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AN APPLICATION OF LEAST AREA SURFACES TO 3-MANIFOLDS

MYOUNGHO MOON

ABSTRACT. We provide a new proof of the following fact using least area surfaces: If the fundamental group of a $\mathbb{P}^2$-irreducible closed 3-manifold M contains a finitely generated nontrivial normal subgroup of infinite index, then M has a finite cover which is a closed surface bundle over S^1, unless N is free.

Introduction

In [4] [5], Jaco and Rubinstein introduced P.L. area of immersed surfaces in 3-manifolds, and provided with the existence of P.L. least area surfaces for various cases. They also pointed out that establishing the existence of appropriate least area surfaces in P.L. cases is easier than in smooth cases. Further they showed that most of the techniques used in smooth cases like exchange and round off trick work for P.L. cases. In this paper we use P.L. least area surfaces to give another proof of Hempel and Jaco's result ([3]) for $\mathbb{P}^2$-irreducible closed 3-manifolds (See the Main Theorem). This paper consists of three sections. In the first section we provide some of the basic facts in 3-manifold theory and group theory. In the second section P.L. least area will be introduced and the existence theorem of P.L. least area surface will be shown for a particular case. The main theorem will be proved in the third section. The main idea was suggested by P. Scott. He also has been a great help in filling out the details of the proof of the Main Theorem.

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1. Preliminaries

we review some basic material for the main theorem. Recall that G is indecomposable if $G = G_1 * G_2$ implies that either $G_1 = 1$ or $G_2 = 1$. The following proposition is well known.

PROPOSITION 1.1. *Suppose that a finitely generated subgroup U of a group G contains a nontrivial normal subgroup N of G and U is of infinite index in G. Then G is indecomposable.*

We need the following lemma from the covering space theory. The proof of it can be found in [1].

PROPOSITION 1.2. *Let X be a locally connected, locally compact, pathwise connected space. Suppose we have covering spaces ,*

$$X_N \xrightarrow{\alpha} X_U \longrightarrow X$$

corresponding to the subgroups $N < U < G = \pi_1(X)$, and N is normal in G. If $\mid G : U \mid$, the index of U in G, is infinite, and $A \subset X_N$ and $B \subset X_U$ are compact, then for all but a finite number of $\overline{g} \in G/N \backslash U/N$,

$$\alpha(\overline{g}A) \cap B = \emptyset.$$

Recall that a 3-manifold M is irreducible if every 2-sphere in M bounds a 3-cell in M, and that M is $\mathbb{P}^2$-irreducible if M is irreducible and no 2-sided projective plane is embedded in M.

DEFINITION 1.3. A 2-manifold F in a 3-manifold M is incompressible in M provided that whenever D is a 2-cell in M with $D \cap S = \partial D$, then ∂D bounds a 2-cell in F.

REMARK. If F is 2-sided in M or $F \subset \partial M$, then the loop theorem and Dehn's lemma imply that F is incompressible in M if and only if the inclusion induced map $i_* : \pi_1(F) \longrightarrow \pi_1(M)$ is a monomorphism (See [2]).

PROPOSITION 1.4,[8]. *Let M be a compact irreducible 3-manifold. Suppose that there is a finitely generated subgroup K of $\pi_1(M)$ satisfying the following short exact sequence:*

$$1 \longrightarrow K \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z} \longrightarrow 1.$$

If $K \neq 1$ or $\mathbb{Z}_2$, then M is a bundle over S^1 with fiber F a compact surface and $\pi_1(F) = K$.

PROPOSITION 1.5,[9]. *Let M be a 3-manifold and let U be a subgroup of $\pi_1(M)$ such that*

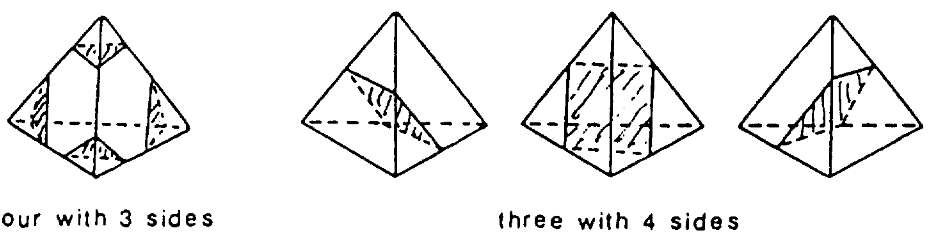
(1) *U is finitely generated,*
(2) *U is indecomposable and is not isomorphic to $\mathbb{Z}$.*

Then there is a compact incompressible submanifold W of M such that the image of $\pi_1(W)$ in $\pi_1(M)$ contains a conjugate of U.

REMARK. In [9], the condition (1) was that U was finitely presented. However, it can be replaced by the condition (1), considering P. Scott's result which says that every finitely generated 3-manifold group is finitely presented(See [7]).

PROPOSITION 1.6, [3]. *Let M be a compact, $\mathbb{P}^2$-irreducible 3-manifol d. Let F be a compact, connected, incompressible 2-manifold in ∂M. Then the following hold.*

(1) *If $i_* : \pi_1(F) \longrightarrow \pi_1(M)$ is an isomorphism, then there is a homeomorphism $h : F \times I \longrightarrow M$ such that $h(x,0) = x$ for all $x \in F$.*
(2) *If $i_*(\pi_1(F))$ has index two in $\pi_1(M)$, then there is a homeomorphism between M and a twisted I-bundle over a compact 2-manifold which takes F to the corresponding 0-sphere bundle.*



FIGURE

2. P.L. Least Area Surfaces

Let M be a 3-manifold with a fixed triangulation $\mathcal{T}$ and a Riemannian metric on the 2-skeleton $\mathcal{T}^{(2)}$. For example, put the hyperbolic metric on each 2-simplex so that each 2-simplex can be considered an ideal triangle in $\mathbb{H}^2$. For an immersed surface F in M transverse to the triangulation $\mathcal{T}$, F meets the 1-skeleton $\mathcal{T}^{(1)}$ transversely in a finite number of points(with multiplicity), and $F \cap \mathcal{T}^{(2)}$ is a finite union of arcs with a finite total length L. We define the P.L. area of F by (ω, L) with the dictionary order. ω is called the weight of F. A properly embedded surface F in a 3-manifold M is defined to be a normal surface with respect to a triangulation $\mathcal{T}$ of M if F meets each 3-simplex of $\mathcal{T}$ in a pairwise disjoint collection of disk types of $\mathcal{T}$ shown in the Figure.

Given a normal surface $f : F \mapsto M$, the normal homotopy class $\mathcal{N}(f)$ is defined to be the set of all normal surfaces $g : F \mapsto M$ which are normally homotopic to f. It can be shown that if M is $\mathbb{P}^2$-irreducible and a closed normal surface $f : F \mapsto M$ is π_1-injective with $\pi_1(F) \neq 0$, then the normal homotopy class of f contains a normal map of least P.L. area. Moreover, if M is $\mathbb{P}^2$-irreducible and a P.L. least area surface F is π_1-injective with $\pi_1(F) \neq 0$, then F is normal. Hence to find a P.L. least area surface, we only need to look into normal surfaces.

LEMMA 2.1. *Let M be a 3-manifold and Y be a compact subset of M. Then there are only a finite number of normal homotopy classes of surfaces of a given weight which meet Y.*

PROOF. Let N be a given weight. Since Y is compact, any normal surface of weight N which meet Y is contained in a finite number of 3-simplices in M. Let $\sigma_1, \cdots, \sigma_k$ be the 1-simplices in the union of those 3-simplices. Assign a nonnegative integer n_i to each σ_i so that $\sum n_i = N$. Note that there are only a finite number of ways of doing this. Next, in each 2-simplex, choose points on 1-simplices and join by a path. Each normal surface is obtained by doing the previous two steps up to normal homotopy. Since there are only a finite number of ways of doing each step, there are only a finite number of normal homotopy classes of a given weight.

THEOREM 2.2. *Let M be a $\mathbb{P}^2$-irreducible closed 3-manifold with $\pi_1(M) = G$, and let M_N be a covering space of M with $\pi_1(M_N) = N$, where N is a nontrivial normal subgroup of G. If M_N contains a closed surface F representing a nontrivial element in $H_2(M_N; \mathbb{Z}_2)$, then there is a least possible P.L. area surface L among the closed surfaces representing nontrivial elements in $H_2(M_N; \mathbb{Z}_2)$.*

PROOF. Let $p : M_N \mapsto M$ be the regular covering map and Q be the covering transformation. Since M is compact, there exists a compact fundamental region Y in M_N and any surface F in M_N can be translated by an element of Q so as to meet Y. By Lemma 2.1, there are only a finite number of normal homotopy classes of surfaces of given weight which meet Y, as Y is compact. Hence there is a least area surface L among the closed surfaces representing nontrivial elements in $H_2(M_N; \mathbb{Z}_2)$.

3. Proof of the Main Theorem

First, we state the main theorem.

MAIN THEOREM. *Let M be a closed, $\mathbb{P}^2$-irreducible 3-manifold satisfying the following exact sequence:*

$$1 \longrightarrow N \longrightarrow \pi_1(M) \xrightarrow{\eta} Q \longrightarrow 1$$

where N is a finitely generated normal subgroup of $\pi_1(M)$ with infinite quotient group Q. If N is not free, then the following are true:

(1) *N is isomorphic to the fundamental group of a closed surface,*
(2) *M is either a fiber bundle with fiber F a closed surface, or the union of two twisted I-bundles over a closed 2-manifold F which meet in the corresponding 0-sphere bundles.*
(3) *N is a subgroup of finite index of $\pi_1(F)$.*

PROOF. Since N is not free, N has an indecomposable free factor A which is not isomorphic to $\mathbb{Z}$. Consider the cover $p : \tilde{M} \longrightarrow M$ with $\pi_1(\tilde{M}) = N$. Since $|\pi_1(M) : N| = \infty$, $\tilde{M}$ is an open manifold. By Proposition 1.5, there is a compact connected submanifold W of $\tilde{M}$ such that each component of ∂W is incompressible in $\tilde{M}$ and $\pi_1(W)$ contains A. Since $\tilde{M}$ is $\mathbb{P}^2$-irreducible, no component of ∂W is S^2 or $\mathbb{P}^2$. If $\partial W = \emptyset$, then $W = \tilde{M}$, so $\tilde{M}$ is compact. This is a contradiction to the fact that $\tilde{M}$ is an open manifold. Hence $\partial W \neq \emptyset$. Let S be a component of ∂W. If $[S] = 0$ in $H_2(\tilde{M}, \mathbb{Z}_2)$, S bounds a compact submanifold V of $\tilde{M}$. If $\pi_1(S) = \pi_1(V)$, then by Proposition 1.6, $V \cong S \times I$, so ∂V has two components, which is a contradiction. Thus $\pi_1(S) \subsetneq \pi_1(V)$.

Since S is 2-sided and the only boundary of V, S separates $\tilde{M}$. There are infinitely many disjoint copies of V in $\tilde{M}$, as Q is an infinite group acting properly discontinuously on $\tilde{M}$. These copies give rise to a decomposition of $\pi_1(\tilde{M})$ amalgamated along their boundaries. Since $\pi_1(S) \subsetneq \pi_1(V)$, $\pi_1(\tilde{M})$ is not finitely generated. This is a contradiction to the fact that N is finitely generated. Thus $[S] \neq 0$ in $H_2(\tilde{M}, \mathbb{Z}_2)$, so $H_2(\tilde{M}, \mathbb{Z}_2) \neq 0$.

Now consider the set of all 2-sided closed surfaces $S \subset \tilde{M}$ such that $[S] \neq 0$ in $H_2(\tilde{M}, \mathbb{Z}_2)$. Theorem 2.2 guarantees the existence of least area surfaces among all such surfaces. Take L of least area among all such surfaces. We will prove that L is Q-equivariant. First, L separates $\tilde{M}$. In fact, if L does not separate $\tilde{M}$, then there is a closed curve C in $\tilde{M}$ intersecting L in 1 point transversally. Choose $g \in Q$ such that $g(L \cup C) \cap (L \cup C) = \emptyset$. Then $L \cup gL$ does not separate $\tilde{M}$. By repeating this argument, for each integer n, we can find n pairwise disjoint translates $L_1, \ldots, L_n$ of L whose union does not separate $\tilde{M}$. Thus we can find in $\tilde{M}$ a graph Γ of euler characteristic $1 - n$ dual to $\cup L_i$ and a retraction $\rho : \tilde{M} \longrightarrow \Gamma$. Then $\pi_1(\Gamma)$ is free of rank n. We get a contradiction if n is greater than the cardinality of some finite set of generators for $\pi_1(\tilde{M})$. Now suppose L intersects some translate gL and $gL \neq L$. We may assume that gL intersects L transversally. $gL \cap L$ separates L and gL into $B_1 \cup B_2$ and $C_1 \cup C_2$, respectively, as L separates $\tilde{M}$. Note that B_1, B_2, C_1, C_2 need not be connected. Take the least area one among B_1, B_2, C_1, C_2. Suppose it is B_1. Consider $(gL - C_2) \cup B_1 = C_1 \cup B_1$ and $(gL - C_1) \cup B_1 = C_2 \cup B_1$. Each of these two surfaces is of a disjoint union of embedded 2-sided closed surfaces in $\tilde{M}$ and is of area less than the area of gL, after rounding corners. Hence each surface represents 0 in $H_2(\tilde{M}, \mathbb{Z}_2)$. Now $[gL] = [C_2] - [B_1]$ and $[gL] = [C_1] - [B_1]$. Thus

$$0 = [gL] - [gL] = [C_2] - [B_1] - [C_1] + [B_1].$$

It follows that $[C_2] = [C_1]$. Since $(gL - C_2) \cup B_1 = C_1 \cup B_1$, $[gL] = [C_2] + [C_1] = 2[C_1]$. Hence $[gL] = 0$ in $H_2(\tilde{M}, \mathbb{Z}_2)$. This is impossible as $[L] \neq 0$ in $H_2(\tilde{M}, \mathbb{Z}_2)$. Therefore, if L intersects some translate gL, then $gL = L$. In other words, L is Q-equivariant.

By the main Theorem in [6], $\tilde{M}$ has a $\mathbb{P}^2$-irreducible compact core $\tilde{W}$. Since both $\tilde{M}$ and $\tilde{W}$ are aspherical, $i : \tilde{W} \to \tilde{M}$ is a homotopy equivalence. Note that all but a finite number of gL miss $\tilde{W}$. If $gL \cap \tilde{W} = \emptyset$, then gL is homotopic into $\tilde{W}$. Hence gL is homologous into W, and so there is a compact submanifold $\tilde{W}_1$ of $\tilde{M}$ such that $\partial \tilde{W}_1$ is the union of gL and some surfaces in $\tilde{W}$. Note that $\partial \tilde{W}_1 \neq gL$, as $[L] \neq 0$ in $H_2(\tilde{M}, \mathbb{Z}_2)$. Hence there is a path in $\tilde{W}_1$ from L into $\tilde{W}$. This path must meet a component S of $\partial \tilde{W}$, so $S \cap \tilde{W}_1 \neq \emptyset$. Also, $S \cap \partial \tilde{W}_1 = \emptyset$, as $\partial \tilde{W}_1 = gL \cup$ (some surfaces in $\tilde{W}$). Both $\tilde{W}_1$ and S are compact, so $S \cap \tilde{W}_1$ is compact. Since $S \cap \partial \tilde{W}_1 = \emptyset$, $S \cap \tilde{W}_1$ is open in S. It follows that $S \cap \tilde{W}_1 = S$, as S is connected. Hence $S \subset \tilde{W}_1$. This implies that S is homologous to gL, as $S \cup gL$ bounds a submanifold of $\tilde{W}_1$. Therefore, gL is homologous to a component of $\partial \tilde{W}$ if $gL \cap \tilde{W} = \emptyset$. It follows that the set of all gL's forms only a finite number of homology classes, as $\tilde{W}$ is compact. Since Q is an infinite group, we can find $\tau \in Q$ such that $\tau L \neq L$ but $[\tau L] = [L]$. It follows that τL and L bound a compact submanifold Y of $\tilde{M}$. There are infinitely many disjoint copies of Y, so there are infinitely many separating 2-sided closed surfaces. These surfaces give rise to the decomposition of $\pi_1(\tilde{M})$ amalgamated along them. This is impossible unless $\pi_1(Y) = \pi_1(L)$, as $\pi_1(\tilde{M})$ is finitely generated. Hence, $\pi_1(Y) = \pi_1(L)$. By Proposition 1.6, $Y \cong L \times I$ and $\partial Y = L \cup \tau L$. Clearly, $\cup_k \tau^k Y$ is an open connected subset of $\tilde{M}$. Also, it is closed in $\tilde{M}$. In fact, consider the cover $\alpha : \tilde{M} \to \tilde{M}/\langle \tau \rangle$. Then, $\alpha(\cup_k \tau^k Y) = Y/L \sim \tau L$, which is compact, so closed. Since $\alpha^{-1}(Y/L \sim \tau L) = \cup_k \tau^k Y$, $\cup_k \tau^k Y$ is closed. It follows that $\tilde{M} = \cup_k \tau^k Y$, as $\tilde{M}$ is connected. If τ is of finite order, then $\tilde{M}$ must be compact, which is a contracdiction. Thus τ is of infinite order. Therefore $\tilde{M} \cong L \times \mathbb{R}$, so $H_2(\tilde{M}, \mathbb{Z}_2) \cong \mathbb{Z}_2$, which implies that $[gL] = [L]$ for any $g \in Q$.

Now since L is Q-equivariant, L finitely covers a closed surface $F = L/\text{stabilizer of } L$. Pick σ so that L and σL bound a compact submanifold X and the interior of X does not intersect any translate of L. Then $\tilde{M} = \cup_{\eta \in Q} \eta X$ and $X \cong L \times I$, so $\tilde{M} \cong L \times I$. It follows that $N = \pi_1(L)$, and

$$M = X/Q = (X/\text{stabilizer of } X)/(L \sim \sigma L).$$

Case 1, in which $\gamma L = L$ for all $\gamma \in$ stabilizer of X.

$$(X/\text{stabilizer of } X) \cong F \times I.$$

Hence $M \cong F \times I/(F \times 0 \sim F \times 1)$, which is a bundle over S^1 with fiber F. Furthermore, N is of finite index in $\pi_1(F)$, as $N = \pi_1(L)$.

Case 2, in which there is a covering transformation γ such that $\gamma X = X$ and $\gamma L \neq L$. It follows that $\gamma L = \sigma L$ and $\gamma(\sigma L) = L$. Thus,

$$X/(\text{stabilizer of } X) = F\tilde{\times}I,$$

where $F\tilde{\times}I$ denotes a twisted I-bundle over F. Let Z be $X/$stabilizer of X. Then

$$M = Z/\text{free involution on } F_0,$$

where F_0 is the boundary of $F\tilde{\times}I$. Note that we may regard Z as the union of a collared neighborhood Σ of ∂Z and $\overline{(Y \setminus \Sigma)}$. $\overline{(Y \setminus \Sigma)}$ is homeomorphic to $F\tilde{\times}I$, and

$$(\Sigma/\text{free involution on } F_0) \cong F\tilde{\times}I.$$

Furthermore the two pieces meet in F. Since $N = \pi_1(L)$, N is of finite index in $\pi_1(F)$. This completes the proof.

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Mathematics Education Department
Konkuk University
Seoul 143-701, Korea
E-mail address: mhmoon@kkucc.konkuk.ac.kr