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ON CAUSALITY IN A SPACE-TIME

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ABSTRACT. We study some causality characterizations in a space-time by considering chronological common future (or past) sets and indecomposable sets that are important roles in studying causal boundary structure of the space-time.

1. Introducation and Preliminaries

A space-time represents a time-oriented Lorentzian manifold M. Causality structure is regarded to be a basic structure in any discussion of the general theory of relativity. Various causality conditions have been considered in many physical and mathematical situations and characterized in terms of various methods by many authors. As usual, the chronological (causal) relation is written as $\ll$ $(\prec)$, $I^+(I^-), J^+(J^-)$ shall denote the chronological future(past), causal future(past) in a given space-time. The *chronological common future* $\uparrow U$ of an open subset U of M is defined by

$$\uparrow U = I^+\{p \in M \mid u \ll p \text{ for all } u \in U\}$$

The *chronological common past* $\downarrow U$ of U is defined dually. A nonempty subset of a space-time is called *indecomposable* if it cannot be expressed as a union of either two proper future subsets or proper past subsets. The indecomposable sets so defined are divided into two classes, consisting of those which are the chronological futures or pasts of a single point (these sets are called *proper indecomposable sets*) and those which are not (these sets are called *terminal indecomposable sets*).

Those sets play important roles in construction of causal boundary and the boundary structure of a space-time. (*cf*. Beem [2], Budic and Sachs [4],

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Geroch, Kronheimer and Penrose [6], etc.). Rübe [10] showed that for a general causally simple space-time the chronological common past of a terminal indecomposable future set need not be an indecomposable set. In this paper, we investigate some interesting results characterizing certain causality by chronological common past(future) sets and indecomposable sets and we show that in a strongly causal space-time if the chronological common past of a proper indecomposable future set is indecomposable then it is proper, but this result does not hold for general space-times satisfying weaker causality conditions (for example, even distinguishing space-times).

We now state some concepts of causality conditions that we shall use later. A space-time M is *chronological* if it contains no closed timelike curve, equivalently $p \notin I^+(p)$ for all $p \in M$. M is *distinguishing* if $I^+(p) = I^+(q)$ or $I^-(p) = I^-(q)$ implies $p = q$, equivalently, for every point $p \in$M, every neighborhood of p contains a neighborhood of p which no nonspacelike curve through p intersects more than once. M is *strongly causal* if for every point p in M and any neighborhood U of p there is a neighborhood V of p contained in U such that no nonspacelike curve intersects V more than once. We say that M is *causally continuous* if it is distinguishing and $I^+(p) \subseteq I^+(q)$ iff $I^-(q) \subseteq I^-(p)$ for all $p, q \in M$. M is *causally simple* if it is distinguishing and $J^+(x)$ and $J^-(x)$ are closed for all $x \in M$. A strongly causal space-time M is *globally hyperbolic* if for each pair of points $p, q \in M$, the set $J^+(p) \cap J^-(q)$ is compact. The causality conditions as above can be arranged in the following implications: global hyperbolicity $\Rightarrow$ causal simplicity $\Rightarrow$ causal continuity $\Rightarrow$ strong causality $\Rightarrow$ distinguishing $\Rightarrow$ chronology condition. The respective converse implications are all false.

2. Main Results

In this section, we give some results related to causality of space-time by considering chronological common future or past sets and indecomposable sets.

For a point p in a space-time M, define

$$\Gamma^+(p) = \{q \in M \mid \uparrow I^-(p) = \uparrow I^-(q)\}$$

$$\Gamma^-(p) = \{q \in M \mid \downarrow I^+(p) = \downarrow I^+(q)\}$$

We first observe that it is possible to characterize the strong causality condition in terms of these sets, which is obtained from Proposition 3.1 of Racz [9].

THEOREM 1. *A space-time M is strongly causal if and only if $\Gamma^+(p) = \{p\}$ for all $p \in M$ (or $\Gamma^-(p) = \{p\}$ for all $p \in M$).*

We show the chronologyness of space-time can be characterized by $\Gamma^\pm$ set, in particular by topological size of the region of strong causality violation at each point in the following result. In what follows, let $intA$ mean the interior of a subset A of M and clA the closure of A for the manifold topology of M.

THEOREM 2. *The following are equivalent.*

(1) *A space-time M is chronological,*
(2) *$\Gamma^+(p) \cap \uparrow I^-(p) = \emptyset$ for all $p \in M$ (or $\Gamma^-(p) \cap \downarrow I^+(p) = \emptyset$ for all $p \in M$),*
(3) *$int\ \Gamma^+(p) = \emptyset$ for all $p \in M$ (or $int\ \Gamma^-(p) = \emptyset$ for all $p \in M$).*

PROOF. (1) $\Rightarrow$ (2): If $q \in \Gamma^+(p) \cap \uparrow I^-(p)$ for some $p, q \in M$, then $\uparrow I^-(q) \cap I^-(q) \neq \emptyset$, say $x \in \uparrow I^-(q) \cap I^-(q)$. Then we have $x \in I^+(x)$ and hence M is not chronological.

(2) $\Rightarrow$ (3): Suppose that there is a point $p \in M$ such that $int\Gamma^+(p) \neq \emptyset$. Then $int\Gamma^+(p)$ is a neighborhood of some q, so that $\Gamma^+(p) \cap I^+(q) \neq \emptyset$. Thus there is a point $z \in M$ such that $\uparrow I^-(z) = \uparrow I^-(p) = \uparrow I^-(q)$ and $q \ll z$. Hence $\Gamma^+(z) \cap \uparrow I^-(z) \neq \emptyset$.

(3) $\Rightarrow$ (1): Assume that $p \ll p$ for some $p \in M$. Then for any $q \in M$ with $p \ll q \ll p$, $\uparrow I^-(p) = \uparrow I^-(q)$. Thus $int\Gamma^+(p) \neq \emptyset$.

Now we consider some results in which indecomposable sets are related in an interesting manner with strong causality. The following lemma was shown by Geroch, Kronheimer and Penrose ([6] Theorem 2.1 and 2.3), which is useful for us. Dual results will often be taken for granted.

LEMMA 3.

(1) *A subset P of a space-time M is an indecomposable past set if and only if it is the form $I^-(\gamma)$ for some future directed timelike (or nonspacelike) curve γ in M.*
(2) *If M is strongly causal, then an indecomposable past set $I^-(\gamma)$ is terminal if and only if γ is future inextendable.*

THEOREM 4. *Let M be a strongly causal space-time and let $p \in M$. If $\downarrow I^+(p)$ is an indecomposable set, then $\downarrow I^+(p) = I^-(p)$.*

PROOF. If $\downarrow I^+(p)$ is indecomposable, by Lemma 3 there is a timelike curve γ in M such that $\downarrow I^+(p) = I^-(\gamma)$. Then either p is in $I^-(\gamma)$ or p is in the boundary $\partial I^-(\gamma)$ of $I^-(\gamma)$.

If p was in $I^-(\gamma)$, then there is a point q in γ such that $p \ll q$. Taking $x, y \in M$ with $p \ll x \ll y \ll q$, we have $y \in I^-(\gamma) = \downarrow I^+(p)$. Since also $x \in I^+(p)$, $y \in I^-(x) \subseteq I^-(y)$. Thus chronology condition is violated at y. Hence $p \in \partial I^-(\gamma)$.

Now suppose that p is not the future endpoint of γ. By Theorem 3.20 of Penrose [8] there is a future directed null geodesic α with past endpoint p such that α is contained in $\partial I^-(\gamma)$. Choose a point q in α different to p. Let U, V be neighborhoods of p, q respectively such that $U \cap V = \emptyset$. Now, we take a future directed timelike curve β defined on $(-1, 1)$ such that $\beta(0) = p$ and β is contained in U. Since

$$\beta(-1,0) \subseteq I^-(p) \subseteq I^-(q) \subseteq I^-(\gamma) = \downarrow I^+(p) \text{ and } \beta(0,1) \subseteq I^+(p),$$

there is a sequence $\{v_n\}$ in V with

$$\beta(-\frac{1}{n}) \ll v_n \ll \beta(\frac{1}{n})$$

for all positive integers n. Then if given any neighborhood W of p contained in U there is a positive integer n_w such that

$$\beta(-\frac{1}{n_w}),\ \beta(\frac{1}{n_w}) \in W.$$

But here $v_{n_w} \in V$ and so $v_{n_w} \notin W$. Thus strong causality condition is violated at p. Hence p is the future endpoint of γ and by Lemma 3, $\downarrow I^+(p) = I^-(p)$.

From the above theorem and Lemma 2.3 in Kim [7], we have the following corollary.

COROLLARY 5. *Let M be a strongly causal space-time. Then the following are equivalent.*

(1) *M is causally continuous,*
(2) *every chronological common future of a proper indecomposable past set is indecomposable and every chronological common past of a proper indecomposable future set is indecomposable,*
(3) *$\downarrow I^+(p) = I^-(p)$ and $\uparrow I^-(p) = I^+(p)$ for all $p \in M$,*
(4) *$p \in \downarrow I^+(q)$ if and only if $q \in \uparrow I^-(p)$.*

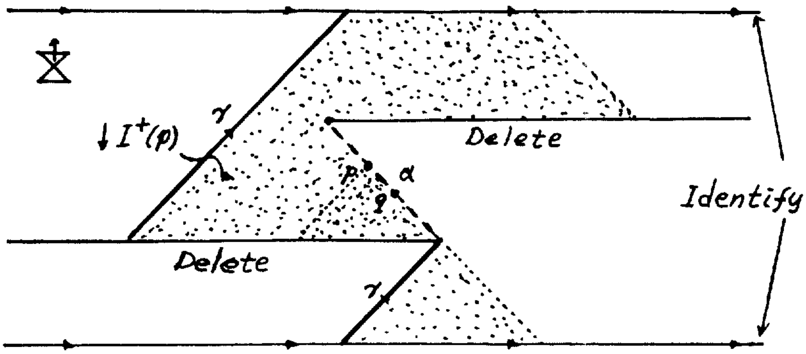

FIGURE 1. Let M be the space-time obtained from the cylinder $S^1 \times \mathbb{R}$ with the metric $ds^2 = -d\theta^2 + dt^2$ by deleting two spacelike half-lines whose endpoints were the endpoints of a short null geodesic α. This space-time M is distinguishing but not strongly causal. In particular, if two points p, q are in α, then $\downarrow I^+(p) = \downarrow I^+(q)$. We note that $\downarrow I^+(p)$, the darked region, is the indecomposable set $I^-(\gamma)$ where γ is the null geodesic (bolded line) in the figure. Moreover, $\downarrow I^+(p) \neq I^-(p)$.

REMARK. (1) Theorem 4 does not hold for arbitrary space-time (even though M is distinguishing); This is illustrated by Figure 1.

(2) Even for a causally simple space-time the chronological common future (or past) of a terminal indecomposable past (or future) set need not be indecomposable; See Rübe [10].

THEOREM 6. *Let M be a globally hyperbolic. Then any proper indecomposable past set contains no terminal indecomposable past set, equivalently, $\uparrow P = \emptyset$ for all terminal indecomposable past sets P.*

PROOF. Suppose that $I^-(\gamma) \subseteq I^-(q)$ where $\gamma : [0,1) \to M$ is a future directed timelike curve in M and $q \in M$. Then $\gamma([0,1)) \subseteq clI^-(q) = J^-(q)$, $\gamma([0,1)) \subseteq J^+(\gamma(0)) \cap J^-(q)$. Since $J^+(\gamma(0)) \cap J^-(q)$ is compact and M is strongly causal, γ is not future inextendible from Proposition 2.9 of Beem and Ehrlich [3]. Therefore by Lemma 3, any proper indecomposable past set cannot contain a terminal indecomposable past set.

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