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HOMOTOPICAL TRIVIALITY OF ENTIRE RATIONAL MAPS TO EVEN DIMENSIONAL SPHERES

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ABSTRACT. Let $G = \mathbb{Z}_2$. Let X be any compact connected orientable nonsingular real algebraic variety of $\dim X = k$ =odd with the trivial G action, and let Y be the unit sphere S^{2n-k} with the antipodal action of G. Then we prove that any G invarinat entire rational map $f : X \times Y \to S^{2n}$ is G homotopically trivial. We apply this result to prove that any entire rational map $g : X \times \mathbb{R}P^{2n-k} \to S^{2n}$ is homotopically trivial.

1. Introduction

A real algebraic variety is the common zero set of polynomials $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, k$. A map $f : X \to Y$ between two algebraic varieties $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is called a regular map if there exists a polynomial $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f = P|_X$. A map $f : X \to Y$ is entire rational if there are polynomials $P : \mathbb{R}^n \to \mathbb{R}^m$ and $Q : \mathbb{R}^n \to \mathbb{R}$ such that $Q^{-1}(0) \cap Y = \emptyset$ and $f = P/Q$ on X. One of the interesting problems concerning entire rational maps is to determine homotopy types of entire rational maps between given two real algebraic varieties. Bocknak and Kucharz study the problem in [BoKu79], [BoKu87] and [BoKu88] for the case when the target spaces are spheres. The following theorem of Bocknak and Kucharz shows a striking difference between homotopy types of entire rational maps and those of smooth or continuous maps.

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THEOREM 1.1. [2] *Let X be a nonsingular real algebraic variety of odd dimension k. Assume that X is compact connected and orientable as a smooth manifold. If $k < 2n$, then every entire rational map from $X \times S^{2n-k}$ to S^{2n} is homotopically trivial.*

In this paper we prove the following equivariant analogue of the above theorem.

THEOREM 1.2. *Let $G = \mathbb{Z}_2$ be the order two group. Let X be a nonsingular real algebraic variety of odd dimension k with the trivial G action. Assume that X is compact connected and orientable as a smooth manifold. Let Y be the sphere S^{2n-k} on which the nontrivial element of G acts as the antipodal map. If $k < 2n$, then every entire rational G map from $X \times Y$ to S^{2n} is G homotopically trivial.*

Using Theorem 1.2 we are able to prove the following nonequivariant result.

THEOREM 1.3. *Let X be a nonsingular real algebraic variety of dimension k with odd k. Assume that X is compact connected and orientable as a smooth manifold. If $k < 2n$, then every entire rational map from $X \times \mathbb{R}P^{2n-k}$ to S^{2n} is homotopically trivial.*

In [3] Bochnak and Kucharz have introduced a subring $H^*_{\mathbb{C}-alg}(Z : \mathbb{Z})$ of the cohomology ring of a real algebraic variety Z, which is defined using concepts from complex algebraic geometry such as Chow cohomology ring. They also find several properties of $H^*_{\mathbb{C}-alg}(Z : \mathbb{Z})$. Using these properties one can prove Theorem 1.3 . However in this paper we give a different proof of Theorem 1.3 using only Theorem 1.2 .

2. Proof of main results

Let $G = \mathbb{Z}_2$, and let $0 \neq g \in G$ act on $Y = S^{n-1}$ as the antipodal map. Topologically the orbit space Y/G is homeomorphic to the real projective space $\mathbb{R}P^{n-1}$ and the orbit map $\pi : Y \to Y/G$ is continuous. The following lemma shows that we can give a similar argument algebraically. Note that the real projective space $\mathbb{R}P^{n-1}$ is a nonsingular real algebraic variety by the the following identification.

$$\mathbb{R}P^{n-1} = \{L \in M(n, \mathbb{R}) \mid L^2 = L = L^T, \operatorname{tr} L = 1\}$$

where $M(n,\mathbb{R})$ is the variety of all $n \times n$ real matrices.

LEMMA 2.1. *Let $G = \mathbb{Z}_2$. Let $0 \neq g \in G$ act on $Y = S^{n-1}$ as the antipodal map. Then there exists a surjective G invariant entire rational map $p : Y \to \mathbb{R}P^{n-1}$ and a homeomorphism $\psi : Y/G \to \mathbb{R}P^{n-1}$ such that $\psi \circ \pi = p$.*

PROOF. Let $\mathbb{R}[x_1, \dots, x_n]$ denote the $\mathbb{R}$-algebra of the real polynomials on n variables, and let $\mathbb{R}[S^{n-1}]^G$ denote the $\mathbb{R}$-algebra of G invariant regular maps on S^{n-1}. Then $\mathbb{R}[S^{n-1}]^G$ is generated by the polynomials $t_{ij} := x_i x_j \in \mathbb{R}[x_1, \dots, x_n]$ for $0 \leq i \leq j \leq n$. The polynomial relations of t_{ij} are generated by the following equations:

$$\sum_{i=1}^{n} t_{ii} = 1$$
$$(t_{ij})^2 = t_{ii}t_{jj} \qquad \text{for } 1 \leq i < j \leq n$$

We now consider the map $p' : Y \to \mathbb{R}^{\frac{n^2+n}{2}}$ defined by

$$p'(x_1, \dots, x_n) = (t_{11}, t_{12}, \dots, t_{1n}, t_{23}, \dots, t_{2n}, \dots, t_{nn}).$$

Let $(z_{11}, z_{12}, \dots, z_{1n}, z_{23}, \dots, z_{2n}, \dots, z_{nn})$ denote the coordinate system of $\mathbb{R}^{\frac{n^2+n}{2}}$. Let $Z \subset \mathbb{R}^{\frac{n^2+n}{2}}$ be the algebraic variety defined by the polynomials

$$\sum_{i=1}^{n} z_{ii} = 1$$
$$(z_{ij})^2 = z_{ii}z_{jj} \qquad \text{for } 1 \leq i < j \leq n$$

Let $\phi : Z \to \mathbb{R}P^{n-1}$ be the map which maps the point

$$(z_{11}, z_{12}, \dots, z_{1n}, z_{23}, \dots, z_{2n}, \dots, z_{nn})$$

to the symmetric matrix L whose (i,j)-entries are z_{ij} for $1 \leq i \leq j \leq n$. Namely,

$$\phi(z_{11}, \dots, z_{nn}) = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{12} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{1n} & z_{2n} & \cdots & z_{nn} \end{pmatrix}$$

Then it is an elementary check to see that the map ϕ is an isomorphism. Note that p' is clearly a G invariant map onto Z, and hence there is a homeomorphism $\rho : Y/G \to Z$ such that $p' = \rho \circ \pi$. Let $p = \phi \circ p'$ and $\psi = \phi \circ \rho$. Then they are the desired maps.

LEMMA 2.2. *Let G, Y, p, and ψ be as in Lemma 2.1. For any real algebraic varieties X and Z with the trivial G action and a G invariant entire rational map $f : X \times Y \to Z$ there exists a unique entire rational map $g : X \times \mathbb{R}P^{n-1} \to Z$ such that $f = g \circ (\mathrm{Id} \times p)$.*

PROOF. It is enough to prove the theorem when $Z = \mathbb{R}$. We first claim that any G invariant regular map $f : X \times Y \to \mathbb{R}$ can be expressed as $f = \sum f_i g_i$ where $f_i : X \to \mathbb{R}$ are regular maps on X and $g_i : Y \to \mathbb{R}$ are G invrariant regular maps on Y. Since f is G invariant

$$f(x, y) = \frac{1}{2}(f(x, y) + f(x, -y))$$

for $(x, y) \in X \times Y$. On the other hand it is clear that $f = \sum \overline{f_i}\overline{g_i}$ for some regular maps $\overline{f_i} : X \to \mathbb{R}$ and $\overline{g_i} : Y \to \mathbb{R}$. Thus we have

$$\begin{aligned} f(x, y) &= \frac{1}{2}(f(x, y) + f(x, -y)) \\ &= \frac{1}{2}\sum(\overline{f_i}(x)\overline{g_i}(y) + \overline{f_i}(x)\overline{g_i}(-y)) \\ &= \sum \overline{f_i} \cdot \frac{1}{2}(\overline{g_i}(y) + \overline{g_i}(-y)) \\ &= \sum f_i(x) g_i(y) \end{aligned}$$

where $f_i = \overline{f_i}$ and $g_i(y) = \overline{g_i}(y) + \overline{g_i}(-y)$. It is clear that g_i are G invariant regular maps. This proves the first claim. We now prove the special case of the lemma with all entire rational maps replaced by regular maps. Since $\mathbb{R}[Y]^G$ is generated by t_{ij} (see the proof of Lemma 2.1) for any G invariant regular map $g : Y \to \mathbb{R}$ there exists a polynomial h such that $g(y) = h(t_{11}(y), t_{12}(y), \ldots, t_{nn}(y))$. Let $f : X \times Y \to \mathbb{R}$ be a given G invariant polynomial map. Then for $(x,y) \in X \times Y$ we have

$$\begin{aligned} f(x,y) &= \sum f_i(x)g_i(y) \\ &= \sum f_i(x)h_i(t_{11}(y), t_{12}(y), \dots, t_{nn}(y)) \\ &= \sum f_i(x)h_i(p(y)) \\ &= (\sum f_ih_i) \circ (\mathrm{Id} \times p)(x,y) \end{aligned}$$

Thus if we let $g = \sum f_ih_i$ then g is a regular map such that $f = g \circ (\mathrm{Id} \times p)$. This proves the special case. We now prove the general case. Let $f : X \times Y \to \mathbb{R}$ be a G invariant entire rational map. From the definition of entire rational map there exists polynomials P and Q such that Q does not vanish on $X \times Y$ and $f = P/Q$ on $X \times Y$. Since f is G invariant we may assume that both P and Q are G invariant, see Proposition 2.5 of [DMS94]. From the previous special case there exists polynomials $\overline{P}$ and $\overline{Q}$ such that $P = \overline{P} \circ (\mathrm{Id} \times p)$ and $Q = \overline{Q} \circ (\mathrm{Id} \times p)$. Since $Q^{-1}(0) = (\mathrm{Id} \times p)^{-1} \circ \overline{Q}^{-1}(0)$, if there exists $z \in \overline{Q}^{-1}(0) \cap (X \times \mathbb{R}P^{n-1})$, then there exists $w \in (\mathrm{Id} \times p)^{-1}(z)$. This shows that $Q(w) = 0$ for some $w \in X \times Y$, which is a contradiction. Therefore $\overline{Q}$ does not vanish on $X \times \mathbb{R}P^{n-1}$. Now let $g = \overline{P}/\overline{Q}$. Then clearly g is an entire rational map such that $f = g \circ (\mathrm{Id} \times p)$. The uniqueness of g is obvious.

We are now ready to prove Theorem 1.2. We first prove the following special case.

LEMMA 2.3. *Let X be a nonsingular real algebraic varieties of dimension $2n-1$. Assume X is compact connected and orientable as a smooth manifold. Let $G = \mathbb{Z}_2$ act trivially on X and S^{2n}. Let Y be the circle S^1 where $0 \neq g \in G$ acts as the mulitplication by -1. Then any G invariant entire ratinal map $f : X \times Y \to S^{2n}$ is G homotopically trivially.*

PROOF. Note that $\mathbb{R}P^1$ is isomorphic to S^1. In fact the isomorphism is given as follows: Since

$$\mathbb{R}P^1 = \begin{pmatrix} \alpha & \beta \\ \beta & 1-\alpha \end{pmatrix} \mid \alpha \in [0,1], \beta^2 = \alpha(1-\alpha)\}$$

define $\phi : \mathbb{R}P^1 \to S^1$ by

$$\phi(\begin{pmatrix} \alpha & \beta \\ \beta & 1-\alpha \end{pmatrix}) = (1-2\alpha, 2\beta).$$

The inverse map $\psi : S^1 \to \mathbb{R}P^1$ is defined by

$$\psi(a,b) = \frac{1}{2}\begin{pmatrix} 1-a & b \\ b & 1+a \end{pmatrix}.$$

Let $p : S^1 \to \mathbb{R}P^1$ be the entire rational map as in Lemma 2.1. Lemma 2.2 implies that for a given G invariant entire rational map $f : X \times Y \to S^{2n}$ there exists an entire rational map $g : X \times \mathbb{R}P^1 \to S^{2n}$ such that $f = g \circ (\mathrm{Id} \times p)$. Since $\mathbb{R}P^1$ is isomorphic to S^1 we can apply Theorem 1.1 to show that g is null homotopic. This null homotopy composed with $\mathrm{Id} \times p$ induces a null homotopy for f.

We now prove the general case of Theorem 1.2.

PROOF OF THEOREM 1.2. Let S be the unit circle $S^1 \subset \mathbb{R}^2$ on which $0 \neq g \in G$ acts as the antipodal map. For simplicity let $m := 2n-k-1$, hence m is even. Consider the entire rational map $\phi : S^m \times S \to Y$ defined by

$$\phi(x_0, \ldots, x_m, y_0, y_1) = (x_0 y_0, \ldots, x_m y_0, y_1).$$

Then this map is a well defined G equivariant map. We claim that $\deg \phi = \pm 2 \neq 0$. It is easy to see that for almost all $B \in Y$ the inverse image $\phi^{-1}(B)$ consists of two points A and A'. Therefore $\deg \phi$ is either ± 2 or 0 depending on whether ϕ preserves (or reverses) the local orientation at both A and A' or not. Let $t : S^m \times S \to S^m \times S$ be the map defined by

$$t(x_0, \ldots, x_m, y_0, y_1) = (-x_0, \ldots, -x_m, -y_0, y_1).$$

Then $\phi \circ t = \phi$, $t(A) = A'$, and $\deg(t) = (-1)^{m+2} = 1$ because m is even. Therefore the local degree of ϕ at A and A' are either both $+1$ or both -1. This shows that $\deg \phi = \pm 2 \neq 0$. Let $g : X \times Y \to S^{2n}$ be a given G invariant enitre rational map. Consider the composition

$$g \circ (\mathrm{Id} \times \phi) : X \times S^m \times S \to X \times Y \to S^{2n}.$$

By Lemma 2.3 this map is G-homotopically trivial. Now let $\pi : Y \to Y/G \cong \mathbb{R}P^{2n-k}$ be the orbit map. Since the map g is G invariant there exists a smooth map $\overline{g} : X \times (Y/G) \to S^{2n}$ such that $g = \overline{g} \circ (\mathrm{Id} \times \pi)$. Then the degree of the composition $(\mathrm{Id} \times \pi) \circ \mathrm{Id} \times \phi$ is ± 4. Indeed, we consider the composition

$$\phi \circ \pi : S^m \times S \to Y \to Y/G.$$

For each point $[B] \in Y/G$ the inverse image $\pi^{-1}([B])$ consists of two points B and $-B$. For almost all $B \in Y$ the inverse image $\phi^{-1}(B)$ consists of two points

$$A_1 = (x_0, \dots, x_m, y_0, y_1), \qquad A_1' = (-x_0, \dots, -x_m, -y_0, y_1),$$

and $\phi^{-1}(-B)$ consists of two points

$$A_2 = (x_0, \dots, x_m, -y_0, -y_1), \qquad A_2' = (-x_0, \dots, -x_m, y_0, -y_1).$$

Since the local degree of $\phi \circ \pi$ at these four points are equal, the degree of $\phi \circ \pi$ is ± 4. This shows that the degree of the composition $(\mathrm{Id} \times \pi) \circ (\mathrm{Id} \times \phi)$ is ± 4. Since $\overline{g} \circ (\mathrm{Id} \times \pi) \circ (\mathrm{Id} \times \phi) = g \circ (\mathrm{Id} \times \phi)$ is homotopically trivial, we have $4 \cdot \deg \overline{g} = 0$. Therefore $\deg \overline{g} = 0$, which implies that $\overline{g}$ is homotopically trivial. Thus g is G homotopically trivial.

We now prove Theorem 1.3

PROOF OF THEOREM 1.3. This follows immediately from Theorem 1.2 and Lemma 2.2. Indeed, for any entire rational map $g : X \times \mathbb{R}P^{2n-k} \to S^{2n}$ the map $f = g \circ (\mathrm{Id} \times p) : X \times Y \to S^{2n}$ is a G invariant entire rational map. Thus Theorem 1.2 gives null homotopy $f_t : X \times Y \to S^{2n}$, $0 \le t \le 1$, of f. By Lemma 2.2 there exists null homotopy $g_t : X \times \mathbb{R}P^{2n-k} \to S^{2n}$ of g.

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