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ESTIMATION FOR THE DERIVATIVES OF MEAN PERFORMANCE MEASURES IN A MARKOVIAN QUEUE WITH BATCH ARRIVALS

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ABSTRACT. This article finds smoothed perturbation analysis estimates for the derivatives of mean performance measures in a Markovian queue with batch arrivals. We show that those estimates can be observed from a single sample path.

1. Introduction

In this paper, we consider a batch arrival queueing process defined by the distribution functions $F_i(x, \lambda_i) = 1 - e^{-\lambda_i x}$, $i = 1, 2, \ldots, n$ and $G(y, \theta) = 1 - e^{-\frac{1}{\theta} y}$ as follows.

At each state, except state 0, X_i , $i = 1, 2, \ldots, n$ and Y are generated according to $F_i(x, \lambda_i)$ $i = 1, 2, \ldots, n$ and $G(y, \theta)$ respectively. If $\min\{X_1, \ldots, X_n, Y\} = X_i$, X_i becomes a sojourn time of the state, and the process jumps up i-steps. If $\min\{X_1, \ldots, X_n, Y\} = Y$, Y becomes a sojourn time of the state and the process jumps down one step. At state 0, $\min\{X_1, \ldots, X_n\}$ becomes a sojourn time after which the process always jumps up.

In the following section, we derive a method to estimate right derivatives of the mean performancd measures from a single sample path in the above mentioned batch arrival queueing process. First, we find an estimate for the derivative of expected total system time during a busy cycle with respect to the mean service time and, after that, we do the

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same thing for the expected length of busy cycle so that we can find an estimate for the derivative of steady state mean system time. In [1,2], we derived estimates for those derivatives with respect to the mean inter-arrival time in an M/M/1 queue. Here, by using appropriate conditional expectations, we derive estimates for those derivatives with respect to the mean service time in a batch arrival queue. The result also clearly shows that IPA estimate does not work in general. We begin with defining notations.

If $X_1, X_2, \ldots, X_n$ and Y are generated at the end of (i-1)-th sojourn time, we denote it by $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ and Y_i respectively. Hence, the i-th sojourn time will be $X_{i,k}$ if $min\{X_{i,1}, X_{i,2}, \ldots, X_{i,n}, Y_i\} = X_{i,k}$.

$C_{0,0}(\theta)$ denotes the length of a busy cycle i.e. the recurrence time of state 0, $C_{i,0}(\theta)$ denotes the time for the first transition from state i to state 0, and $C_{i,0}(\theta, j)$ denotes $C_{i,0}(\theta)$ in the j-th busy cycle.

$R_{i,0}(\theta)$ represents the area under the graph of the process from state i to state 0. We note that $R_{0,0}(\theta)$ is the sum of system times during a busy cycle for the above mentioned queueing process.

Let U denote the set of indices at which the process jumps up during $C_{0,0}(\theta)$,D be the set of indices at which the process jumps down during the same busy cycle $C_{0,0}(\theta)$. Let D_i be the subset of indices in D which are greater than i.

Suppose that when the parameter θ is increased to $\theta + \Delta\theta$, Y_i is correspondingly increased to $Y_i + \Delta Y_i$. In this case, we let A_i be the event of a corresponding change from jump-down to jump-up at the end of sojourn time Y_i and B_i be the event that any change from jump-down to jump-up does not occur, i.e.,

$$A_i = \{Y_i < \min_j X_{i,j},\ Y_i + \Delta Y_i \geq \min_j X_{i,j}\},$$

$$B_i = \{Y_i < \min_j X_{i,j},\ Y_i + \Delta Y_i < \min_j X_{i,j}\}.$$

IC denotes the set of indices at which interchanges occur when there are two or more interchanges in a sample path during $C_{0,0}(\theta)$.

For convenience, we introduce the notation W, as follows

$$W_j = \begin{cases} A_j, & \text{if } j \notin IC \\ B_j, & \text{if } j \in IC. \end{cases}$$

$\alpha(i)$ denotes the state of process at the i-th transition.

Finally, $k(i)$ denotes the index of minimum interarrival time generated at the i-th transition. Hence, if $\min_j X_{i,j} = X_{i,t}$, $k(i) = t$.

2. Estimate for the expected total system time of customers during a busy cycle

In this section, we derive a method to estimate the expected total system time of customers from a single sample path.

THEOREM 1. *In a batch arrival queueing process defined in section 1, assume the stability condition* $\theta \sum_{k=1}^{n} k\lambda_k < 1$. *Then*

$$\begin{aligned}
\frac{dE[R_{0,0}(\theta)]}{d\theta^+} =& \frac{1}{\theta}E[\sum_{i\in D} \alpha(i)Y_i] \\
&+ \frac{\sum_{k=1}^{n}(k+1)\lambda_k}{\theta\lambda_1 + \cdots + \theta\lambda_n + 1}E[\sum_{i\in D} C_{\alpha(i)-1,0}(\theta)] \\
&+ \frac{\sum_{k=1}^{n}(k+1)\lambda_k}{\theta\lambda_1 + \cdots + \theta\lambda_n + 1}E[|D|]E[R_{1,0}(\theta)] \\
&+ \frac{\sum_{k=1}^{n}k(k+1)\lambda_k}{2(\theta\lambda_1 + \cdots + \theta\lambda_n + 1)}E[|D|]E[C_{1,0}(\theta)].
\end{aligned}$$

PROOF. Considering the effects on a given sample path of a small change in the parameter θ [1, 2], we have

$$\begin{aligned}
&R_{0,0}(\theta+\Delta\theta) - R_{0,0}(\theta) = I(\text{no interchange})\sum_{i\in D}\alpha(i)\Delta Y_i \\
&+ \sum_{i\in D} I(\text{only one interchange at } Y_i) * \\
&\{ \sum_{j\in D-i} \alpha(j)\Delta Y_j + (k(i)+1)C_{\alpha(i)-1,0}(\theta) + (X_{i,k(i)} - Y_i)\alpha(i) \\
&\qquad + \sum_{t\in D_i}(k(i)+1)\Delta Y_t + R_{k(i)+1,0}(\theta+\Delta\theta)\} \\
&+ I(\text{two or more interchanges})\{R_{0,0}(\theta+\Delta\theta) - R_{0,0}(\theta)\}
\end{aligned}$$

$$= \sum_{i\in D} \alpha(i)\Delta Y_i + \sum_{i\in D} I(\text{only one interchange at } Y_i)*$$
$$[(k(i)+1)C_{\alpha(i)-1,0}(\theta) + \{X_{i,k(i)} - (Y_i + \Delta Y_i)\}\alpha(i) + \sum_{t\in D_i}(k(i)+1)\Delta Y_t + R_{k(i)+1,0}(\theta+\Delta\theta)]$$
$$+ I(\text{two or more interchanges})\{R_{0,0}(\theta+\Delta\theta) - R_{0,0}(\theta) - \sum_{i\in D}\alpha(i)\Delta Y_i\}$$

Using notations from section 1 and noting that $\Delta Y = \frac{Y}{\theta}\Delta\theta$ for exponential random variable, we have the following expectation from the above expression.

(1)

$$\frac{E[R_{0,0}(\theta+\Delta\theta) - R_{0,0}(\theta)]}{\Delta\theta}$$
$$= E[\sum_{i\in D}\alpha(i)\frac{Y_i}{\theta}] + \frac{1}{\Delta\theta}E[\sum_{i\in D} I(A_i)\{\prod_{j\in D-i} I(B_j)\}\{(k(i)-1)C_{\alpha(i)-1,0}(\theta)$$
$$+ \{X_{i,k(i)} - (Y_i + \frac{Y_i}{\theta}\Delta\theta)\}\alpha(i) + \sum_{t\in D_i}(k(i)+1)\frac{Y_t}{\theta}\Delta\theta + R_{k(i)+1,0}(\theta+\Delta\theta)\}]$$
$$+ \frac{1}{\Delta\theta}E[\sum_{|IC|>1}\{\prod_{i\in D} I(W_i)\}\{R_{0,0}(\theta+\Delta\theta) - R_{0,0}(\theta) - \sum_{i\in D}\alpha(i)\frac{Y_i}{\theta}\Delta\theta\}]$$

In the above expression, $|IC| > 1$ denotes all the possible combinations of interchanges when there are two or more interchanges.

When $\Delta\theta$ goes to zero, we compute the limit value of the second term in the right hand side of (1) and show the limit value of the third term is equal to zero. Similarly as in [1,2], we consider events of the following type

$$\{min_j x_{2,j} \le Y_2, min_j X_{3,j} > Y_3, \cdots, min_j x_{l,j} > Y_l,$$
$$min_j x_{1,j} = x_{1,5}, min_j x_{2,j} = x_{2,1}, \ldots, min_j x_{l,j} = X_{l,3}\}$$

Let l be the number of sojourn times during a busy cycle. If l is given, by selecting all the possible choices of inequality signs and considering all the possible jump sizes obtained from $\min_j X_{i,j}$, $i = 1, 2, \ldots, l$, we have finite events of this type. Let us denote these events by $H_{l,m}$, $l = 2, 3, \ldots,$; $m = 1, 2, \ldots,$. Let H be the smallest σ-algebra generated by these events. Now, conditioning on H, we compute the limit value of the second term in the right hand side of (1)

$$E[\sum_{i\in D} I(A_i)\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)]$$

(2)
$$= E[\ E[\sum_{i\in D} I(A_i)\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)|H]\ ]$$

$$= \sum_l \sum_m E[\sum_{i\in D} I(A_i)\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)|H_{l,m}]P(H_{l,m})$$

Since $I(A_i)$ is conditionally independent of $I(B_j)$ and $C_{\alpha(i)-1,0}(\theta)$.

(3)
$$E[I(A_i)\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)|H_{l,m}]$$
$$= E[I(A_i)|H_{l,m}]E[\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)|H_{l,m}]$$

Then

(4)
$$E[I(A_i)|H_l,m]$$
$$= E[I(Y_i < \min_j X_{i,j}\ , Y_i + \Delta Y_i \le \min_j X_{i,j})|H_{l,m}]$$
$$= P\{Y_i < X_{i,t} < (1+\frac{\Delta\theta}{\theta})Y_i | Y_i < \min_j X_{i,j}, \min_j X_{i,j} = X_{i,t}\}$$
$$= P\{Y_i < X_{i,t} < (1+\frac{\Delta\theta}{\theta})Y_i | Y_i < X_{i,t} < \min_{j\neq t} X_{i,j}\}$$
$$= \frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)\Delta\theta}{(\lambda_1+\lambda_2+\cdots+\lambda_n)(\theta+\Delta\theta)+1}$$

From (3) and (4), expression (2) becomes

(5)
$$\frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)\Delta\theta}{(\lambda_1+\lambda_2+\cdots+\lambda_n)(\theta+\Delta\theta)+1} *$$
$$\sum_l \sum_m E[\sum_{i\in D}\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)|H_{l,m}]P(H_{l,m})$$
$$= \frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)\Delta\theta}{(\lambda_1+\lambda_2+\cdots+\lambda_n)(\theta+\Delta\theta)+1} *$$
$$E[E[\sum_{i\in D}\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)]$$

Since $\sum_{i\in D} C_{\alpha(i)-1,0}(\theta) \leq |D|C_{0,0}(\theta)$ and $E[|D|C_{0,0}(\theta)] < \infty$, by the Lebesgue convergence theorem[3],

$$
\begin{aligned}
&\lim_{\Delta\theta\to 0+} \frac{1}{\Delta\theta} E[\sum_{i\in D} I(A_i)\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)] \\
&= \lim_{\Delta\theta\to 0+} \frac{1}{\Delta\theta} \frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)\Delta\theta}{(\lambda_1+\lambda_2+\cdots+\lambda_n)(\theta+\Delta\theta)+1} * \\
&\qquad \cdot E[\sum_{i\in D}\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)] \\
&= \frac{\lambda_1+\lambda_2+\cdots+\lambda_n}{(\lambda_1+\lambda_2+\cdots+\lambda_n)\theta+1} * \\
&\qquad E[\lim_{\Delta\theta\to 0+}\sum_{i\in D}\{\prod_{j\in D-i} I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)] \\
&= \frac{\lambda_1+\lambda_2+\cdots+\lambda_n}{(\lambda_1+\lambda_2+\cdots+\lambda_n)\theta+1} E[\sum_{i\in D}(k(i)+1)C_{\alpha(i)-1,0}(\theta)]
\end{aligned} \tag{6}
$$

Let $HR_{l,m}$ be an event of the following type

$$\{\min_j X_{2,j} \leq Y_2, \min_j X_{3,j} > Y_3, \ldots min_j X_{l,j} > Y_l\}$$

Also, let HR be the smallest $\sigma-$algebra generated by all events of the form $HR_{l,m}$. Then, for any i in D

$$
\begin{aligned}
E[k(i)+1|HR_{l,m}] &= \sum_{k=1}^{n}(k+1)P\{\min_j X_{i,j} = X_{i,k}|Y < \min_j X_{i,j}\} \\
&= \sum_{k=1}^{n}(k+1)\frac{\lambda_k}{\lambda_1+\lambda_2+\cdots+\lambda_n}
\end{aligned} \tag{7}
$$

In other words, $E[k(i)+1|HR]$ is a constant for all $i \in D$. Since

$$
\begin{aligned}
&E[(k(i)+1)C_{\alpha(i)-1,0}(\theta)|HR_{l,m}] \\
&\quad = E[(k(i)+1)|HR_{l,m}]E[C_{\alpha(i)-1,0}(\theta)|HR_{l,m}],
\end{aligned}
$$

$$
\begin{aligned}
& E[\sum_{i\in D}(k(i)+1)C_{\alpha(i)-1,0}(\theta)] \\
=& E[E[\sum_{i\in D}(k(i)+1)C_{\alpha(i)-1,0}(\theta)|HR]] \\
=& E[E[k(i)+1|HR]E[\sum_{i\in D}C_{\alpha(i)-1,0}(\theta)|HR]] \\
=& E[k(i)+1|HR]E[E[\sum_{i\in D}C_{\alpha(i)-1,0}(\theta)|HR]] \\
=& \frac{\sum_{k=1}^{n}(k+1)\lambda_k}{\lambda_1+\lambda_2+\cdots+\lambda_n}E[\sum_{i\in D}C_{\alpha(i)-1,0}(\theta)]
\end{aligned} \tag{8}
$$

Hence, from the expression (6) and (8),

$$
\begin{aligned}
& \lim_{\Delta\theta\to 0+}\frac{1}{\Delta\theta}E[\sum_{i\in D}I(A_i)\{\prod_{j\in D-i}I(B_j)\}(k(i)+1)C_{\alpha(i)-1,0}(\theta)] \\
=& \frac{\sum_{k=1}^{n}(k+1)\lambda_k}{(\lambda_1+\lambda_2+\cdots+\lambda_n)\theta+1}E[\sum_{i\in D}C_{\alpha(i)-1,0}(\theta)]
\end{aligned} \tag{9}
$$

On the other hand,

$$
\begin{aligned}
& E[I(A_i)\{\prod_{j\in D-i}I(B_j)\}R_{k(i)+1,0}(\theta+\Delta\theta)|H_{l,m}] \\
&= E[I(A_i)|H_{l,m}]E[\prod_{j\in D-i}I(B_j)|H_{l,m}]E[R_{k(i)+1,0}(\theta+\Delta\theta)|H_{l,m}] \\
&= E[I(A_i)|H_{l,m}]E[\prod_{j\in D-i}I(B_j)|H_{l,m}]* \\
&\qquad E[(k(i)+1)R_{1,0}(\theta+\Delta\theta)+\frac{(k(i)+1)k(i)}{2}C_{1,0}(\theta+\Delta\theta)|H_{l,m}] \\
&= E[I(A_i)|H_{l,m}]\langle E[(k(i)+1)\{\prod_{j\in D-i}I(B_j)\}|H_{l,m}]E[R_{1,0}(\theta+\Delta\theta)] \\
&\qquad + E[\frac{(k(i)+1)k(i)}{2}\{\prod_{j\in D-i}I(B_j)\}|H_{l,m}]E[C_{1,0}(\theta+\Delta\theta)]\rangle
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&E[\sum_{i\in D} I(A_i)\{\prod_{j\in D-i} I(B_j)\}R_{k(i)+1,0}(\theta+\Delta\theta)] \\
&= E[E[\sum_{i\in D} I(A_i)\{\prod_{j\in D-i} I(B_j)\}R_{k(i)+1,0}(\theta+\Delta\theta)|H]] \\
(10)\qquad &= E[I(A_i)|H_{l,m}]\langle E[\sum_{i\in D}(k(i)+1)\{\prod_{j\in D-i} I(B_j)\}]E[R_{1,0}(\theta+\Delta\theta)] \\
&\quad + E[\sum_{i\in D}\frac{(k(i)+1)k(i)}{2}\{\prod_{j\in D-i} I(B_j)\}]E[C_{1,0}(\theta+\Delta\theta)]\rangle
\end{aligned}
$$

By the Lebesgue convergence theorem,

$$
\begin{aligned}
&\lim_{\Delta\theta\to 0+}\frac{1}{\Delta\theta}E[\sum_{i\in D} I(A_i)\{\prod_{j\in D-i} I(B_j)\}R_{k(i)+1,0}(\theta+\Delta\theta)] \\
&= \lim_{\Delta\theta\to 0+}\frac{1}{\Delta\theta}\frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)\Delta\theta}{(\lambda_1+\lambda_2+\cdots+\lambda_n)(\theta+\Delta\theta)+1}* \\
&\quad \langle E[\sum_{i\in D}(k(i)+1)\{\prod_{j\in D-i} I(B_j)\}]E[R_{1,0}(\theta+\Delta\theta)] \\
&\quad + E[\sum_{i\in D}\frac{(k(i)+1)k(i)}{2}\{\prod_{j\in D-i} I(B_j)\}]E[C_{1,0}(\theta+\Delta\theta)]\rangle \\
(11)\qquad &= \frac{\lambda_1+\lambda_2+\cdots+\lambda_n}{(\lambda_1+\lambda_2+\cdots+\lambda_n)\theta+1}* \\
&\quad \langle E[\lim_{\Delta\theta\to 0+}\sum_{i\in D}(k(i)+1)\{\prod_{j\in D-i} I(B_j)\}]E[\lim_{\Delta\theta\to 0+} R_{1,0}(\theta+\Delta\theta)] \\
&\quad + E[\lim_{\Delta\theta\to 0+}\sum_{i\in D}\frac{(k(i)+1)k(i)}{2}\{\prod_{j\in D-i} I(B_j)\}]E[\lim_{\Delta\theta\to 0+} C_{1,0}(\theta+\Delta\theta)]\rangle \\
&= \frac{\lambda_1+\lambda_2+\cdots+\lambda_n}{(\lambda_1+\lambda_2+\cdots+\lambda_n)\theta+1}* \\
&\quad \langle E[\sum_{i\in D}(k(i)+1)]E[R_{1,0}(\theta)] + E[\sum_{i\in D}\frac{(k(i)+1)k(i)}{2}]E[C_{1,0}(\theta)]\rangle
\end{aligned}
$$

By the similar method as in the statements following (6),

$$
\begin{aligned}
E[\sum_{i\in D}(k(i)+1)] &= E[E[\sum_{i\in D}(k(i)+1)|HR]] \\
&= E[k(i)+1|HR]E[|D|] \\
&= \frac{\sum_{k=1}^{n}(k+1)\lambda_k}{\lambda_1+\lambda_2+\cdots+\lambda_n}E[|D|]
\end{aligned}
$$

and

$$
\begin{aligned}
E[\sum_{i\in D}\frac{k(i)(k(i)+1)}{2}] &= E[E[\sum_{i\in D}\frac{k(i)(k(i)+1)}{2}|HR]] \\
&= E[\frac{k(i)(k(i)+1)}{2}|HR]E[|D|] \\
&= \frac{\sum_{k=1}^{n}k(k+1)\lambda_k}{2(\lambda_1+\lambda_2+\cdots+\lambda_n)}E[|D|]
\end{aligned}
$$

From this, expression (11) becomes

$$
\begin{aligned}
&= \frac{\sum_{k=1}^{n}k(k+1)\lambda_k}{(\lambda_1+\lambda_2+\cdots+\lambda_n)\theta+1}\theta E[|D|]E[R_{1,0}(\theta)] \\
&\quad + \frac{\sum_{k=1}^{n}k(k+1)\lambda_k}{2\{(\lambda_1+\lambda_2+\cdots+\lambda_n)\theta+1\}}E[|D|]E[C_{1,0}(\theta)]
\end{aligned}
\tag{12}
$$

On the other hand,

$$
\begin{aligned}
&\lim_{\Delta\theta\to 0+}\frac{1}{\Delta\theta}E[\sum_{i\in D}I(A_i)\{\prod_{j\in D-i}I(B_j)\}\sum_{i\in D}(k(i)+1)\frac{Y_i}{\theta}\Delta\theta] \\
&= E[\lim_{\Delta\theta\to 0+}\sum_{i\in D}I(A_i)\{\prod_{j\in D-i}I(B_j)\}\sum_{i\in D}(k(i)+1)\frac{Y_i}{\theta}] = 0
\end{aligned}
\tag{13}
$$

and
(14)

$$
\begin{aligned}
&\lim_{\Delta\theta\to 0+}\frac{1}{\Delta\theta}E[\sum_{i\in D}I(A_i)\{\prod_{j\in D-i}I(B_j)\}\{-X_{i,k(i)}+(Y_i+\Delta Y_i)\}\alpha(i)] \\
&\le \lim_{\Delta\theta\to 0+}\frac{1}{\Delta\theta}E[\sum_{i\in D}I(A_i)\{\prod_{j\in D-i}I(B_j)\}\{-Y_i+(Y_i+Y_i\frac{\Delta\theta}{\theta})\}\alpha(i)] \\
&= E[\lim_{\Delta\theta\to 0+}\sum_{i\in D}I(A_i)\{\prod_{j\in D-i}I(B_j)\}\sum_{i\in D}\frac{Y_i}{\theta}\alpha(i)] = 0
\end{aligned}
$$

Finally, we show that the last limit value in (1) is equal to zero. If we assume that the first interchange among two or more interchange occured at i,

$$E[\sum_{|IC|>1}\{\prod_{i\in D} I(W_i)\}\{R_{0,0}(\theta+\Delta\theta)-R_{0,0}(\theta)-\sum_{i\in D}\alpha(i)\frac{Y_i}{\theta}\Delta\theta\}]$$

(15)

$$= E[\sum_{i\in D} I(A_i)\{\prod_{j\in D-i} I(W_j)\}\{R_{0,0}(\theta+\Delta\theta)-R_{0,0}(\theta)-\sum_{k\in D}\alpha(k)\frac{Y_k}{\theta}\Delta\theta\}]$$

Similarly as in (2), by using conditional expectation, the above expression becomes

$$E[E[I(A_i)|H_{l,m}]E[\sum_{i\in D}\{\prod_{j\in D-i} I(W_j)\}*$$
$$\{R_{0,0}(\theta+\Delta\theta)-R_{0,0}(\theta)-\sum_{k\in D}\alpha(k)\frac{Y_k}{\theta}\Delta\theta\}|H_{l,m}]]$$

(16)

$$= [E[I(A_i)|H_{l,m}]E[E[\sum_{i\in D}\{\prod_{j\in D-i} I(W_j)\}*$$
$$\{R_{0,0}(\theta+\Delta\theta)-R_{0,0}(\theta)-\sum_{k\in D}\alpha(k)\frac{Y_k}{\theta}\Delta\theta\}|H_{l,m}]]$$
$$= \frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)\Delta\theta}{(\lambda_1+\lambda_2+\cdots+\lambda_n)(\theta+\Delta\theta)+1}E[\sum_{i\in D}\{\prod_{j\in D-i} I(W_j)\}*$$
$$\{R_{0,0}(\theta+\Delta\theta)-R_{0,0}(\theta)-\sum_{k\in D}\alpha(k)\frac{Y_k}{\theta}\Delta\theta\}]$$

Trivially,

$$R_{0,0}(\theta+\Delta\theta)-R_{0,0}(\theta)-\sum_{k\in D}\alpha(k)\frac{Y_k}{\theta}\Delta\theta \le R_{0,0}(\theta+\Delta\theta)$$

Hence, the expression (16) becomes less than or equal to

(17)

$$\frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)\Delta\theta}{(\lambda_1+\lambda_2+\cdots+\lambda_n)(\theta+\Delta\theta)+1}E[\sum_{i\in D}\{\prod_{j\in D-i} I(W_j)\}R_{0,0}(\theta+\Delta\theta)]$$

If $\Delta\theta$ is small enough to satisfy the stability condition $\theta\sum_{k=1}^{n} k\lambda_k < 1$,

$$E[|D|R_{0,0}(\theta+\Delta\theta)] < \infty \text{ [5]}.$$

By the Lebesgue convergence theorem,
(18)

$$\lim_{\Delta\theta\to 0^+} \frac{1}{\Delta\theta} E[\sum_{|IC|>1} \{\prod_{i\in D} I(W_i)\}\{R_{0,0}(\theta+\Delta\theta) - R_{0,0}(\theta) - \sum_{i\in D} \alpha(i)\frac{Y_i}{\theta}\Delta\theta\}]$$

$$\leq \lim_{\Delta\theta\to 0^+} \frac{1}{\Delta\theta} \frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)\Delta\theta}{(\lambda_1+\lambda_2+\cdots+\lambda_n)(\theta+\Delta\theta)+1} E[\sum_{i\in D}\{\prod_{j\in D-i} I(W_j)\}R_{0,0}(\theta+\Delta\theta)]$$

$$\leq \frac{\lambda_1+\lambda_2+\cdots+\lambda_n}{(\lambda_1+\lambda_2+\cdots+\lambda_n)\theta+1} E[\lim_{\Delta\theta\to 0^+} \sum_{i\in D}\{\prod_{j\in D-i} I(W_j)\}R_{0,0}(\theta+\Delta\theta)] = 0$$

Hence, from (1),(9),(12),(13),(14), and (18), the theorem is proved. □

3. Estimate for the derivative of expected busy cycle

Using a similar method as in section 2, the result for expected busy cycle can be obtained. We omit the proof and simply state the result in theorem2.

THEOREM 2. *Under the same assumption as in theorem 1,*

$$\frac{dE[C_{0,0}(\theta)]}{d\theta^+} = \frac{1}{\theta}E[\sum_{i\in D} Y_i] + \frac{\sum_{k=1}^n (k+1)\lambda_k}{(\lambda_1+\lambda_2+\cdots+\lambda_n)\theta+1} E[|D|]E[C_{1,0}(\theta)]$$

4. Verification for a simple queue

Here, we will verify theorem 2 for a batch arrival queue defined by the distribution functions $F_i(x,\lambda_i) = 1 - e^{-\lambda_i x}, i = 1,2$ and $G(y,\theta) = 1 - e^{-\frac{1}{\theta}y}$ as in section 1. Let N be a batch size, B be a busy period, and $S_N = Y_1 + \cdots + Y_N$ be a batch service time.

$$\begin{aligned} E[C_{0,0}(\theta)] &= E[\min(X_1,X_2)] + E[B] \\ &= \frac{1}{\lambda_1+\lambda_2} + E[S_N]\frac{1}{1-\rho} \\ &= \frac{1}{\lambda_1+\lambda_2} + \frac{\theta(\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}\frac{1}{1-\theta(\lambda_1+2\lambda_2)} \\ &= \frac{1}{(\lambda_1+\lambda_2)\{1-\theta(\lambda_1+2\lambda_2)\}} \end{aligned}$$

$$\frac{dE[C_{0,0}(\theta)]}{d\theta^+} = \frac{\lambda_1 + 2\lambda_2}{\lambda_1 + \lambda_2} \frac{1}{(1-\rho)^2} \tag{19}$$

On the other hand,

$$E[|D|] = \frac{1}{1-\rho}E[N] = \frac{1}{1-\rho}\frac{\lambda_1 + 2\lambda_2}{\lambda_1 + \lambda_2}$$
$$E[C_{1,0}(\theta)] = \frac{1}{1-\rho}E[Y] = \frac{1}{1-\rho}\theta \quad \text{[see, [5]]}$$
$$E[Y|Y < \min(X_1, X_2)] = \frac{1}{\frac{1}{\theta} + \lambda_1 + \lambda_2}$$
$$E[\sum_{i \in D} Y_i] = E[|D|]E[Y|Y < \min(X_1, X_2)]$$

Hence,

$$\begin{aligned} &\frac{1}{\theta}E[\sum_{i\in D} Y_i] + \frac{\sum_{k=1}^{2}(k+1)\lambda_k}{(\lambda_1+\lambda_2)\theta + 1}E[|D|]E[C_{1,0}(\theta)] \\ &= E[|D|]\{\frac{1}{\theta}E[Y|Y < \min(X_1, X_2)] + \frac{2\lambda_1 + 3\lambda_2}{(\lambda_1+\lambda_2)\theta + 1}E[C_{1,0}(\theta)]\} \\ &= \frac{\lambda_1 + 2\lambda_2}{\lambda_1 + \lambda_2}\frac{1}{(1-\rho)^2} \end{aligned} \tag{20}$$

From (19) and (20), the result follows.

5. Remarks

All the quantities in theorem 1 and 2 can be estimated while we observe a simulated sample path of the system. For example, if m busy cycles of our queueing process were simulated, the following estimation can be obtained by simply observing a single sample path.

$$E[C_{1,0}(\theta)] \approx \frac{\sum_{i=1}^{m} C_{1,0}(\theta, i)}{m}$$

The other quantities in theorem 1 and 2 can be estimated in a similar way.

$\frac{dE[C_{0,0}(\theta)]}{d\theta}$ can be estimated also by the conventional finite difference method, that is,

$$\frac{dE[C_{0,0}(\theta)]}{d\theta} \approx \frac{1}{dt}\{\frac{\sum_{i=1}^{m} C_{0,0}(\theta+\Delta\theta, i) - \sum_{i=1}^{m} C_{0,0}(\theta, i)}{m}\}$$

It is known that the methods in theorem 1 and 2 are more economical and more accurate than the conventional finite difference method[6].

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