Comm. Korean Math. Soc. **11** (1996), No. 3, pp. 789–796

HARMONIC MAPS INTO OPEN MANIFOLDS WITH NONNEGATIVE CURVATURE

YOUNG-HEON KIM AND JIN-WHAN YIM*

ABSTRACT. A complete open manifold with nonnegative curvature is diffeomorphic to the normal bundle of the soul, and the projection map is a Riemannian submersion. Under certain circumstances, we prove that a harmonic map from a compact manifold followed by the projection is again harmonic. Therefore we obtain a harmonic map onto the soul when there is a harmonic map into an open manifold.

1. Introduction

The harmonic maps between two Riemannian manifolds have long been a central topic in differential geometry. The basic existence problem is concerned with deforming a given map into a harmonic map within its homotopy class. It is known that a smooth map from a compact manifold to a manifold with non-positive curvature can be deformed into a harmonic map using the heat flow method (see [2], [3]). However if the target manifold has positive curvature, this is no longer true. As a simple case of positive curvature, it is known that for $m \leq 7$ every homotopy class of maps $S^m \to S^m$ a has harmonic representative (cf. [9]), but no general existence theorem is known even in these dimensions. In fact, there are not many nontrivial examples of harmonic maps into positively curved manifolds.

In this paper we are concerned with the harmonic maps from a compact manifold into complete open manifolds with nonnegative curvature. By Cheeger and Gromoll's soul theorem (cf. [1]), it is known that a complete open manifold M with nonnegative curvature has a compact totally

Received May 10, 1996. Revised June 27, 1996.
1991 AMS Subject Classification: Primary 53C20, 58E20.
Key words and phrases: harmonic map, soul.
*This research was partially supported by GARC-KOSEF.

geodesic submanifold Σ, called a soul, and M is diffeomorphic to the normal bundle of Σ. This fact has been an essential tool in the study of the structure of such manifolds. Recently, Perel'man has proved that there exists a C^1 Riemannian submersion π from M onto Σ (cf. [7]), and it was shown by Luis [5] that this projection is in fact C^∞.

Let N be a compact complete Riemannian manifold, and suppose there exists a harmonic map f into an open manifold M with nonnegative curvature. We study when there exists a harmonic map from N to the soul Σ of M. Since Σ is totally geodesic in M, in fact this implies that there exists a harmonic map into M with its image in Σ. This can be used for nonexistence theorem of harmonic maps in open manifolds with nonnegative curvature since there are some of such theorems for compact target spaces. In general, the projection π is not a harmonic map, and the composition of two harmonic maps is not necessarily harmonic. Therefore there is no reason to believe that the composition map $\pi \circ f : N \to \Sigma$ is harmonic. We will show in this paper that in some cases this composition is again harmonic. We do not know what is the most general conditions we can put on f or M to guarantee the harmonicity of the composition. In one of the cases we will consider, we will have that $\pi \circ f$ is energy minimizing. There are however several results (cf, [3]) about the instability of harmonic maps between compact manifolds. Therefore, our result can be used for nonexistence in this case.

2. Preliminary

In this section we will review some necessary results about the structure of complete open manifolds with nonnegative curvature and harmonic maps between Riemannian manifolds. For the details, we refer to [1], [2], [3].

Let M be a complete open manifold with nonnegative curvature. Then M is homotopic to a compact manifold and its structure is characterized by the following theorem due to Cheeger and Gromoll:

THEOREM. *Let M be as above. Then there exists a compact totally convex submanifold $\Sigma \subset M$ such that M is diffeomorphic to the normal bundle $\nu(\Sigma)$.*

A subset Σ is called totally convex if for any two points in Σ every geodesic in M connecting these two points is contained in Σ. In particular, a totally convex submanifold is totally geodesic. This theorem was proved by showing that there exists an exhaustion of M by totally convex compact subsets and the soul is at the center of this exhaustion.

It was then shown by Sharafutdinov (cf. [8], [11]) that there exists a distance nonincreasing deformation retraction h from M onto Σ, which means there exists a continuous map $H : M \times [0,1] \to M$ such that $H(p,0) = p$, $H(p,1) = h(p)$ and for each $t \in [0,1]$ the map $H(\cdot,t) : M \to M$ is distance nonincreasing. In [7], Perelman showed that this map actually coincides with the projection map $\pi : \nu(\Sigma) \to \Sigma$, which means that for any normal vector $v \in \nu_p(\Sigma)$

$$h(\exp_p tv) = p.$$

It is also known that this map is in fact a Riemannian submersion from M to Σ(see [5], [7]).

Let (N, g_N) and (M, g_M) be two Riemannian manifolds with the corresponding Riemannian metrics. A C^1 map $f : N \to M$ is called harmonic if it is a critical point of the energy functional $E(f) = \int_N \frac{1}{2}||df||^2 dvol$. The energy density $e(f) = \frac{1}{2}||df||^2$ can be computed by the formula

$$\begin{aligned} e(f) &= \frac{1}{2}\mathrm{Trace}_{g_N}(f^* g_M) \\ &= \frac{1}{2}\sum_{i=1}^{n} g_M(df(e_i), df(e_i)), \end{aligned}$$

where n is the dimension of N and $\{e_i\}$ is an orthonormal basis of the tangent space of N.

A harmonic map is characterized by the differential equation $\mathrm{Trace}(\nabla^2 f) = 0$, and it is equivalent to the fact that for each orthonormal basis $\{e_i\}$ of a tangent space T_pN,

$$\sum_{i=1}^{n} \nabla^*_{e_i} df(e_i) - df(\nabla^N_{e_i} e_i) = 0,$$

where $\nabla^*_{e_i} df(e_i)$ is the covariant derivative with respect to the induced connection of the pull-back bundle $f^{-1}(TM)$ and it is same as $\nabla^M_{df(e_i)} df(e_i)$

when these vector fields are properly defined. In particular, if we choose a geodesic γ_i tangent to each e_i and put $c_i = f \circ \gamma_i$, then we have

$$\sum_{i=1}^{n} \nabla^M_{\dot{c}_i} \dot{c}_i = 0.$$

If $f : N \to M$ is a harmonic map from a compact Riemannian manifold N to a complete open manifold M with nonnegative curvature, it is easy to see by the maximum principle that the image $f(N)$ is contained in the boundary of a totally convex set we described above. This image can be projected down to the soul by the Riemannian submersion, and we are interested whether this set is again a harmonic image of N. In the next section, we will use these basic facts to prove our main theorem.

3. Main theorem

For the statement of our main theorem, we need to explain some notation and concepts.

The normal bundle $\nu(\Sigma)$ is called flat if the holonomy of this fibre bundle with respect to the normal connection is locally trivial. In particular, there exists locally a parallel normal frame field along Σ.

For the Riemannian submersion $\pi : M \to \Sigma$ we denote by $\mathcal{H}$ and $\mathcal{V}$ the horizontal and the vertical distributions respectively. For any vector $X \in TM$, X^H and X^V will indicate the horizontal and the vertical components of X. A map $f : N \to M$ is called horizontal if $df(TN) \subset \mathcal{H}$, and a vertical map is defined in the same manner.

We now state our main theorem.

THEOREM 3.1. *Let N be a compact Riemannian manifold and M a complete open manifold with nonnegative curvature. If $\pi : M \to \Sigma$ is the projection map, then for any harmonic map $f : N \to M$, the composition $\pi \circ f$ is also a harmonic map in the following cases:*

1) *The normal bundle $\nu(\Sigma)$ is flat.*
2) *$f : N \to M$ is either vertical or horizontal.*
3) *$f : N \to M$ is an energy minimizing map in its homotopy class.*

PROOF. Assume that $f : N \to M$ is a harmonic map, and we will show that the composition $f \circ \pi : N \to \Sigma$ is a harmonic map in each cases.

1) For any three Riemannian manifolds N, M, Σ and two smooth maps $f : N \to M$ and $\pi : M \to \Sigma$, we have the following formula for the tension field:

$$\mathrm{Trace}(\nabla^2(\pi \circ f)) = d\pi(\mathrm{Trace}(\nabla^2 f)) + \mathrm{Trace}(\nabla d\pi(df, df)).$$

In particular, if π is a totally geodesic map, then the second term in the right hand side of the above formula vanishes and hence the harmonicity of f will imply the harmonicity of $\pi \circ f$. It therefore suffices to show that the projection π is totally geodesic.

In [10] it is shown that if the normal bundle $\nu(\Sigma)$ is trivial then M is isometric to a product manifold $P \times \Sigma$, where P is a complete open manifold with nonnegative curvature which is diffeomorphic to $\mathbb{R}^k$. Therefore, in this case, each fibre of the Riemannian submersion π is isometric to P and totally geodesic in M. In general, when the normal bundle is locally trivial, we consider the universal covering spaces $\tilde{M}$, $\tilde{\Sigma}$ of M and Σ. Then it is easy to see that $\tilde{M}$ is a trivial bundle over $\tilde{\Sigma}$ and by the same method as the trivial case we see that if the normal bundle $\nu(\Sigma)$ is flat, then M is locally a product, $P \times \Sigma$. In particular, the projection is a totally geodesic map, and hence the composition $\pi \circ f$ is a harmonic map.

2) If f is vertical, then $df(T_pN) \subset \mathcal{V} = \mathrm{Ker}\, d\pi$. Therefore $d(\pi \circ f)$ vanishes on N, and the energy is identically zero, which is clearly a harmonic map.

Let $X, Y \in T_pM$ be any two horizontal vectors, we extend them in a neighborhood of p to basic vector fields $\bar{X}$ and $\bar{Y}$ respectively, where basic vector fields means horizontal lifts of vector fields on Σ. Since the second fundamental form $\nabla d\pi$ is a tensor, we have

$$\begin{aligned}\nabla d\pi(X, Y) &= \nabla d\pi(\bar{X}, \bar{Y}) \\ &= \nabla^{\Sigma}_{d\pi(\bar{X})} d\pi(\bar{Y}) - d\pi(\nabla^{M}_{\bar{X}} \bar{Y})\end{aligned}$$

Since $\pi : M \to \Sigma$ is a Riemannian submersion, we can apply O'neill's formula[6] to obtain

$$d\pi(\nabla^{M}_{\bar{X}} \bar{Y}) = \nabla^{\Sigma}_{d\pi(\bar{X})} d\pi(\bar{Y}).$$

So we see that $\nabla d\pi$ vanishes for horizontal vectors. In the case when f is horizontal, we have $df(TN) \subset \mathcal{H}$ and hence $\nabla d\pi(df, df)$ vanishes on N, which means $\pi \circ f$ is harmonic.

3) Suppose $f : N \to M$ is an energy minimizing harmonic map in its homotopy class. We will show that there exists a map $f_1 : N \to M$ in the same homotopy class as f such that $f_1(N) \subset \Sigma$ and the energy of f_1 is not greater than that of f. Then we can conclude that the energy is actually same and f_1 is also energy minimizing and hence harmonic.

Let $h : M \to \Sigma$ and $H : M \times [0,1] \to M$ be the distance nonincreasing deformation retraction and the homotopy we discussed in Section 2. Define a homotopy $F : N \times [0,1] \to M$ by $F(x,t) = H(f(x),t)$. Then $F(x,0) = f(x)$ and $f_1(x) = F(x,1) = h \circ f(x) \in \Sigma$. Therefore $f_1 : N \to M$ is homotopic to f and has its image in Σ. Furthermore, since h is distance nonincresing, for any $v \in TN$ we have

$$\begin{aligned} ||df_1(v)|| &= ||dh(df(v))|| \\ &\leq ||df(v)||, \end{aligned}$$

and equality occurs if and only if dh preserves the length of the vector. Therefore for any orthonormal basis $\{e_i\}$ of the tangent space of N we have

$$\begin{aligned} e(f_1) &= \sum_{i=1}^{n} ||df_1(e_i)||^2 \\ &\leq \sum_{i=1}^{n} ||df(e_i)||^2 = e(f). \end{aligned}$$

Hence the energy $E(f_1)$ of f_1 is not greater than $E(f)$, and they must be same. In fact, $dh : T_pM \to T_{h(p)}M$ is an isometry on $f(N)$, and we see that f_1 and f have isometric images in M. □

As we mentioned in Section 1, we know that Σ is totally convex in M, and hence it is also totally geodesic in M. Since a harmonic followed by a totally geodesic map is again harmonic, it is now easy to see:

COROLLARY 3.2. *Let M, N,and f be as above. Then $i\circ\pi\circ f : N \to M$ is a harmonic map with its image in $\Sigma \subset M$, where $i : \Sigma \to M$ is the inclusion map.*

In the proof of our main theorem, we have not used the full strength of the assumptions and certainly it is conceivable that the theorem is still true under weaker conditions, perhaps without the flatness of the normal bundle. We however do not know the answer yet. In fact, there are not many nontrivial examples of noncompact manifolds of nonnegative curvature, and it seems difficult to find nontrivial examples for the weaker conditions.

In many cases, for the existence of a harmonic map in a given homotopy class of maps, we find an energy minimizing map, which certainly is harmonic. We will see that this technique fails in the case of noncompact manifolds as well as the compact case if the curvature is nonnegative. Let N be an arbitrary compact manifold and let M be a noncompact manifolds with nonnegative curvature with S^n as its soul. We know that for $n \geq 3$, there is no nonconstant stable harmonic map from S^n to N, or from N to S^n (cf. [3]). By Theorem 3.1, any energy minimizing harmonic map f from N to M induces an energy minimizing map $f_1 : N \to S^n$ with the same energy. Since f_1 must be a constant map, we see that f is also trivial. In fact, there are many nonexistence theorems of this sort between compact manifolds, and we can apply the same technique to obtain similar results in noncompact case.

References

1. J. Cheeger, D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. Math. **96** (1972), 413–443.
2. J. Eells, L. Lemire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1977), 1–68.
3. ———, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), 385–524.
4. S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian Geometry 2nd edition*, Springer-Verlag, 1990.
5. G. Luis, *private communication.*
6. B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. **13** (1966), 459–469.
7. G. Perel'man, *Proof of the soul conjecture of Cheeger and Gromoll*, J. Diff. Geom. **40** (1994), 209–212.

8. V. A. Sharafutdinov, *Convex sets in a manifold of nonnegative curvature curvature*, Mat. Zametki. **26** (1979), 129–136.
9. R. Smith, *Harmonic mappings of spheres*, Amer. J. Math. **97** (1975), 503–524.
10. J.-W. Yim, *Space of souls in a complete open manifold of nonnegative curvature*, J. Diff. Geom. **32** (1990), 429–455.
11. ———, *Distance nonincreasing retraction on a complete open manifold of nonnegative curvature*, Ann. Global Anal. Geom. **6** (1988), 191–206.

Department of Mathematics
Korea Advanced
Institute of Science and Technology
Taejon 305-701, Korea