Comm. Korean Math. Soc. **11** (1996), No. 3, pp. 815–823

THE KONTSEVICH CONJECTURE ON MAPPING CLASS GROUPS

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ABSTRACT. M. Kontsevich posed a problem on mapping class groups of 3-manifold that is if M is a compact 3-manifold with nonempty boundary, then $BDiff(M \text{ rel } \partial M)$ has the homotopy type of a finite complex. Here, $Diff(M \text{ rel } \partial M)$ is the group of diffeomorphisms of M which restrict to the identity on ∂M, and $BDiff(M \text{ rel } \partial M)$ is its classifying space. In this paper we resolve the problem affirmatively in the case when M is a Haken 3-manifold.

1. Introduction

A 3-manifold M is irreducible if each 2-sphere in M bounds a 3-cell in M. The restriction to irreducible manifolds has its main reason in the Poincaré conjecture.

By a surface, we will mean a compact, connected 2-manifold. Let M be a 3-manifold and F a surface which is either properly embedded in M or contained in ∂M. We say F is incompressible in M if none of the following conditions is satisfied.

(1) F is a 2-sphere which bounds a homotopy 3-cell in M, or
(2) F is a 2-cell and either $F \subset \partial M$ or there is a homotopy 3-cell $X \subset M$ with $\partial X \subset F \cup \partial M$, or
(3) there is a 2-cell $D \subset M$ with $D \cap F = \partial D$ and with ∂D not contractible in F.

Received April 11, 1996. Revised June 21, 1996.
1991 AMS Subject Classification: 57M99.
Key words and phrases: mapping class group, Haken manifold, boundary pattern.
Partially supported by BSRI-94-1422, Ministry of Education and NON DIRECTED RESEARCH FUND, Korea Research Foundation.

A 3-manifold M is said to be sufficiently large if M contains a properly embedded, 2-sided, incompressible surface. An irreducible 3-manifold M is called Haken if it is sufficiently large.

A *boundary pattern* $\underline{\underline{m}}$ for an n-manifold M is a finite set of compact, connected $(n-1)$-manifolds in ∂M, such that the intersection of any i of them is either empty or consists of $(n-i)$-manifolds. Thus when $n=3$, the components of the intersections of pairs of elements of the boundary pattern are arcs or circles, and if three elements meet, their intersection consists of a finite collection of points at which three intersection arcs meet. The symbol $|\underline{\underline{m}}|$ will mean the set of points of ∂M that lie in some element of $\underline{\underline{m}}$. It is important in arguments to distinguish between elements of $\underline{\underline{m}}$, which are surfaces in ∂M, and the points of M which lie in these surfaces, and we will always be precise in this distinction. When $|\underline{\underline{m}}| = \partial M$, $\underline{\underline{m}}$ is said to be *complete*. Provided that ∂M is compact, we define the *completion* of $\underline{\underline{m}}$ to be the complete boundary pattern $\overline{\underline{\underline{m}}}$ which is the union of $\underline{\underline{m}}$ and the collection of free sides. In particular, the set of boundary components of M is the boundary pattern $\overline{\underline{\underline{\emptyset}}}$.

Maps which respect boundary pattern structures are called admissible. Precisely, a map f from $(M, \underline{\underline{m}})$ to $(N, \underline{\underline{n}})$ is called *admissible* when $\underline{\underline{m}}$ is the disjoint union

$$\underline{\underline{m}} = \coprod_{G \in \underline{\underline{n}}} \{ \text{ components of } f^{-1}(G) \}.$$

Suppose $(X, \underline{\underline{x}})$ is an admissibly imbedded codimension-zero submanifold of $(M, \underline{\underline{m}})$, which is admissibly imbedded in $(M, \overline{\underline{\underline{m}}})$. The latter assumption guarantees that $X \cap \partial M = |\underline{\underline{x}}|$, and that an element of $\underline{\underline{x}}$ which does not meet any other element of $\underline{\underline{x}}$ must be imbedded in the manifold interior of an element of $\underline{\underline{m}}$. Let $\underline{\underline{x}}''$ denote the collection of components of the frontier of X in M. To *split M along X* means to construct the manifold with boundary pattern $(\overline{M - X}, \underline{\underline{\widetilde{m}}} \cup \underline{\underline{x}}'')$, where the elements of $\underline{\underline{\widetilde{m}}}$ are the closures of the components of $F - (X \cap F)$ for $F \in \underline{\underline{m}}$. The boundary pattern $\underline{\underline{\widetilde{m}}} \cup \underline{\underline{x}}''$ is called the *proper boundary pattern* on $\overline{M - X}$.

The group of admissible isotopy classes of admissible homemorphisms from $(M, \underline{\underline{m}})$ to $(M, \underline{\underline{m}})$ is denoted by $\mathcal{H}(M, \underline{\underline{m}})$. Suppose $\langle h \rangle \in \mathcal{H}(M, \underline{\underline{m}})$.

Since $h^{-1}(|\underline{\underline{m}}|)=|\underline{\underline{m}}|$, h must carry each free side of $(M,\underline{\underline{m}})$ homeomorphically to a free side of $(M,\underline{\underline{m}})$. Therefore h is admissible for $(M,\overline{\underline{\underline{m}}})$. Thus when working with mapping class groups of manifolds with boundary pattern, the requirement that the boundary pattern be complete is not at all restrictive.

An *i-faced disc* is a 2-disc whose boundary pattern is complete and has i components. Observe that each element of $\underline{\underline{m}}$ is incompressible if and only if whenever D is an admissibly imbedded 1-faced disc in $(M,\underline{\underline{m}})$, the boundary of D bounds a disc in $|\underline{\underline{m}}|$ which is contained in a single element of $\underline{\underline{m}}$. For most of Johannson's theory, a somewhat stronger condition is needed. The boundary pattern $\underline{\underline{m}}$ is called *useful* when the boundary of every admissibly imbedded i-faced disc in $(M,\underline{\underline{m}})$ with $i \leq 3$ bounds a disc D in ∂M such that $D \cap (\cup_{F\in\underline{\underline{m}}}\partial F)$ is the cone on $\partial D \cap (\cup_{F\in\underline{\underline{m}}}\partial F)$. Notice that $\overline{\emptyset}$ is a useful boundary pattern on M if and only if ∂M is incompressible.

Assume that $(M,\underline{\underline{m}})$ has a fixed structure as an I-bundle or Seifert fibered space, with projection map $p: M \to F$. The following definition is from 5.3 of [**J**]. Let G be a manifold. A map $g: G \to M$ is called *vertical* if its image is a union of nonexceptional fibers. It is called *horizontal* if $g^{-1}(\partial M)=\partial G$ and g is transverse to the fibers. In general, an essential surface in a fibered manifold is isotopic to one which is horizontal or vertical. Proposition 5.6 of [**J**] is:

THEOREM 1.1. (Vertical-horizontal Theorem) *Let $(M,\underline{\underline{m}})$ be an I-bundle or Seifert fiber space, with fixed admissible fibration, and let $p: M \to F$ be the fibre projection. Suppose $(M,\underline{\underline{m}})$ is not one of the exceptional fibered manifolds (EF1)-(EF5). Let G be an essential surface imbedded in $(M,\underline{\underline{m}})$ such that $\partial G \subset |\underline{\underline{m}}|$ and such that no component of G is a 2-sphere or an i-faced disc, $1 \leq i \leq 3$. Then G is admissibly isotopic either to a vertical surface or to a horizontal surface. If in addition B is any element of $\underline{\underline{m}}$ which is not a lid of $(M,\underline{\underline{m}})$, such that $B \cap G$ is either horizontal or vertical, then the admissible isotopy of G may be chosen constant on $B \cap G$.*

In most cases, the fibering of a fibered manifold is unique up to isotopy. The exceptions are determined in corollary 5.9 of [**J**]:

THEOREM 1.2. (Unique Fibering Theorem): *Suppose each of $(M_1,\underline{\underline{m_1}})$*

and $(M_2, \underline{\underline{m_2}})$ is an I-bundle or Seifert fibered space with a fixed admissible fibration, but neither is a solid torus with $\overline{\underline{\underline{m_i}}} = \overline{\emptyset}$, nor one of the exceptional fibered manifolds (EF3)-(EF5), (EIB) or (ESF). Then every admissible homeomorphism $h\colon (M_1, \underline{\underline{m_1}}) \to (M_2, \underline{\underline{m_2}})$ is admissibly isotopic to a fiber-preserving homeomorphism. Moreover,

(1) *the conclusion holds if M_i is one of the exceptions (EIB) and h and h^{-1} map lids to lids, and*
(2) *if M_1 is an I-bundle and $h\colon M_1 \to M_1$ is the identity on one lid, then the isotopy may be chosen to be constant on this lid.*

THEOREM 1.3. (Parallel Surfaces Theorem): *Let M be an irreducible 3-manifold with complete and useful boundary pattern, and let $(F, \underline{\underline{f}})$ and $(G, \underline{\underline{g}})$ be connected essential surfaces in $(M, \underline{\underline{m}})$, with $F \cap \partial M = \partial F$ and $G \cap \partial M = \partial G$. Assume that $(G, \underline{\underline{g}})$ is admissibly homotopic into $(F, \underline{\underline{f}})$. Then $(G, \underline{\underline{g}})$ is admissibly isotopic into $(F, \underline{\underline{f}})$. Moreover, if F and G are disjoint, then $(G, \underline{\underline{g}})$ is admissibly parallel to $(F, \underline{\underline{f}})$.*

One of the strongest properties of the characteristic submanifold is proposition 13.1 of [**J**].

THEOREM 1.4. (Enclosing Theorem): *Let $(M, \underline{\underline{m}})$ be a Haken 3-manifold with useful boundary pattern, and let V be its characteristic submanifold. Let $(X, \underline{\underline{x}})$ be an I-bundle or Seifert fiber space whose complete boundary pattern is useful. Suppose that $(X, \underline{\underline{x}})$ is not one of the exceptional cases (EF1)-(EF5). Then every essential map $f\colon (X, \underline{\underline{x}}) \to (M, \underline{\underline{m}})$ is admissibly homotopic into V.*

2. Almost geometric finiteness

We say that a group G is *almost geometrically finite* if it acts smoothly and properly discontinuously on a contractible manifold W containing a simplicial complex L, such that there is a G-equivariant deformation retraction from W onto L, the restricted action of G on L is simplicial, and L/G is finite. Note that any subgroup of finite index in an almost geometrically finite group is also almost geometrically finite.

PROPOSITION 2.1. *Let* $1 \to V \to G \to Q \to 1$ *be an exact sequence of groups, such that* V *contains a finitely generated abelian group of finite index and* Q *is almost geometrically finite. Then every torsion-free subgroup of finite index in* G *is geometrically finite.*

PROOF. Let H be such a group. Since $V \cap H$ contains a finitely generated subgroup of finite index, it must contain a finitely generated free abelian normal subgroup. Since $V \cap H$ is also torsion-free, section 2 of [**K-L-R**] shows that it is a Bieberbach group. The image R of H in Q has finite index, so is almost geometrically finite and acts properly dicontinously on a contractible manifold W containing an invariant contractible simplicial complex L such that L/R is finite. From [**C-R**], there exists a Seifert-fibering $p: \Sigma \to W/R$ with compact fiber and fundamental group H. Moreover, the universal covering of Σ is $W \times \mathbb{R}^n$, where n is the rank of the free abelian subgroup. The covering transformations corresponding to elements of $V \cap H$ take each $\mathbb{R}^n$-fiber in $W \times \mathbb{R}^n$ to itself, so H preserves $L \times \mathbb{R}^n$. The quotient $(L \times \mathbb{R}^n)/H$ is $p^{-1}(L/R)$, and since L/R is a finite complex and the fiber of p is a compact flat manifold, $(L \times \mathbb{R}^n)/H$ is a $K(H,1)$ which is a finite complex.

Let $(M, \underline{\underline{m}})$ be a Haken manifold with a complete and useful boundary pattern. We allow the possibility that ∂M is empty. From proposition 2.1 and similar arguments used in section 3 and section 4 of [**M**], we can deduce the following thereom. The detailed proofs of this theorem and its corollary can be found in [**H-M**].

THEOREM 2.2. *Let* M *be a Haken manifold and* $\underline{\underline{m}}$ *a boundary pattern on* M *whose completion is useful. Let* G *be a torsion-free subgroup of* $\mathcal{H}(M, \underline{\underline{m}})$. *Then* G *is geometrically finite.*

We remark that by theorem 4.3.1 of [**M**], $\mathcal{H}(M, \underline{\underline{m}})$ always contains a geometrically finite subgroup of finite index. Therefore the theorem 2.2 has the content for every Haken manifold.

COROLLARY 2.3. *Let* M *be a Haken 3-manifold with nonempty incompressible boundary. Then any torsion-free subgroup of finite index in* $\mathcal{H}(M \text{ rel } \partial M)$ *is geometrically finite.*

To see how this corollary applies to the Kontsevich conjecture, we first recall the following theorem of A. Hatcher [**H**].

THEOREM 2.4. *If M is a Haken 3-manifold with nonempty boundary then the components of $Diff(M \text{ rel } \partial M)$ are contractible.*

Thus $\pi_q(Diff(M \text{ rel } \partial M)) = 0$ for $q \geq 1$ and $\pi_1(BDiff(M \text{ rel } \partial M)) \cong \pi_0(Diff(M \text{ rel } \partial M))$, the mapping class group of $Diff(M \text{ rel } \partial M)$. It also implies that $BDiff(M \text{ rel } \partial M)$ is a $K(\pi_0(Diff(M \text{ rel } \partial M)), 1)$-space. Applying corollary 2.3 shows that whenever $\pi_0(Diff(M \text{ rel } \partial M))$ does not contain torsion, $BDiff(M \text{ rel } \partial M)$ is homotopy equivalent to a finite complex.

In the next section, we show that $\pi_0(Diff(M \text{ rel } \partial M))$ is torsion free.

3. The Kontsevich Conjecture

LEMMA 3.1. *Let M be a Haken manifold containing an incompressible surface G. Let f and g be two homeomorphisms of M which are homotopic relative to ∂M. Then f and g are isotopic relative to ∂M. If f and g agree on G, then they are isotopic relative to $G \cup \partial M$.*

PROOF. Replacing f by $g^{-1}f$, we may assume that f is orientation-preserving and g is the identity. By theorem 7.1 of [**W**], f is isotopic to the identity relative to ∂M. If f and g agree on G, then by Laudenbach theorem (see for example, page 31 of [**M**]), we may assume that the isotopy is relative to $G \cup \partial M$.

PROPOSITION 3.2. *Let M be a Haken manifold with nonempty incompressible boundary. Assume that each component of ∂M is a torus. If g is a map from M to itself such that $g^n \simeq 1_M$ relative to ∂M, then g is isotopic relative to ∂M to a homeomorphism h with $h^n = 1_M$.*

PROOF. If M is a Seifert fiber space the proposition follows directly from theorem 1 of [**H-T**] and lemma 3.1. In the remainder of the proof, lemma 3.1 must be used in similar fashion to strengthen conclusions from [**H-T**], but we will no longer mention these individually.

Assuming that M is not a Seifert fiber space and let Σ be the characteristic submanifold of $(M, \overline{\emptyset})$. Since Σ is unique up to isotopy, we may assume that $g(\Sigma) = \Sigma$. Let F be a component of the frontier of Σ. Let $\widehat{F}$ be $\cup g^i(F)$, a union of components of the frontier of Σ. By lemma 9(ii) of [**H-T**], there is a homeomorphism h isotopic to g, such that $h(\widehat{F}) = \widehat{F}$

and h^n is isotopic to 1_M relative to $\widehat{F} \cup \partial M$. Repeating this for all components of the frontier of Σ, we may assume that g^n is isotopic to 1_M relative to the union of ∂M and the frontier of Σ. Consider the components of M cut along the frontier of Σ. Since the boundary of M consists of tori, these components are either Seifert-fibered or simple, and have only torus boundary components. On each of them, we can use theorem 1 of [**H-T**] (for the Seifert-fibered ones) or the corollary of [**H-T**] (for the simple ones) to change g by isotopy (relative to ∂M and the frontier of Σ) to have order n.

Here is the main result of this section. This theorem together with corollary 2.3 implies the Kontsevich Conjecture for Haken 3-manifolds.

THEOREM 3.3. *Let M be a Haken 3-manifold such that ∂M is non-empty and incompressible. Then $\mathcal{H}(M \text{ rel } \partial M)$ is torsionfree.*

PROOF. We must prove that if f is a homeomorphism which is the identity on the boundary and $f^n \simeq 1_M$ for some $n > 0$ then $f \simeq 1_M$, where here and throughout the proof the symbol $f \simeq g$ means that f is *isotopic* to g (rather than just homotopic, as is more common in the literature).

Let T be the union of the torus boundary components of M. Suppose first that $T = \partial M$. By proposition 3.2, $f \simeq g$ relative to ∂M with $g^n = 1_M$. Since g is the identity on ∂M, this implies that $g = 1_M$. Now suppose that T is not empty but $T \neq \partial M$. Form N by gluing two copies of M together along $\partial M - T$, and let F be the homeomorphism of N defined by taking f on each copy of M. Since $F^n \simeq 1_N$ relative to ∂N, proposition 3.2 applies as before to show that $F \simeq 1_N$. If we can show that $F \simeq 1_N$ relative to $\partial M - T$, then $f \simeq 1_M$ relative to ∂M, and this will complete the case when $T \neq \partial M$.

Let $H\colon N \times I \to N$ be an isotopy from F to 1_N. Let G be a component of $\partial M - T$ and let m_0 be a basepoint in G. Consider the trace of H at m_0, that is, the element α of $\pi_1(M, m_0)$ represented by the restriction of H to $m_0 \times I$. Suppose that α does not lie in the subgroup $\pi_1(G)$. Let τ be any loop in G based at m_0. Since F is the identity on G, the composition $H \circ (\tau \times 1_I)\colon S^1 \times I \to N$ shows that $\alpha\tau\alpha^{-1}$ equals τ in $\pi_1(M, m_0)$, that is, α centralizes $\pi_1(G, m_0)$. Let S be the subgroup of $\pi_1(M, m_0)$ generated by $\pi_1(G, m_0)$ and α. Let $\widetilde{N}$ be the covering space of N corresponding to the subgroup S. By [**S**], there exists a compact core C of $\widetilde{N}$, so that $\pi_1(C) \to \pi_1(\widetilde{N})$ is an isomorphism. By [**M-Y**], $\widetilde{N}$ is irreducible, so we may fill in any 2-sphere boundary components with 3-balls in $\widetilde{N}$ in order to assume that C is irreducible. Let $\widetilde{G}$ be a lift of G to an imbedded incompressible surface in C. Since $\pi_1(C)$ has nontrivial center, it admits a Seifert fibering $C \to Q$, where Q is the quotient orbifold. By theorem 1.1, $\widetilde{G}$ is isotopic to a surface which is either vertical or horizontal, but since G is a closed surface not a torus, this surface must be horizontal. Since C contains a closed horizontal surface, C must be closed and therefore $C = \widetilde{N}$. This implies that N is closed, contradicting the fact that T is not empty. We conclude that the trace of H lies in $\pi_1(G, m_0)$.

There is an isotopy of G that moves m_0 around a loop representing α. Use this to obtain an isotopy, relative to ∂N and preserving G, to a homeomorphism F' such that F' is isotopic to 1_N, relative to ∂N, by an isotopy H' having trivial trace. By Laudenbach theorem, there is an isotopy from F' to 1_N, relative to $\partial N \cup G$. Repeating for each component G of $\partial M - T$, we obtain the desired isotopy from F to 1_N relative to ∂M and hence from f to 1_M relative to ∂M. This completes the case when M has a torus boundary component.

Now suppose that no component of ∂M is a torus. Let G be a boundary component, and choose an essential simple closed curve γ in G. Let G_1 be a regular neighborhood of γ in G. Let W be $S^1 \times S^1 \times I$, and let G_2 be a regular neighborhood of $S^1 \times \{s_0\} \times \{0\}$ in $S^1 \times S^1 \times \{0\}$ for some $s_0 \in S^1$. Form N by identifying G_1 with G_2 and let G_0 be the incompressible surface in N obtained from G_1 and G_2. Since G_0 is incompressible in M and W, N is Haken. Extend f to a homeomorphism F in N using the identity on W. The isotopy $f^n \simeq 1_M$ relative to ∂M extends using the identity on W to an isotopy $F^n \simeq 1_N$ relative to ∂N. Since N has a torus boundary component, the previous case implies that $F \simeq 1_N$ relative to ∂N. By Laudenbach theorem, $F \simeq 1_N$ relative to $G \cup \partial N$, and therefore $f \simeq 1_M$ relative to ∂M.

ACKNOWLEDGEMENTS. Dedicated to my father.

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