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WEAK CONVERGENCE OF PROCESSES IN A GENERALIZED CURIE-WEISS MODEL AND ITS DUAL

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ABSTRACT. In this paper, we establish the weak convergences of processes occurring in a generalized Curie-Weiss model and its dual.

1. Introduction

Let $\{X_j^{(n)} : j = 1, 2, \cdots, n\}$ be a triangular array of random variables with the joint distribution given by

$$(1.1) \qquad d_n^{-1} \exp[-\beta H_n(x_1, \cdots, x_n)] \prod_{j=1}^{n} dP(x_j),$$

where P is a probability measure on R^1 and d_n is a normalizing constant. The model (1.1) is often considered in a study of ferromagnetic system in statistical mechanics. There, $X_i^{(n)}$ is the magnetic spin at the i-th site, $\beta(> 0)$ inverse temperature and H_n Hamiltonian which represents the energy of the system. When H_n takes the form $H_n(x_1, x_2, \cdots, x_n) = -(\sum_{i=1}^n x_i)^2/2n$, the model(1.1) is usually called the Curie-Weiss model and a number of results on the asymptotic distribution of the total magnetization $S_n = \sum_{i=1}^n X_i^{(n)}$ for this model have been established. See [3], [4] and [7]. A generalized Curie-Weiss model considered in this paper is the model (1.1) in which Hamiltonian H_n takes the form

$$(1.2) \qquad H_n(x_1, x_2, \cdots, x_n) = -n\Psi_Q\left(\sum_{i=1}^{n} x_i/n\right),$$

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where Ψ_Q is the cumulant generating function(c.g.f.) of some suitable probability measure Q. Obviously the Curie-Weiss model is a special case of the model (1.2) when Q is standard normal. We obtained some limit theorems of this model (1.2). See [5] and [6].

The purpose of this paper is to establish weak convergences of the processes based on the appropriate partial sums considered in model (1.2) and its dual in which the role of P is interchanged with that of Q in (1.1). In Section 2, we summarize the result of [5] as preliminaries. The main results are stated and proved in Section 3.

2. Preliminaries

For a given probability measure Q with $\Phi_Q(t) = \int_R \exp(tx)Q(dx) < \infty$, $|t| < h$, $h > 0$, let L_Q be a class of probability measures P such that for $|t| < k$, $k > 0$

$$\Phi_P(t) = \int_R \exp(tx)P(dx) < \infty \tag{2.1}$$

and

$$\int_R \Phi_Q(x)dP(x) < \infty. \tag{2.2}$$

Let $\{X_j^{(n)} : j = 1, 2, \cdots, n\}, n = 1, 2, \cdots$ be a triangular array with the joint distribution given by

(2.3)

$$d\mu_n(x_1, x_2, \cdots, x_n) = z_n^{-1} \exp[n\Psi_Q\{(x_1 + \cdots + x_n)/n\}] \prod_{j=1}^n dP(x_j),$$

where $P \in L_Q, \Psi_Q(t) = \log \Phi_Q(t)$, and z_n is a normalized constant,

$$z_n = \int_{R^n} [n\Psi_Q\{(x_1 + \cdots + x_n)/n\}] \prod_{j=1}^n dP(x_j). \tag{2.4}$$

Assume that $D_Q = (a, b) = \{x|0 < F_Q(x) < 1\}$, where F_Q is the distribution function of Q, and define $G_{QP}(u) = \gamma_Q(u) - \Psi_P(u)$ for all $u \in D_Q$, where γ_Q is the large deviation rate of Q and Ψ_P is the c.g.f. of P.

DEFINITION 2.1. A real number $m(\in D_Q)$ is said to be a global minimum for G_{QP} if

$$G_{QP}(m) \leq G_{QP}(u) \qquad \text{for all } u \in D_Q.$$

DEFINITION 2.2. A global minimum m for G_{QP} is said to be of type k if

$$G_{QP}(m+u) - G_{QP}(m) = \frac{c_{2k}u^{2k}}{(2k)!} + o(u^{2k}) \qquad \text{as } u \to 0, \tag{2.5}$$

where $c_{2k} = G_{QP}^{(2k)}$ is strictly positive.

Since $P \in L_Q$ implies $Q \in L_P$, interchanging the roles of P and Q in the model (2.3) is possible and the dual model is defined as
(2.6)

$$d\mu_n^D(x_1, x_2, \cdots, x_n) = d_n^{-1} \exp\left[n\Psi_P\{(x_1 + \cdots + x_n)/n\}\right] \prod_{i=1}^{n} dQ(x_i),$$

where d_n is a normalizing constant.

For the dual model, we also assume that $D_P = (c, d) = \{x | 0 < F_P(x) < 1\}$ where F_P is the distribution function of P and $G_{PQ}(u) = \gamma_P(u) - \Psi_Q(u)$ for all $u \in D_P$, as in the original model. The following theorems reveal some relationships between two models.

THEOREM 2.3. *Assume that G_{QP} has the unique global minimum of type k at m and*

$$\inf_{t \in D_Q} G_{QP}(t) < \min\left\{ \liminf_{t \to a} G_{QP}(t), \liminf_{t \to b} G_{QP}(t) \right\}, \tag{2.7}$$

Then G_{PQ} has the unique global minimum of type k at $m^D = \Psi'_P(m)$ and in this case $G_{PQ}^{2k}(m^D) = c_{2k}[\Psi''_P(m)]^{-2k}$. Furthermore

$$\inf_{t \in D_P} G_{PQ}(t) < \min\left\{ \liminf_{t \to c} G_{PQ}(t), \quad \liminf_{t \to d} G_{PQ}(t)) \right\}. \tag{2.8}$$

THEOREM 2.4. *Let P and Q be probability measures satisfying the uniform local limit theorem of Daniels [2]. Let $\{X_1^{(n)}, X_2^{(n)}, \cdots, X_n^{(n)}\}$ and $\{X_1^{D(n)}, X_2^{D(n)}, \cdots, X_n^{D(n)}\}$ be triangular arrays of random variables with joint distributions μ_n and μ_n^D given by (2.3) and (2.6) respectively. Assume that m is the unique global minimum of type k for G_{QP} and the condition (2.7) holds. Then*

$$\frac{S_n - nm^D}{m_1^D n^{1-1/2k}} \xrightarrow{d} \begin{cases} N(0,\ 1/m_1^D + 1/c_2) & \text{if } k = 1 \\ \exp\{-c_{2k}y^{2k}/(2k)!\} & \text{if } k \geq 2, \end{cases} \tag{2.9}$$

where $m^D = \Psi'_P(m), m_1^D = \Psi''_P(m), c_{2k} = G_{QP}^{(2k)}(m)$ *and* $S_n = \sum_{i=1}^n X_i^{(n)}$ *and*

$$\frac{S_n^D - nm}{m_1 n^{1-1/2k}} \xrightarrow{d} \begin{cases} N(0,\ 1/m_1 + 1/c'_2) & \text{if } k = 1 \\ \exp\{-c'_{2k}y^{2k}/(2k)!\} & \text{if } k \geq 2, \end{cases} \tag{2.10}$$

where $m = \Psi'_Q(m^D), m_1 = \Psi''_Q(m^D), c'_{2k} = G_{PQ}^{(2k)}(m^D)$ *and* $S_n^D = \sum_{i=1}^n X_i^{D(n)}$.

3. Main Results

For model (2.3), define a process

$$W_n(t) = \frac{(S_{[nt]} - [nt]m^D) + (nt - [nt])(X_{[nt]+1}^{(n)} - m^D)}{m_1^D n^{1-1/2k}}, \quad 0 \leq t \leq 1,$$

where $S_{[nt]} = \sum_{i=1}^{[nt]} X_i^{(n)}$ and $[nt]$ denotes the largest integer not exceeding nt. We also define, for the dual model (2.6),

$$W_n^D(t) = \frac{(S_{[nt]}^D - [nt]m) + (nt - [nt])(X_{[nt]+1}^{D(n)} - m)}{m_1 n^{1-1/2k}}, \quad 0 \leq t \leq 1,$$

where $S_{[nt]}^D = \sum_{i=1}^{[nt]} X_i^{D(n)}$. Note that $W_n(1) = (S_n - nm^D)/m_1^D n^{1-1/2k}$ and $W_n^D(1) = (S_n^D - nm)/m_1 n^{1-1/2k}$. Thus $W_n(1)$ and $W_n^D(1)$ have the limiting distributions stated in Theorem 2.4. In this section, we first prove weak convergence of the process $\{W_n(t), 0 \leq t \leq 1\}$ and, from this result together with the relationships between two models, we establish weak convergence of the process $\{W_n^D(t), 0 \leq t \leq 1\}$ in the dual model. For the rest of this section, we assume that the probability measures P and Q satisfy the uniform local limit theorem of Daniel. [2]

3.1 Conditional weak convergence

Let $\{Z_n; n \geq 1\}$ be a sequence of iid random variables with common distribution Q and let f_n be the probability density function of $\sum_{i=1}^n Z_i/n$. Then, to apply conditioning technique, we express the joint distribution μ_n in (2.3) as follows:

$$
\begin{aligned}
&d\mu_n(x_1, x_2, \cdots, x_n) \\
&= z_n^{-1} \exp\left\{n\Psi_Q\Big(\sum_{i=1}^n x_i/n\Big)\right\} \prod_{i=1}^n dP(x_i) \\
&= z_n^{-1}\left[\int_R \exp\left\{\sum_{i=1}^n x_i\right\} f_n(z)dz\right] \prod_{i=1}^n dP(x_i).
\end{aligned}
$$

Applying Daniels' uniform local limit theorem for f_n with $\Psi'_Q(t_n) = m + yn^{-1}$ and $\Psi''_Q(t_n) = \sigma^2(t_n)$, we have

(3.1)

$$
\begin{aligned}
&z_n^{-1} n^{-1/2k} \int_R \left[\prod_{i=1}^n \exp\{x_i(m + yn^{-1/2k})\} dP(x_i)\right] \\
&\times \sqrt{\frac{n}{2\pi}} \sigma^{-1}(t_n) \exp\{-n\gamma_Q(m + yn^{-1/2k}\}[1 + o(1)]dy \\
&= z_n^{-1/2} n^{k-1/2k} \int_R \left[\prod_{i=1}^n \exp\{x_i(m + yn^{-1/2k}) - \Psi_P(m + yn^{-1/2k})\} dP(x_i)\right] \\
&\times \exp\{-n(\gamma_Q(m + yn^{-1/2k}) - \Psi_P(m + yn^{-1/2k}))\}\sigma^{-1}(t_n)[1 + o(1)]dy \\
&\times \int_R \left[\prod_{i=1}^n \exp\{x_i(m + yn^{-1/2k}) - \Psi_P(m + yn^{-1/2k})\} dP(x_i)\right] \\
&\times \exp\{-n(G_{QP}(m + yn^{-1/2k}) - G_{QP}(m))\}\sigma^{-1}(t_n)[1 + o(1)]dy \\
&= K_n^{-1} \int_R \left[\prod_{i=1}^n dM_{n,y}(x_i)\right] h_n(y)dy,
\end{aligned}
$$

where

$$dM_{n,y}(x) = \exp\{x(m + yn^{-1/2} - \Psi_P(m + yn^{-1/2})\}dP(x),$$
$$h_n(y) = \exp\{-nG_{QP}(m + yn^{-1/2}) - G_{QP}(m))\}\sigma^{-1}(t_n)[1 + o(1)]$$
$$\text{and} \quad K_n = z_n\sqrt{2\pi}n^{-(k-1)/2k}\exp\{nG_{QP}(m)\}$$

Since $\int_{R^n} d\mu(x_1, x_2, \cdots, x_n) = 1$ and $\int_R dM_{n,y} = 1$ for each y and i, we have $K_n = \int_R h_n(y)\,dy$. Thus $h_n^*(y) = K_n^{-1}h_n(y)$ is a probability density function for each n. The representation (3.1) of $d\mu_n$ therefore shows that we can introduce a new random variable V_n with the probability density function $h_n(y)$ such that, given $V_n = y$, the $X_n^{(n)}$'s are independent and identically distributed random variables with the distribution $dM_{n,y}(x)$. Thus, weak convergence of $W_n(\cdot)$, given $V_n = y$, follows from the standard arguments on the proof of weak convergence of the process based on normalized partial sums of iid random variables.

THEOREM 3.1. *Let m be the unique global minimum of type k for G_{QP}. then, under $M_{n,y}$, $\{W_n(t),\ 0 \geq t \geq\}$ is tight.*

PROOF. We will show that for each $\varepsilon > 0$, there exist $\lambda > 1$ and an integer n_0 such that for $n > n_0$,

$$Pr\left\{\max_{i\leq n}\left|S_i - im^D\right| \geq \lambda n^{1-1/2k}m_1^D\right\} \leq \frac{\varepsilon}{\lambda^2} \tag{3.2}$$

Now, since

$$\begin{aligned} & n^{-1+1/2k}\max_{i\leq n}\left|i\Psi_P'(yn^{-1/2k} + m) - im^D\right| \\ =& n^{1/2k}|\Psi_P'(yn^{-1/2k} + m) - m^D| \\ =& n^{1/2k}|\Psi_P''(m)yn^{-1/2k} + o(n^{-1/2k})| \\ =& |\Psi_P''(m)y + o(1)|, \end{aligned}$$

there exists n_1 such that for $n > n_1$,

$$\max_{i\leq n}\left|i\Psi_P'(yn^{-1/2k} + m) - im^D\right| \leq \frac{3}{2}m_1^D|y|n^{1-1/2k}. \tag{3.3}$$

Next, note that, under $M_{n,y}$, the mean and variance of $X_i^{(n)}$ are Ψ'_P $(ny^{-1/2k} + m)$ and Ψ''_P $(ny^{-1/2k} + m)$, respectively. Thus, it follows pp. 69, [1] that

$$\begin{aligned}
&Pr\left\{ \max_{i \le n} \left| S_i - i\Psi'_P(yn^{-1/2k} + m) \right| \ge \frac{\lambda}{2} n^{1-1/2k} m_1^D \right\} \\
\le& 2Pr\left\{ \left| S_n - n\Psi'_P(yn^{-1/2k} + m) \right| \ge \left(\frac{\lambda n^{1-1/2k} m_1^D}{2\sqrt{n\Psi''_P(yn^{-1/2k} + m)}} - \sqrt{2} \right) \right. \\
&\left. \times \sqrt{n\Psi''_P(yn^{-1/2k} + m)} \right\} \\
=& 2Pr\left\{ \left| \frac{S_n - n\Psi'_P(yn^{-1/2k} + m)}{\sqrt{n\Psi''_P(yn^{-1/2k} + m)}} \right| \ge \frac{\lambda n^{1-1/2k} m_1^D}{2\sqrt{n\Psi''_P(yn^{-1/2k} + m)}} - \sqrt{2} \right\}.
\end{aligned}$$

By the central limit theorem, there exists n_2 such that if $n \ge n_2$, then

$$\begin{aligned}
(3.4) \qquad & Pr\left\{ \max_{i \le n} \left| S_i - i\Psi'_P(yn^{-1/2k} + m) \right| \ge \frac{\lambda}{2} n^{1-1/2k} m_1^D \right\} \\
&\le 3Pr\left\{ |N(0,1)| \ge \frac{\lambda n^{(k-1)/2k} \sqrt{m_1^D}}{2} - \sqrt{2} \right\} \\
&\le 3Pr\left\{ |N(0,1)| \ge \frac{\lambda n^{(k-1)/2k} \sqrt{m_1^D}}{2 \cdot 2} \right\} \quad \text{for } \lambda > 4\sqrt{2} \\
&\le 3 \times \frac{2^4 \times 2^4}{\lambda^4 n^{2(k-1)/k} (m_1^D)^2} \times E|N(0,1)|^4 \\
&= \frac{48^2}{\lambda^4 n^{2(k-1)/k} (m_1^D)^2}.
\end{aligned}$$

Let $\varepsilon > 0$ be given and choose $\lambda > \max\{3|y|,\ 48/m_1^D\sqrt{\varepsilon}\}$ and $n_0 =$

$\max\{n_1, n_2\}$. Then, for all $n \geq n_0$, we have

$$\begin{aligned}
&Pr\left\{\max_{i\leq n}\left|S_i - im^D\right| \geq \lambda n^{1-1/2k} m_1^D\right\}\\
\leq &Pr\left\{\max_{i\leq n}\left|S_i - i\Psi'_P(yn^{-1/2k}+m)\right| \geq \frac{\lambda}{2} n^{1-1/2k} m_1^D\right\}\\
&\qquad + Pr\left\{\max_{i\leq n}\left|i\Psi'_P(yn^{-1/2k}+m) - im_1^D\right.\right\}\\
\leq &\frac{48^2}{\lambda^4 n^{2(k-1)/k}(m_1^D)} \qquad\qquad \text{by (3.3) and (3.4)}\\
\leq &\frac{48^2}{\lambda^4 (m_1^D)^2} \times \frac{(m_1^D)^2\varepsilon}{48^2}\\
= &\frac{\epsilon}{\lambda^2}.
\end{aligned}$$

The proof is completed.

THEOREM 3.2. *Assume that m is the unique global minimum of type k for G_{QP} and the condition (2.7) holds. Then under $M_{n,y}$, for $0 \leq s \leq t \leq 1$,*

$$(W_n(s), W_n(t) - W_n(s)) \quad \xrightarrow{d}$$
$$\begin{cases} N\left((s^{1/2}y, (t-s)^{1/2}y), \begin{pmatrix} s/m_1^D & 0 \\ 0 & (t-s)/m_1^D \end{pmatrix}\right) & \text{if } k = 1 \\ \delta(s^{1-1/2k}y,\ (t-s)^{1-1/2k}y) & \text{if } k \geq 2 \end{cases}$$

where $\delta(\cdot)$ denotes degenerate distribution.

PROOF. Since, for each $0 \leq t \leq 1$,

$$\left|\frac{W_n(t) - S_{[nt]} - [nt]m^D}{m_1^D n^{1-1/2k}}\right| \leq \left|\frac{X_{[nt]}^{(n)} - m^D}{m_1^D n^{1-1/2k}}\right| \quad \xrightarrow{d} \quad 0,$$

it suffices to obtain the limiting distribution of

$$\left(\frac{S_{[nt]} - [nt]m^D}{m_1^D n^{1-1/2k}},\ \frac{(S_{[nt]} - S_{[ns]}) - ([nt]-[ns])m^D}{m_1^D n^{1-1/2k}}\right). \tag{3.5}$$

Since $\frac{S_{[nt]} - [nt]m^D}{m_1^D n^{1-1/2k}} = \frac{[ns]^{1-1/2k}}{n^{1-1/2k}} \cdot \frac{S_{[ns]} - [ns]m^D}{m_1^D [ns]^{1-1/2k}}$, we have

$$\frac{S_{[ns]} - [ns]m^D}{m_1^D n^{1-1/2k}} \xrightarrow{d} \begin{cases} N(s^{1/2}y,\ s^{1/2}/m_1^D) & \text{if} \quad k = 1 \\ \delta(s^{1-1/2k}y) & \text{if} \quad k \geq 2, \end{cases}$$

Since the component in (3.5) are independent, the theorem is proved.

From Theorem 3.2, the following conditional weak convergence can be established.

THEOREM 3.3. *Let m be the unique global minimum of type k for G_{QP} and assume that the condition (2.7) holds. Then, under $M_{n,y}$, $W_n(t)$, $0 \leq t \leq 1$, converges weakly to $W_y(\cdot)$, where $W_y(\cdot)$ is a Gaussian process with independent and stationary increments and with $E(W_y(t)) = t^{1/2}$ and $Var(W_y(t)) = t/m_1^D$, if $k = 1$, and $W_y(\cdot)$ is a process degenerate at $t^{1-1/2k}y$, if $k \geq 2$.*

3.2 Weak convergence

Recall that the joint distribution μ_n in (2.3) was expressed in (3.1) in terms of $M_{n,y}$ and $h_n^*(y)$. Weak convergence of $W_n(\cdot)$ therefore follows from Theorem 2 of [8], if $h_n^*(y)$ converges to a probability density function $h^*(y)$ for each y. That is to say, the limiting process $W(\cdot)$ will then be determined as the h^*-mixture of $W_y(\cdot)$ obtained in Theorem 3.3. In fact, by (2.9) for each y, $h_n^*(y) \to h^*(y)$, as $n \to \infty$, where

$$h^*(y) = \frac{\exp\{-c_{2k}y^{2k}/(2k)!\}}{\int_R \exp\{-c_{2k}u^{2k}/(2k)!\}du}. \tag{3.6}$$

Thus we have the following result of weak convergence.

THEOREM 3.4. *Assume that G_{QP} has the unique global minimum of type k at m and condition (2.7) holds. Then the process $W_n(t)$, $0 \leq t \leq 1$, converges weakly to a process $W(t)$ whose finite dimensional distribution is determined as follows: for $0 \leq s \leq t \leq 1$,*

$$(W(s),\ W(t) - W(s)) \quad \sim \quad N(\underline{0}, \textstyle\sum = (\sigma_{ij})), \qquad \text{if } k = 1\ ,$$

and

$$Pr(W(s) \le x_1,\ W(t) - W(s) \le x_2) = \int_{-\infty}^{u(x_1,x_2)} h^*(y)dy, \qquad \text{if } k \ge 2\ ,$$

where $\sigma_{11} = s(m_1^D + c_2)/(m_1^D c_2)$, $\sigma_{22} = (t-s)(m_1^D + c_2)/(m_1^D c_2)$, $\sigma_{12} = \sigma_{21} = s^{1/2}(t-s)^{1/2}/c_2$, $u(x_1, x_2) = \min\{x_1/s^{1-1/2k},\ x_2/(t-s)^{1-1/2k}\}$ *and* h^* *is a probability density function defined in (3.6).*

3.3 Weak convergence in the dual model

If G_{QP} has the unique global minimum of type k at m, then G_{PQ} also has the unique global minimum of type k at $m^D = \Psi'_P(m)$ by Theorem 2.1. Furthermore condition (2.7) implies condition (2.8). Therefore if the probability measure P satisfies the uniform local limit theorem of Daniels, the same argument in Sections 3.1 and 3.2 are available for the dual model. Thus we have the following weak convergence of the process $\{W_n^D(t),\ 0 \le t \le 1\}$.

THEOREM 3.5. *Assume that* G_{QP} *has the unique global minimum of type* k *at* m *and condition (2.7) holds. Then the process* $W_n^D(t)$, $0 \le t \le 1$, *converges weakly to a process* $W^D(t)$ *whose finite dimensional distribution is determined as follows: for* $0 \le s \le t \le 1$,

$$(W^D(s),\ W^D(t) - W^D(s)) \quad \sim \quad N(\underline{0}, \textstyle\sum = (\sigma_{ij})), \qquad \text{if } k = 1\ ,$$

and

$$Pr(W^D(s) \le x_1, W^D(t) - W^D(s) \le x_2) = \int_{-\infty}^{u(x_1,x_2)} h^D(y)dy, \quad \text{if } k \ge 2,$$

where $\sigma_{11} = s(m_1 + c_2')/(m_1 c_2')$, $\sigma_{22} = (t-s)(m_1 + c_2')/(m_1 c_2')$, $\sigma_{12} = \sigma_{21} = s^{1/2}(t-s)^{1/2}/c_2'$, $c_2' = c_2[\Psi''_P(m)]^2$, $u(x_1, x_2) = \min\{x_1/s^{1-1/2k}, x_2/(t-s)^{1-1/2k}\}$, $c_{2k}' = G_{PQ}^{(2k)}(m^d)$ *and* $h^D(y) = \dfrac{\exp\{-c_{2k}' y^{2k}/(2k)!\}}{\int_R \exp\{-c_{2k}' y^{2k}/(2k)!\}du}$.

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