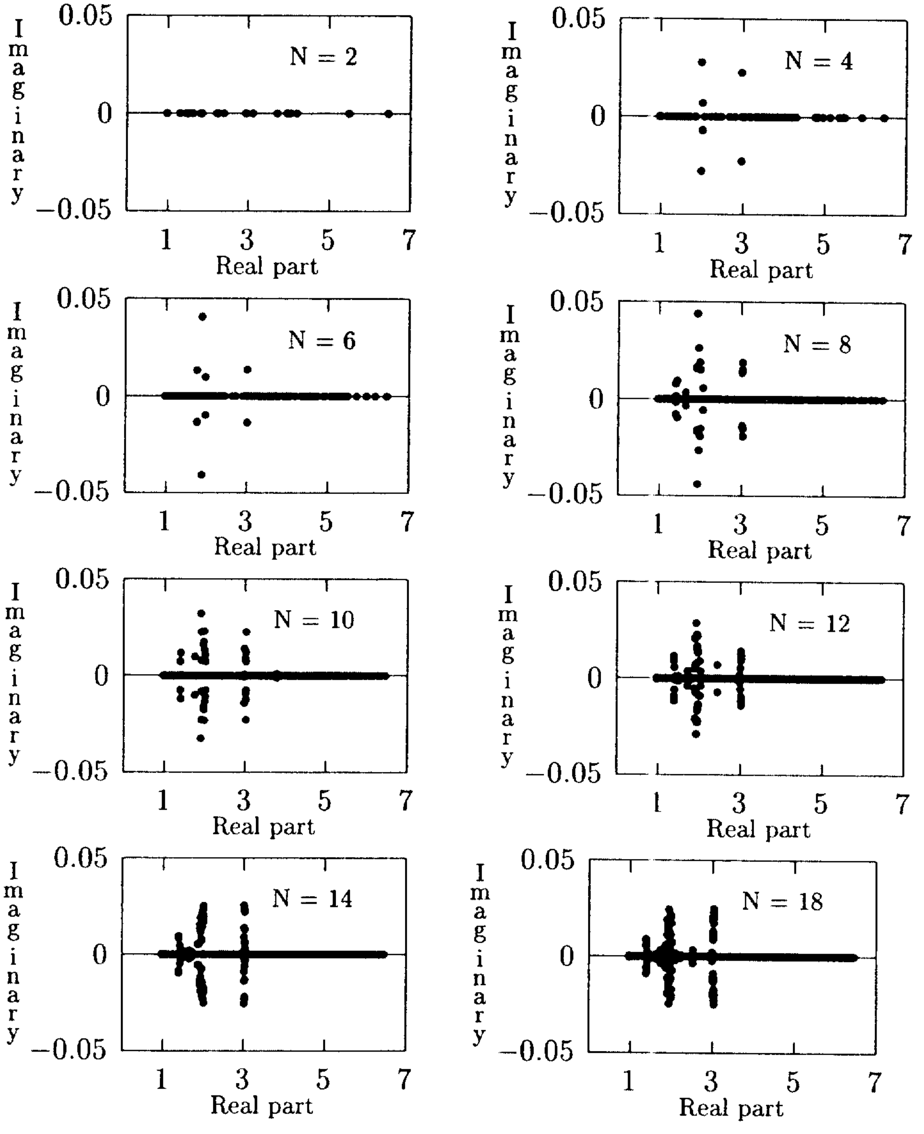

FIGURE 1. The distribution of eigenvalues of FEM preconditioned matrix

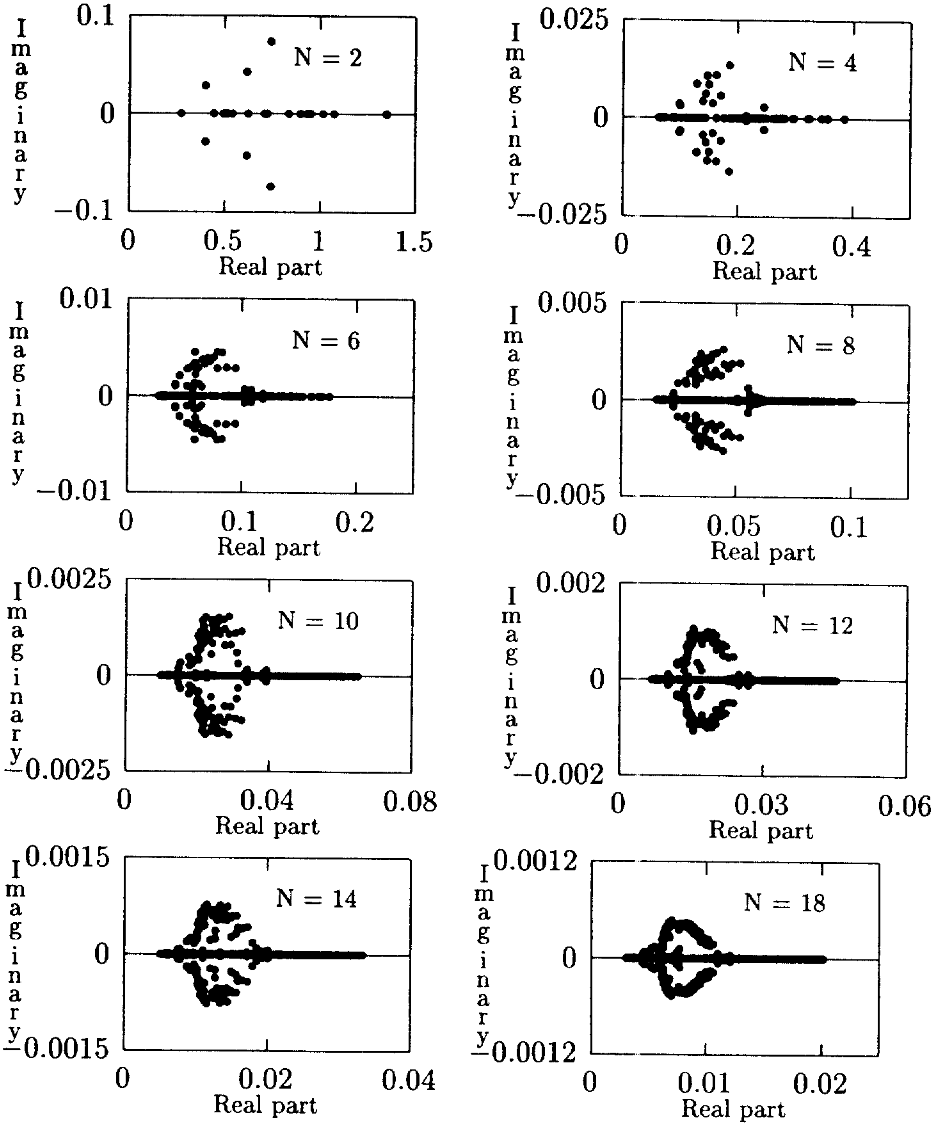

FIGURE 2. The distribution of eigenvalues of FDM preconditioned matrix

5. Numerical Solutions

In this section we investigate the accuracy of solution and the iteration number for Bi-CGSTAB method for the linear elastic model problem as follows. Let the computational domain be $\Omega = (0,2) \times (0,2)$. The data $\mathbf{f}$ and Φ are chosen in such a way that the exact solution is given by

$$u^1(x,y) = -\sin(\pi y) + \cos(\pi x)\sin(\pi y) \tag{5.1a}$$

$$u^2(x,y) = 4\sin(\pi x) - 4\cos(\pi y)\sin(\pi x) \tag{5.1b}$$

in the first case, and by

$$u^1(x,y) = -\sin(\pi y) + \cos(\pi x)\sin(\pi y) \tag{5.2a}$$

$$u^2(x,y) = \begin{cases} \frac{2}{3}xy(y-2), & x \le 1.5, \\ y(y-2)(1-2.0(x-1.5)^4), & x > 1.5, \end{cases} \tag{5.2b}$$

in the second case. Figure 3 shows the solution accuracy for the two case.

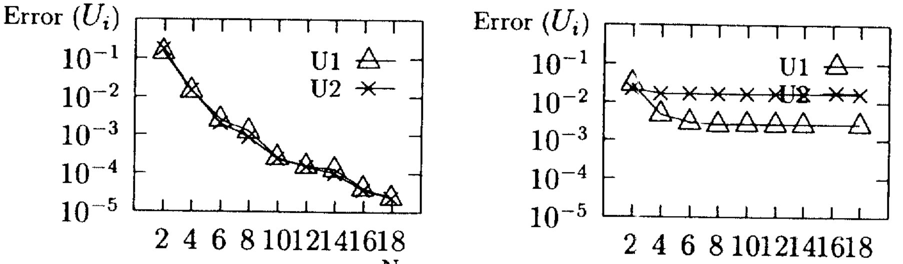

FIGURE 3. THE ACCURACY OF THE SOLUTION OF FEM preconditioning method (left : Case (5.1), right : Case (5.2))

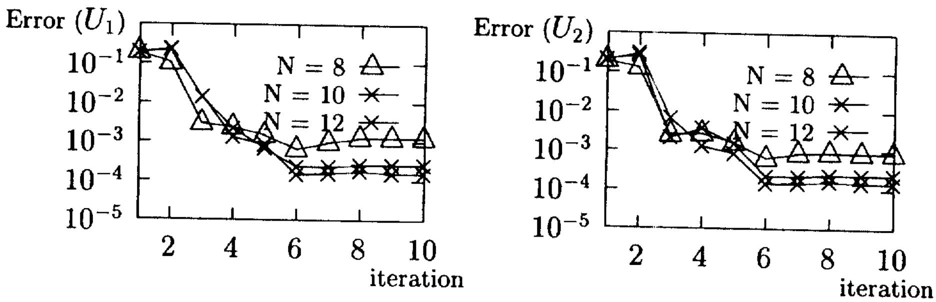

FIGURE 4. THE HISSTORY OF CONVERGENCE OF (5.1) OF FEM preconditioning method

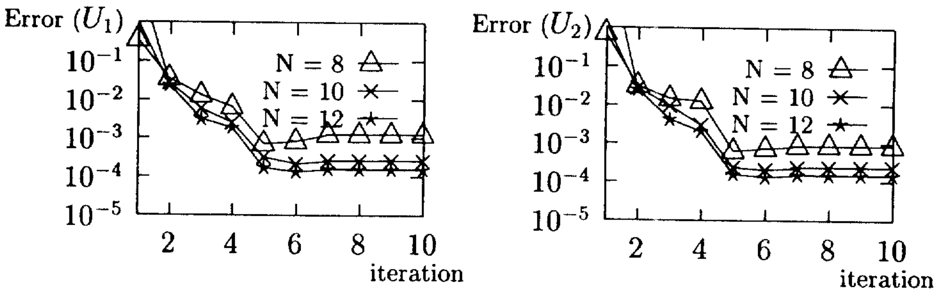

FIGURE 5. THE HISTORY OF CONVERGENCE OF (5.1) OF FDM preconditioning method

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