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A WEAKLY DEPENDENCE CONCEPTS OF BIVARIATE STOCHASTIC PROCESSES

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1. Introduction

In the last years there has been growing interest in concepts of positive(negative) dependence of stochastic processes such that concepts are considerable us in deriving inequalities in probability and statistics. Lehmann[7] introduced various concepts of positive(negative) dependence in the bivariate case. Stronger notions of bivariate positive(negativ e) dependence were later developed by Esary and Proschan[6]. Ahmed et al.[2], and Ebrahimi and Ghosh[5] obtained multivariate versions of various positive (negative) dependence as described by Lehmann[7] and Esary and Proschan[6]. Concepts of positive(negative) dependence for random variables have subsequently been extended to stochastic processes in different directions by many authors. For references of available results see Baek[3] and Ebrahimi[9,10]. When we observed several processes we could study each quantity on its own and treat each as a separate univariate process. Although this would give us some information about each quantity it could never give information about the interrelationship between various quantities. This leads us to introduce some concepts of weakly positive(negative) dependence for multivariate stochastic processes. To introduce the new ideas involving in the study of multivariate processes and to avoid complexity we consider the bivariate processes. The extension to the multivariate case is straightforward.
In this paper, we introduce the notions of weakly positive(negative) dependence of two bivariate processes. Furthermore, the main objective of

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this paper is to survey important results and the weaker properties than positive(negative) quadrant dependence defined over bivariate stochastic processes.
Also, the usefulles of weakly positive(negative) dependence in hypothesis testing, probability inequality theory, confidence estimation and reliability theory is well known. In Section 2, some definitions and properties are given. Some preservation theoretical results are proved in Section 3 and then illustrated with examples in Section 4.

2. Preliminaries

In this section, we present definitions, notations and basic facts used throughout the paper. Suppose that we are given two stochastic processes $\{X_1(t)|t \geq 0\}$ and $\{X_2(t)|t \geq 0\}$. The state space of $(X_1(t), X_2(t))$ will be taken to be a subset, $E = E_1 \times E_2$ of the plane R^2. For any state $a_i \in E_i, i = 1, 2$ we define the random times as follows

$$T_i(a_i) = \inf\{t|X_i(t) \leq a_i, t \geq 0\}, i = 1, 2.$$

In other words, $T_i(a_i)$ is the first time that the process $X_i(t)$ reaches or goes below a_i (see[9]). If we base the dependence between two processes on the dependence of their hitting times, then we have the following definitions.

DEFINITION 2.1 [EBRAHIMI (1987)]. *The processes $\{X_1(t)|t \geq 0\}$ and $\{X_2(t)|t \geq 0\}$ are said to be positive(negative) quadrant dependent (PQD(NQD)) if*

$$(2.1) \qquad P(\cap_{i=1}^{n}(T_i(a_i) > t_i)) \geq (\leq) \prod_{i=1}^{2} P(T_i(a_i) > t_i),$$
$$\textit{for all } t_i \geq 0, a_i \in E_i, i = 1, 2.$$

If equality holds in (2.1), then we say two processes are independent.

DEFINITION 2.2. *The processes* $\{X_1(t)|t \geq 0\}$ *and* $\{X_2(t)|t \geq 0\}$ *are said to be weakly positive(negative) quadrant dependent 1(WPQD1(WN QD1)) if*

(2.2)
$$\int_{t_2}^{\infty}\int_{t_1}^{\infty} P(\cap_{i=1}^{2}(T_i(a_i) > t_i))dt_1dt_2$$
$$\geq (\leq) \int_{t_2}^{\infty}\int_{t_1}^{\infty}\prod_{i=1}^{2} P(T_i(a_i) > t_i)dt_1dt_2, \quad \text{for all } t_i \geq 0, a_i \in E_i, i = 1,2.$$

DEFINITION 2.3. *The processes* $\{X_1(t)|t \geq 0\}$ *and* $\{X_2(t)|t \geq 0\}$ *are said to be weakly positive(negative) quadrant dependent 2 (WPQD2(WN QD2)) if*

$$\begin{aligned}(2.3)\qquad &\int_{0}^{t_2}\int_{0}^{t_1} P(\cap_{i=1}^{2}(T_i(a_i) > t_i))dt_1dt_2 \geq (\leq)\\ &\int_{0}^{t_2}\int_{0}^{t_1}\prod_{i=1}^{2} P(T_i(a_i) > t_i)dt_1dt_2, \text{ for all } t_i \geq 0, a_i \in E_i, i = 1,2.\end{aligned}$$

Moreover, $\{X_1(t)|t \geq 0\}$ and $\{X_2(t)|t \geq 0\}$ are said to be weakly positive (negative) quadrant dependent (WPQD(WNQD)) if they satisfy both WPQD1(WNQD1) and WPQD2 (WNQD2).

DEFINITION 2.4 [EBRAHIMI(1987)]. *The processes* $\{X_1(t)|t \geq 0\}$ *and* $\{X_2(t)|t \geq 0\}$ *are said to be associated if*

$$(2.4)\qquad cov(f(T_1(a_1), T_2(a_2)), g(T_1(a_1), T_2(a_2))) \geq 0,$$

for all non-decreasing, non-negative functions f and g for which the covariance exists and $a_i \in E_i, i = 1,2$.

First, we list below a number of important properties between definitions (2.2)-(2.4). It is easy to prove that

(WP_1) If the processes $\{X_1(t)|t \geq 0\}$ and $\{X_2(t)|t \geq 0\}$ are independent, $\{X_1(t)|t \geq 0\}$ and $\{X_2(t)|t \geq 0\}$ are WPQD(WNQD).

(WP_2) The union of independent set of WPQD(WNQD) is WPQD (WN QD).

(WP_3) Non-decreasing, non-negative convex functions of associated stochastic processes are associated.

Also, as a direct consequence of definitions (2.1)-(2.3), it is easy to verify that PQD(NQD) $\Rightarrow$ WPQD(WNQD).
The following example shows that WPQD1 does not imply PQD.

EXAMPLE 2.5. Consider a stochastic process $\{X_1(t)|t \geq 0\}$ and $\{X_2(t)|t \geq 0\}$ such that $X_1(0)$ and $X_2(1)$ have the following joint probabilities

	0	1	2
0	0.1	0	0.15
1	0.3	0.15	0.05
2	0	0.05	0.2

It is easy to check that $X_1(0)$ and $X_2(1)$ are WPQD1 but not PQD.

3. Some preservation of theoretical results

THEOREM 3.1. *The processes $\{X_1(t)|t \geq 0\}$ and $\{X_2(t)|t \geq 0\}$ are WPQD1 if and only if $\{f(X_1(t))|t \geq 0\}$ and $\{g(X_2(t))|t \geq 0\}$ are WPQD1 for all non-decreasing convex functions f and g. Similarly the processes $\{X_1(t)|t \geq 0\}$ and $\{X_2(t)|t \geq 0\}$ are WPQD2 if and only if $\{f(X_1(t))|t \geq 0\}$ and $\{g(X_2(t))|t \geq 0\}$ are WPQD2 for all non-decreasing concave functions f and g.*

THEOREM 3.2. *Let (a) $\{(X_{i1}(t), X_{i2}(t))|t \geq 0\}$ be WPQD1 processes for each $i = 1, 2, ..., n$ (b) $\{(X_{11}(t), X_{12}(t))|t \geq 0\}, \ldots, \{(X_{n1}(t), X_{n2}(t))|t \geq 0\}$ are independent 2-variate processes with increasing sample paths (c) f and g are non-negative non-decreasing convex functions. Then $Y_1(t) = f(X_{11}(t), \cdots, X_{n1}(t))$, $Y_2(t) = g(X_{12}(t), \cdots, X_{n2}(t))$ are WPQD1.*

PROOF. The proof will be given for the case $n = 2$. For the general n, the proof is similar. Fix $t_i \geq 0, i = 1, 2$ and introduce the variables $V_i = X_{2i}(t_i)$ and $U_i = \sup_{0 \leq s \leq t_i} g_i(X_{1i}(s), X_{2i}(s)), i = 1, 2$, where for simplicity,

t_1, t_2 have been suppressed in V_i and U_i. Consider any hitting times of $Y_i(s) = g_i(X_{1i}(s), X_{2i}(s))$ given by

$$W_i(a_i) = \inf\{s | Y_i(s) \geq a_i\}, i = 1, 2.$$

It suffices to show that

$$\int_{t_1}^{\infty}\int_{t_2}^{\infty} P(W_1(a_1) > t_1, W_2(a_2) > t_2) dt_2 dt_1$$
$$\geq \int_{t_1}^{\infty}\int_{t_2}^{\infty} P(W_1(a_1) > t_1) P(W_2(a_2) > t_2) dt_2 dt_1$$

Note that $U_i = \sup_{0 \leq s \leq t_i} g_i(X_{1i}(s), V_i)$ and that, by hypothesis, V_1 and V_2 are WPQDI random variables(see [1]). Now, we obtain

$$\int_{t_1}^{\infty}\int_{t_2}^{\infty} P(W_1(a_1) > t_1, W_2(a_2) > t_2) dt_2 dt_1$$
$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} P(U_1 < a_1, U_2 < a_2) dt_2 dt_1$$
$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} EP(U_1 < a_1, U_2 < a_2 | V_1, V_2) dt_2 dt_1$$
$$\geq \int_{t_1}^{\infty}\int_{t_2}^{\infty} E[P(U_1 < a_1 | V_1) P(U_2 < a_2 | V_2)] dt_2 dt_1 (\text{by (a), (b),(c) })$$
$$\geq \int_{t_1}^{\infty}\int_{t_2}^{\infty} EP(U_1 < a_1 | V_1) EP(U_2 < a_2 | V_2) dt_2 dt_1 (\text{by (a), (c)})$$
$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} P(W_1(a_1) > t_1) P(W_2(a_2) > t_2) dt_2 dt_1$$

COROLLARY 1. *Let (a) $\{(X_{i1}(t), X_{i2}(t)) | t \geq 0\}$ be WPQD1 processes with for each $i = 1, 2, ..., n$ (b)$\{Z_1(t) | t \geq 0\}$ and $\{Z_2(t) | t \geq 0\}$ are independent stochastic processes and $\{(Z_1(t), Z_2(t)) | t \geq 0\}$ be independent of $(X_{i1}(t), X_{i2}(t)), i = 1, 2, ..., n$ (c) f and g are non-negative non-decreasing convex functions. Then $X_1(t) = f(Z_1(t), X_{11}(t), X_{21}(t), \cdots, X_{n1}(t)$, $X_2(t) = g(Z_2(t), X_{21}(t), X_{22}(t), \cdots, X_{n2}(t))$ are WPQD1.*

The next theorem demonstrates the property of the WPQD1 under limites.

THEOREM 3.3. *Suppose* $\{X_{1n}(t)|t \geq 0\}$ *and* $\{X_{2n}(t)|t \geq 0\}$ *are sequence of non-negative WPQD1 with distribution functions* H_n *such that* $H_n \to H$ *weakly as* $n \to \infty$, *where* H *is the distribution functions of a stochastic processes. If* $E(T_1(a_1)T_2(a_2))$, $E(T_1(a_1))$ *and* $E(T_2(a_2))$ *are finite and* $cov(T_{1n}(a_1), T_{2n}(a_2)) \to cov(T_1(a_1), T_2(a_2))$, *then* $\{(X_1(t), X_2(t))|t \geq 0\}$ *is WPQD1*

PROOF. We show that $\{X_1(t)|t \geq 0\}$ and $\{X_2(t)|t \geq 0\}$ are WPQD1. Consider that

$$
\begin{aligned}
&\int_{t_1}^{\infty}\int_{t_2}^{\infty} \big[P(T_{1n}(a_1) > t_1, T_{2n}(a_2) > t_2) \\
&\quad - P(T_{1n}(a_1) > t_1)P(T_{2n}(a_2) > t_2)\big]\,dt_2 dt_1 \\
&= \operatorname{cov}(T_{1n}(a_1), T_{2n}(a_2)) - \int_0^{t_1}\int_0^{t_2} \big[P(T_{1n}(a_1) > t_1, T_{2n}(a_2) > t_2) \\
&\quad - P(T_{1n}(a_1) > t_1)P(T_{2n}(a_2) > t_2)\big]\,dt_2 dt_1.
\end{aligned}
$$

By taking the limit(as $n \to \infty$) and using the dominated convergence theorem and using the assumption of the theorem concerning the convergence of $\operatorname{cov}(T_{1n}(a_1), T_{2n}(a_2))$, we can obtain the result for hitting time processes.

The following theorem is application of theorem3.3 which is very important in recognizing WPQD1 in compound distributions which arise naturally in stochastic processes

THEOREM 3.4. *Let (a)* $\{(X_i(t), Y_i(t))|t \geq 0\}$ *be WPQD1, for* $i = 1, 2, ..., \infty$ *(b)* $N(t)$ *be a Poisson process which is independent of* $X_i(t)$ *and* $Y_i(t)$ *(c)* $\{(X_i(t), Y_i(t))|t \geq 0\}$ *be a sequence of non-negative independent bivariate random processes. Then* $X(t) = \sum_{i=1}^{N(t)} X_i(t)$ *and* $Y(t) = \sum_{i=1}^{N(t)} Y_i(t)$ *are WPQD1.*

PROOF.

$$\int_{t_1}^{\infty}\int_{t_2}^{\infty} P(T_1(a_1) > t_1, T_2(a_2) > t_2)dt_2dt_1$$

$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} P\big[\{\sum_{i=1}^{N(t)} X_i(t) > a_1, t_1 \leq s < \infty\},$$

$$\{\sum_{j=1}^{N(s)} Y_j(t) > a_2, t_2 \leq s < \infty\}\big]dt_2dt_1$$

$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} P\big(\sum_{i=1}^{N(t_1)} X_i(t) > a_1, \sum_{j=1}^{N(t_2)} Y_j(t) > a_2\big)dt_2dt_1$$

$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P\big(\sum_{i=1}^{k_1} X_i(t) > a_1, \sum_{j=1}^{k_2} Y_j(t) > a_2|$$

$$N(t_1) = k_1, N(t_2) = k_2) \cdot P(N(t_1) = k_1, N(t_2) = k_2)dt_2dt_1$$

$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P\big(\sum_{i=1}^{k_1} X_i(t) > a_1, \sum_{j=1}^{k_2} Y_j(t) > a_2\big)$$

$$P(N(t_1) = k_1, N(t_2) = k_2)dt_2dt_1$$

$$\geq \int_{t_1}^{\infty}\int_{t_2}^{\infty} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P\big(\sum_{i=1}^{k_1} X_i(t) > a_1\big)P\big(\sum_{j=1}^{k_2} Y_j(t) > a_2\big)$$

$$P(N(t_1) = k_1, N(t_2) = k_2)dt_2dt_1$$

$$= \int_{t_1}^{\infty} \sum_{k_1=0}^{\infty} P\big(\sum_{i=1}^{k_1} X_i(t) > a_1|N(t_1) = k_1\big)P(N(t_1) = k_1)dt_1$$

$$\cdot \int_{t_2}^{\infty} \sum_{k_2=0}^{\infty} P\big(\sum_{j=1}^{k_2} Y_j(t) > a_2|N(t_2) = k_2\big)P(N(t_2) = k_2)dt_2$$

$$= \int_{t_1}^{\infty}\int_{t_2}^{\infty} P(T_1(a_1) > t_1)P(T_2(a_2) > t_2)dt_2dt_1$$

4. Examples

EXAMPLE 4.1. Consider a bivariate process $\{(X_n(t), Y_n(t))|t \geq 0, n \geq 1\}$ such that $(X_1(t), Y_1(t)), (X_2(t), Y_2(t)), \cdots$ are independent with WPQD1 respectively. Then it is easy to check that $\{(X_n(t), Y_n(t))|t \geq 0, n \geq 1\}$ is WPQD1.

EXAMPLE 4.2. Let $\{(N_1(t), N_2(t))|t \geq 0\}$ be a bivariate poisson process(cf Barlow and Proschan, p137) joint probability function

$$P(N_1(t) = n_1, N_2(t) = n_2) = e^{-\lambda+(P_{11}+P_{10}+P_{01})}$$
$$\times \sum_{m=0}^{\min(k_1,k_2)} \frac{(\lambda + P_{11})^m(\lambda + P_{10})^{k_1-m}(\lambda + P_{01})^{k_2-m}}{m!(k_1 - m)!(k_2 - m)!}$$

Then it is easy to check that $\{(N_1(t), N_2(t))|t \geq 0\}$ is WPQD1.

EXAMPLE 4.3. Let $\{X(t)|t \geq 0\}$ be a Brownian motion process (cf. Ross(1983)) and let $\{Z(t)|t \geq 0\}$ be a Brownian motion reflected at the origin, that is $Z(t) = |X(t)|, t \geq 0$. Then we can obtain that $\{X(t)|t \geq 0\}$ and $\{Z(t)|t \geq 0\}$ are WPQD1.

EXAMPLE 4.4. Consider a system with two components which is subjected to shocks. Let $N_1(t), N_2(t)$ be the poisson process of shocks received by time t and let $X(t) = \sum_{i=1}^{N_1(t)} X_i$, $Y(t) = \sum_{i=1}^{N_2(t)} Y_i$ be total damages to components 1,2 by shocks, respectively. Then, by Theorem (3.4), $\{(X(t), Y(t))|t \geq 0\}$ is WPQD1.

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