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SOME VANISHING THEOREMS ON KÄHLER FOLIATIONS

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ABSTRACT. We shall prove some vanishing theorems for the transversal Dirac operators on Kähler foliations

1. Introduction

J. Brüning and F. W. Kamber ([1]) studied the transversal Dirac operators on compact foliated Riemannian manifolds and proved some vanishing theorems for the transversal Dirac operators. Also, J.S.Pak and S.D.Jung ([8]) extended the above results to the complete cases. In this paper, we shall prove some vanishing theorems on compact Kähler foliations. Throughout this paper, we shall be in c^∞-class. Manifolds are assumed to be connected, orientable, paracompact and hausdorff spaces. We also adopt the following ranges of indices :

$$1 \leqslant i, j, \cdots \leqslant p; \quad 1 \leqslant a, b, \cdots \leqslant n,$$

$$1 \leqslant \alpha, \beta, \cdots \leqslant q(= 2n), \quad 1 \leqslant A, B, \cdots \leqslant p + q.$$

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2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with an oriented foliation $\mathcal{F}$ of codimension $q(=2n)$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then there exists an exact sequence of vector bundles

$$O \to L \to TM \to Q \to O,$$

where L is the tangent bundle and Q is the normal bundle of $\mathcal{F}$ with respect to g_M ([9]). The foliation is assumed to be transversally Kähler. By a *Kähler foliation* $\mathcal{F}$ we mean a foliation satisfying the following conditions; (i) $\mathcal{F}$ is Riemannian, with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q \cong L^\perp$, (ii) there is a holonomy invariant almost complex structure $J : Q \to Q$, where $dimQ = q(= 2n)$ (real dimension), with respect to which g_Q is Hermitian, i.e., $g_Q(JX, JY) = g_Q(X, Y)$ for $X,Y \in \Gamma(Q)$, and (iii) if ∇ denotes the unique metric and torsion free connection in Q, then ∇ is almost complex, i.e., $\nabla J = 0$. Note that $\Phi(X, Y) = g_Q(X, JY)$ defines a basic 2-form Φ, which is closed as a consequence of $\nabla g_Q = 0$ and $\nabla J = 0$. Let R_∇ be the curvature associated to the unique metric and torsion free connection ∇ in the normal bundle $\Gamma(Q)$ of the Riemannian foliation $\mathcal{F}$. Let similarily S_∇ be the Ricci curvature. For a Kähler foliation we have then the following properties :

$$R_\nabla(X, Y)J = JR_\nabla(X, Y), \tag{2.1}$$

$$R_\nabla(JX, JY) = R_\nabla(X, Y), \tag{2.2}$$

$$S_\nabla(JX, JY) = S_\nabla(X, Y), \tag{2.3}$$

$$R_\nabla(X, Y)Z + R_\nabla(Y, Z)X + R_\nabla(Z, X)Y = 0, \tag{2.4}$$

where X,Y and Z are elements of $\Gamma(Q)$. In the sequal it will be convinient to use the following orthonormal frame on M. For $x \in M$, let $\{e_A\}$ be an oriented orthonormal basis of T_xM with e_i in L_x and e_α in $L_x^\perp$ ($\mathcal{F}$ is of codimension $q = 2n$ on M^{p+2n}). The transversal Kähler property of $\mathcal{F}$ allows then to extend e_a, Je_a to local vector fields E_a,$JE_a \in \Gamma L^\perp$ such that

(2.5)

$$(\nabla_{E_a} E_b)_x = 0, \quad (\nabla_{E_a} JE_b)_x = 0, \quad (\nabla_{JE_a} E_b)_x = 0, \quad (\nabla_{JE_a} JE_b)_x = 0.$$

As a consequence of torsion freeness

$$(2.6)\qquad [E_a, E_b]_x,\quad [E_a, JE_b]_x,\quad [JE_a, JE_b]_x \in L_x.$$

The E_a,JE_a can be chosen as (local) infinitesimal automorphisms of $\mathcal{F}$, so that

$$(2.7)\qquad \nabla_X E_a = \pi[X, E_a] = 0 \quad \text{for} \quad X \in \Gamma L.$$

We can complete E_a,JE_a by the Gram-Schmidt process to a moving local frame by adding $E_i \in \Gamma L$ with $(E_i)_x = e_i$. In terms of such a moving frame the transversal Ricci operator and the scalar curvature are given by

$$(2.8)\qquad \rho_\nabla = \sum JR_\nabla(E_a, JE_a) \quad \text{and}$$

$$(2.9)\qquad \sigma_\nabla = \sum g_Q(\rho_\nabla(E_\alpha), E_\alpha)$$

respectively. Let $\Omega^r_B(\mathcal{F})$ be the space of all basic forms of degree r. The exterior differential d restrcts to $d_B : \Omega^r_B \to \Omega^{r+1}_B$ and let δ_B be the formal adjoint of d_B with respect to the induced scalar product $<,>_B$ on Ω_B ([8]). Now, assume that the mean curvature form k of the foliation $\mathcal{F}$ is isoparametric, i.e., $k \in \Omega^1_B(\mathcal{F})$. It is well known that if $k \in \Omega^1_B(\mathcal{F})$, then $dk = 0$ ([9]).

3. Vanishing Theorems on Kähler foliations

Let $Cl(Q)$ be the transversally Clifford algebra of Q and $\mathbf{C}l(Q) = Cl(Q)\otimes_{\mathbb{R}} \mathbf{C}$ the complexification of $Cl(Q)$. Set

$$(3.1)\qquad \epsilon_a = \frac{1}{2}(E_a - iJE_a),\quad \bar{\epsilon_a} = \frac{1}{2}(E_a + iJE_a),$$

where $\{E_a, JE_a\}$ is an orthonormal basis of Q. Then $\{\epsilon_a, \bar{\epsilon_a}\}$ forms a basis of $Q\otimes\mathbf{C}$, complexification of Q and $\mathbf{C}l(Q)$ is generated by $\{\epsilon_a, \bar{\epsilon_a}\}$ which satisfies the relations

$$(3.2)\qquad \epsilon_a\bar{\epsilon}_b + \bar{\epsilon}_b\epsilon_a = -\delta_{ab},\quad \epsilon_a\epsilon_b = -\epsilon_b\epsilon_a,\quad \bar{\epsilon_a}\bar{\epsilon_b} = -\bar{\epsilon_b}\bar{\epsilon_a}.$$

Here we omitted the Clifford multiplication " · ". Let $E \to M$ be the holomophic foliated bundle of left modules over $\mathbf{C}l(Q)$, i.e., the fiber E_x is a left module over $\mathbf{C}l(Q)_x$ for each $x \in M$, and the multiplication map is smooth. We assume that E carries a hermitian metric (,) such that ;

(i) Module multiplication by unit tangent vectors is unitary, i.e.,

$$(\varphi s, t) + (s, \bar{\varphi} t) = 0 \tag{3.3}$$

for all $\varphi \in \mathbf{C}l(Q)$ and for all $s, t \in \Gamma(E)$.

(ii) With respect to the cannonical hermitian connection, covariant differentiation is a derivation of module multiplication, i.e., for all $\varphi \in \Gamma(\mathbf{C}l(Q))$ and all $s \in \Gamma(E)$, we have

$$\nabla(\varphi, s) = (\nabla\varphi)s + \varphi(\nabla s). \tag{3.4}$$

We now introduce two differential operators $\mathcal{D}$, $\bar{\mathcal{D}} : \Gamma(E) \to \Gamma(E)$ by formulas

$$\begin{aligned} \mathcal{D} &= \sum \epsilon_a \nabla_{\bar{\epsilon_a}} - \frac{1}{4} H, \\ \bar{\mathcal{D}} &= \sum \bar{\epsilon_a} \nabla_{\epsilon_a} - \frac{1}{4} \bar{H}, \end{aligned} \tag{3.5}$$

where $H = \frac{1}{2}\{k - iJk\}, k$ is a mean curvature form of $\mathcal{F}$.

THEOREM 3.1. *The operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are formal adjoints of one another and transversally elliptic.*

PROOF. Fix $x \in M$ and choose a local frame $\epsilon_1, \cdots, \epsilon_n, \bar{\epsilon_1}, \cdots, \bar{\epsilon_n}$ as above such that $(\nabla\epsilon_a)_x = (\nabla\bar{\epsilon_a})_x = 0$. Then for all s, $t \in \Gamma(E)$, we have at the point x that

$$\begin{aligned} (\mathcal{D}s, t)_x &= \sum (\epsilon_a \nabla_{\bar{\epsilon_a}} s - \frac{1}{4} H s, t)_x \\ &= -\sum (\nabla_{\bar{\epsilon_a}} s, \bar{\epsilon_a} t)_x + \frac{1}{4}(s, \bar{H} t)_x \\ &= -\sum \bar{\epsilon_a}(s, \bar{\epsilon_a} t)_x \sum + (s, \bar{\epsilon_a} \nabla_{\epsilon_a} t)_x + \frac{1}{4}(s, \bar{H} t)_x \\ &= (divU)_x + \sum (s, \bar{\epsilon_a} \nabla_{\epsilon_a} t)_x + \frac{1}{4}(s, \bar{H} t)_x, \end{aligned}$$

where U is the complex vector field in $Q \otimes \mathbf{C}$ defined by the condition that $g_Q(V, U) = \frac{1}{4}((V - iJV)s, t)$ for all real vectors $V \in \Gamma(Q)$. Then

$$
\begin{aligned}
(divU)_x &= \sum g_Q(\nabla_{E_a} U, E_a)_x + \sum g_Q(\nabla_{JE_a} U, JE_a)_x \\
&= \sum E_a g_Q(U, e_a)_x + \sum JE_a g_Q(U, JE_a)_x \\
&= \frac{1}{4} \sum \{E_a((E_a - iJE_a)s, t)_x + JE_a((JE_a + ie_a)s, t)_x\} \\
&= \frac{1}{2} \sum \{E_a(\epsilon_a s, t)_x + iJE_a(\epsilon_a s, t)_x\} = \sum \bar{\epsilon_a}(\epsilon_a s, t)_x \\
&= -\sum \bar{\epsilon_a}(s, \bar{\epsilon_a} t)_x.
\end{aligned}
$$

By the Green's theorem ([10]),

$$
\begin{aligned}
\int_M divU = \ll U, k \gg &= \frac{1}{4} \int_M ((k - iJk)s, t) \\
&= \frac{1}{2} \int_M (Hs, t) = -\frac{1}{2} \int_M (s, \bar{H}t),
\end{aligned}
$$

where $\ll U, V \gg = \int_M g_Q(U, V)$. It follows that

$$
\int_M (\mathcal{D}s, t) = \int_M (s, \bar{\mathcal{D}}t)
$$

for all $s, t \in \Gamma(E)$. Hence we have $\mathcal{D}^* = \bar{\mathcal{D}}$. Moreover, by straightforward calculation, $\sigma_{\mathcal{D}}(x, \xi_0) = \xi$ and $\sigma_{\bar{\mathcal{D}}}(x, \xi_0) = \bar{\xi}$ for $\xi_0 \in \Gamma(Q^*) \equiv \Gamma(Q)$, where $\xi = \frac{1}{2}(\xi_0 - iJ\xi_0)$. □

Now, we define the subspace $\Gamma_B(E)$ of basic or holonomy invariant section of E by

(3.6) $$\Gamma_B(E) = \{s \in \Gamma(E) | \quad \nabla_X s = 0, \quad X \in \Gamma(L)\}.$$

If we consider the vector bundle $E = \Lambda Q^* \otimes \mathbf{C}$, then we have

(3.7) $$\Gamma_B(E) = \Omega_B^*(\mathcal{F}) \otimes \mathbf{C}.$$

From (3.5), we see that $\mathcal{D}$ and $\bar{\mathcal{D}}$ leaves $\Gamma_B(E)$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric. Put

$$\mathcal{D}_b \equiv \mathcal{D}_{|\Gamma_B(E)} \quad \text{and} \quad \bar{\mathcal{D}}_b \equiv \bar{\mathcal{D}}_{|\Gamma_B(E)}.$$

Now, let $\Omega_B^{r,s}(\mathcal{F})$ be the standard Dolbealt decomposition of $\Omega_B^*(\mathcal{F})\otimes\mathbf{C}$. Then there are operators

$$\partial : \Omega_B^{r,s}(\mathcal{F}) \otimes \mathbf{C} \to \Omega_B^{r+1,s}(\mathcal{F}) \otimes \mathbf{C},$$
$$\bar{\partial} : \Omega_B^{r,s}(\mathcal{F}) \otimes \mathbf{C} \to \Omega_B^{r,s+1}(\mathcal{F}) \otimes \mathbf{C}$$

are given by the followings :

$$\partial = \bar{\epsilon}_a \wedge \nabla_{\epsilon_a}, \quad \bar{\partial} = \epsilon_a \wedge \nabla_{\bar{\epsilon}_a}, \tag{3.9}$$

where ∇ is the Kähler connection on $Q\otimes\mathbf{C}$ and their formal adjoints of ∂ and $\bar{\partial}$ are

$$\partial^* = -i(\epsilon_a)\nabla_{\bar{\epsilon}_a} + \frac{1}{2}i(H), \tag{3.10}$$
$$\bar{\partial}^* = -i(\bar{\epsilon}_a)\nabla_{\epsilon_a} + \frac{1}{2}i(\bar{H}).$$

These follows from d_B and δ_B by breaking up the formulas of d_B and δ_B into (1,0) and (0,1) components and using $d_B = 2(\partial + \bar{\partial}), \delta_B = 2(\partial^* + \bar{\partial}^*)$. Moreover, if $\mathcal{F}$ is harmonic kähler, by the well known facts; $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$, $\mathcal{D}_b = \bar{\partial} + \partial^*$ and $\bar{\mathcal{D}}_b = \partial + \bar{\partial}^*$, we have

$$\mathcal{D}_b\bar{\mathcal{D}}_b + \bar{\mathcal{D}}_b\mathcal{D}_b = \frac{1}{4}\Delta_B, \tag{3.11}$$

where $\Delta_B = d_B\delta_B + \delta_B d_B$ is the basic Laplacian. Also, we define invariant operators on $\Gamma(E)$ by

$$\nabla_{tr}^*\nabla_{tr}s = -\nabla_{\epsilon_a}\nabla_{\bar{\epsilon}_a}s + \frac{1}{2}\nabla_{\bar{H}}s, \tag{3.12}$$
$$\bar{\nabla}_{tr}^*\nabla_{tr}s = -\nabla_{\bar{\epsilon}_a}\nabla_{\epsilon_a}s + \frac{1}{2}\nabla_{H}s,$$
$$\mathcal{R} = \sum \epsilon_a\bar{\epsilon}_b R^E(\bar{\epsilon}_a, \epsilon_b),$$
$$\bar{\mathcal{R}} = \sum \bar{\epsilon}_a\epsilon_b R^E(\epsilon_a, \bar{\epsilon}_b),$$

where R^E is the curvature tensor field on $\Gamma(E)$. Then we have

PROPOSITION 3.2. *The operators $\nabla_{tr}^*\nabla_{tr}$ and $\bar{\nabla}_{tr}^*\bar{\nabla}_{tr}$ are nonnegative, transversally elliptic, formally self-adjoint differential operators.*

PROOF. Fix $x \in M$. If we choose a local frame $\{\epsilon_a, \bar{\epsilon_a}\}$ such that $(\nabla\epsilon_a)_x = (\nabla\bar{\epsilon_a})_x = 0$, then for $s, t \in \Gamma(E)$, we have

$$\begin{aligned}(\nabla_{tr}^*\nabla_{tr}s, t)_x &= -\sum(\nabla_{\epsilon_a}\nabla_{\bar{\epsilon_a}}s, t)_x + \frac{1}{2}(\nabla_H s, t)_x \\ &= -\sum \epsilon_a(\nabla_{\bar{\epsilon_a}}s, t)_x + \sum(\nabla_{\bar{\epsilon_a}}s, \nabla_{\bar{\epsilon}_a}t)_x + \frac{1}{2}(\nabla_H s, t)_x \\ &= -(divU)_x + \sum(\nabla_{\bar{\epsilon}_a}s, \nabla_{\bar{\epsilon}_a}t)_x + \frac{1}{2}(\nabla_{\bar{H}}s, t)_x \\ &= -(divU)_x + (divW)_x - \sum(s, \nabla_{\epsilon_a}\nabla_{\bar{\epsilon_a}}t) + \frac{1}{2}(\nabla_{\bar{H}}s, t)_x.\end{aligned}$$

Here U is the complex vector field in $Q\otimes\mathbf{C}$ defined by the relation : $g_Q(V, U) = \frac{1}{4}(\nabla_{V+iJV}s, t)$ for all real vectors $V \in \Gamma(Q)$. Also, W is defined as

$$g_Q(V, W) = \frac{1}{4}(s, \nabla_{V+iJV}t).$$

Note that at the point $x \in M$,

$$\begin{aligned}(divU)_x &= \sum\{g_Q(\nabla_{E_a}U, E_a)_x + g_Q(\nabla_{JE_a}U, JE_a)_x\} \\ &= \sum\{E_a g_Q(U, E_a)_x + JE_a g_Q(U, JE_a)_x\} \\ &= \frac{1}{4}\sum\{E_a(\nabla_{E_a+iJE_a}s, t)_x + JE_a(\nabla_{JE_a-iE_a}s, t)_x\} \\ &= \sum \epsilon_a(\nabla_{\bar{\epsilon_a}}s, t)_x.\end{aligned}$$

By the Green's theorem ([10]),

$$\begin{aligned}\int_M divU = \ll U, k \gg &= \frac{1}{4}\int_M(\nabla_{k+iJk}s, t) \\ &= \frac{1}{2}\int_M(\nabla_H s, t)\end{aligned}$$

and similarly

$$(divW)_x = \sum \bar{\epsilon_a}(s, \nabla_{\bar{\epsilon_a}} t)_x.$$

Hence

$$\int_M divW = \frac{1}{2}\int_M (s, \nabla_{\bar{H}} t)$$

Therefore, by integrating

$$\int_M (\nabla_{tr}^* \nabla_{tr} s, t) = \int_M (\nabla_{tr} s, \nabla_{tr} t) = \int_M (s, \nabla_{tr}^* \nabla_{tr} t).$$

where $(\nabla_{tr} s, \nabla_{tr} t) = \sum(\nabla_{\bar{\epsilon_a}} s, \nabla_{\bar{\epsilon_a}} t)$. Hence $\nabla_{tr}^* \nabla_{tr}$ is nonnegative, formally self adjoint operator. Also, by simple calculation, $\nabla_{tr}^* \nabla_{tr}$ and $\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}$ have the same principal symbols. So $\nabla_{tr}^* \nabla_{tr}$ is transversally elliptic. Arguments for $\nabla_{tr}^* \nabla_{tr}$ are similar. □

THEOREM 3.3. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric Kähler foliation $\mathcal{F}$ and a bundle-like metric g_M. Then on $\Gamma(E)$, we have the following identity*

$$2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = \frac{1}{2}\nabla_T^* \nabla_T s + \mathcal{R}^E(s) + \mathcal{K}s,$$

where

$$\nabla_T^* \nabla_T s = 2(\nabla_{tr}^* \nabla_{tr} s + \bar{\nabla}_{tr}^* \bar{\nabla}_{tr} s) = -\sum(\nabla^2_{e_a, e_a} + \nabla^2_{Je_a, Je_a})s,$$

$$\mathcal{R}^E = \frac{1}{4}\sum E_\alpha E_\beta R^E(E_\alpha, E_\beta) \quad \text{and}$$

$$\mathcal{K} = -\frac{1}{2}\{(\partial^* \bar{H} + \bar{\partial}^* H) - \frac{1}{2}|H|^2\}.$$

PROOF. If we choose a local frame $\{\epsilon_a, \bar{\epsilon_a}\}$ such that $(\nabla \epsilon_a)_x = (\nabla \bar{\epsilon_a})_x = 0$, then for any $s \in \Gamma(E)$, using (3.2) and (3.4) we have

$$\begin{aligned}(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = &-\sum \nabla_{\epsilon_a}\nabla_{\bar{\epsilon_a}} s + \mathcal{R}(s) \\ &- \frac{1}{4}\sum\{(\epsilon_a \bar{H} + \bar{H}\epsilon_a)\nabla_{\bar{\epsilon_a}} s + (H\bar{\epsilon_a} + \bar{\epsilon_a} H)\nabla_{\epsilon_a} s\} \\ &- \frac{1}{4}\sum\{\epsilon_a \nabla_{\bar{\epsilon_a}} \bar{H} s + \bar{\epsilon_a}\nabla_{\epsilon_a} H s\} - \frac{1}{8}|H|^2 s.\end{aligned}$$

Since $H\bar{\epsilon_a} + \bar{\epsilon_a}H = -2g_Q(H, \bar{\epsilon_a})$, we have $\sum(\bar{\epsilon_a}H + H\bar{\epsilon_a})\nabla_{\epsilon_a}s = -\nabla_H s$. Similarily, $\sum(\epsilon_a\bar{H} + \bar{H}\epsilon_a)\nabla_{\bar{\epsilon_a}}s = -\nabla_{\bar{H}}s$. From (3.9) and (3.10), we have

$$\begin{aligned}(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = &-\sum \nabla_{\epsilon_a}\nabla_{\bar{\epsilon_a}}s + \mathcal{R}(s) + \frac{1}{4}(\nabla_H s + \nabla_{\bar{H}}s)\\ &-\frac{1}{4}\{(\bar{\partial} + \partial^*)\bar{H} + (\partial + \bar{\partial}^*)H\}s + \frac{1}{8}|H|^2 s\end{aligned}$$

because of $i(H)\bar{H} = g_Q(H, \bar{H}) = |H|^2$. and by another calculation, we obtain

$$\begin{aligned}(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = &-\sum \nabla_{\bar{\epsilon_a}}\nabla_{\epsilon_a}s + \bar{\mathcal{R}}(s) + \frac{1}{4}(\nabla_H s + \nabla_{\bar{H}}s)\\ &-\frac{1}{4}\{(\bar{\partial} + \partial^*)\bar{H} + (\partial + \bar{\partial}^*)H\}s + \frac{1}{8}|H|^2 s.\end{aligned}$$

Summing up the above two equations, we have

$$\begin{aligned}2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = &(\nabla_{tr}^*\nabla_{tr} + \bar{\nabla}_{tr}^*\nabla_{tr})s + (\mathcal{R} + \bar{\mathcal{R}})s\\ &-\frac{1}{4}\{(\bar{\partial} + \partial^*)\bar{H} + (\partial + \bar{\partial}^*)H\}s + \frac{1}{4}|H|^2 s.\end{aligned}$$

Since $dk = 0$ and $dJk = 0$. we have $\partial H = \bar{\partial}\bar{H} = 0$. Moreover, by straight calculation, we have

$$\mathcal{R} + \bar{\mathcal{R}} = \frac{1}{4}\sum E_\alpha E_\beta R^E(E_\alpha, E_\beta).$$

Hence this proof is completed. □

COROLLARY 3.4. *Let $(M, g_M, \mathcal{F})$ be as in Theorem3.3. Then on $\Gamma_B(E)$, we have*

$$2(\mathcal{D}_b\bar{\mathcal{D}}_b + \bar{\mathcal{D}}_b\mathcal{D}_b) = \Delta|_{\Gamma_B}(E),$$

where $\Delta = \frac{1}{2}\nabla^*\nabla + \mathcal{R}^E + \mathcal{K}$ is a Laplace type operator.

Corollary 3.4 may be used to prove vanishing theorems for Ker $\mathcal{D}_b$ provided one is able to control the divergence term $\delta\mathcal{K}$ in the above expression for $\mathcal{K}$. In fact, we assume that $\delta k = 0$. Then we have $\mathcal{K} =$

$\frac{1}{4}|H|^2$ and the resulting equation is given as following from Corollary 3.4 :

$$g_E((\mathcal{D}_b\bar{\mathcal{D}}_b + \bar{\mathcal{D}}_b\mathcal{D}_b)s, s) = \frac{1}{2}\|\nabla s\|^2 + \frac{1}{2}g_E(\mathcal{R}^E(s), s) + \frac{3}{8}|H|^2\|s\|^2$$

for any $s \in \Gamma_B(E)$, where g_E is a pointwise inner product on E. Thus we have

THEOREM 3.5. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric Kähler foliation $\mathcal{F}$ and a bundle-like metric g_M. Suppose that the mean curvature form k of $\mathcal{F}$ satisfies $\delta k = 0$. If $\mathcal{R}^E$ is non-negative, then every $s \in Ker\ \mathcal{D}_b \cap Ker\ \overline{\mathcal{D}}_6$ is parallel. Moreover, If $\mathcal{R}^E$ is non-negative and positive at some point of M, then every $s \in Ker\ \mathcal{D}_b \cap Ker\ \bar{\mathcal{D}}_b$ vanishes.*

Moreover, since $\mathbf{C}l(Q)$ is a left module of itself, we can calculate $\mathcal{R}^E$ on $\mathbf{C}l(Q)$ as following : for any $s \in \Gamma(Q)$

(3.13)

$$\begin{aligned}
\mathcal{R}^E(s) &= \frac{1}{4}\sum E_\alpha E_\beta R_\nabla(E_\alpha, E_\beta)s \\
&= \frac{1}{4}\sum E_\alpha E_\beta g_Q(R_\nabla(E_\alpha, E_\beta)s, E_\gamma)E_\gamma \\
&= \frac{1}{4}\sum\{E_a E_b g_Q(R_\nabla(E_a, E_b)s, E_c)E_c \\
&\quad + E_a E_b g_Q(R_\nabla(E_a, E_b)s, JE_c)JE_c \\
&\quad + JE_a JE_b g_Q(R_\nabla(JE_a, JE_b)s, E_c)E_c \\
&\quad + JE_a JE_b g_Q(R_\nabla(JE_a, JE_b)s, JE_c)JE_c \\
&\quad + JE_a E_b g_Q(R_\nabla(JE_a, E_b)s, E_c)E_c \\
&\quad + JE_a E_b g_Q(R_\nabla(JE_a, E_b)s, JE_c)JE_c \\
&\quad + E_a JE_b g_Q(R_\nabla(E_a, JE_b)s, E_c)E_c \\
&\quad + E_a JE_b g_Q(R_\nabla(E_a JE_b)s, JE_c)JE_c\}.
\end{aligned}$$

By using the first Bianchi identity, we have

(3.14)
$$\begin{aligned}
&\sum E_a E_b g_Q(R_\nabla(E_a, E_b)s, E_c)E_c \\
&= -\sum g_Q(R_\nabla(E_a, E_b)E_c, s)E_a E_b E_c \\
&= -\frac{1}{3}\sum_{a\neq b\neq c\neq a} g_Q(R_\nabla(E_a, E_b)E_c \\
&\quad + R_\nabla(E_b, E_c)E_a + R_\nabla(E_c, E_a)E_b, s)E_a E_b E_c \\
&\quad + \sum(E_a, E_b)E_b, s)E_a - \sum g_Q(R_\nabla(E_a, E_b)E_a, s)E_b \\
&= -2\sum g_Q(R_\nabla(E_a, s)E_a, E_b)E_b.
\end{aligned}$$

Similarily, we have

$$\begin{aligned}
&\sum JE_a JE_b g_Q(R_\nabla(JE_a, JE_b)s, JE_c)JE_c \\
&= -2\sum g_Q(R_\nabla(JE_a, s)JE_a, JE_b)JE_b.
\end{aligned} \tag{3.15}$$

Also, by straight calculation, we have

$$\begin{aligned}
&\sum\{E_a E_b g_Q(R_\nabla(E_a, E_b)s, JE_c)JE_c \\
&+ JE_a E_b g_Q(R_\nabla(JE_a, E_b)s, E_c)E_c \\
&+ E_a JE_b g_Q(R_\nabla(E_a, JE_b)s, E_c)E_c\} \\
&= -2\sum g_Q(R_\nabla(E_a, JE_b)E_a, s)JE_b \\
&= -2\sum g_Q(R_\nabla(E_a, s)E_a, JE_b)JE_b,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\sum\{JE_a JE_b g_Q(R_\nabla(JE_a, JE_b)s, E_c)E_c \\
&+ JE_a E_b g_Q(R_\nabla(JE_a, E_b)s, JE_c)JE_c \\
&+ E_a JE_b g_Q(R_\nabla(E_a, JE_b)s, JE_c)JE_c\} \\
&= -2\sum g_Q(R_\nabla(JE_a, s)JE_a, E_b)E_b.
\end{aligned} \tag{3.17}$$

Substituting (3.14),(3.15),(3.16) and (3.17) into (3.13), we have

$$\begin{aligned}
\mathcal{R}^E(s) &= \frac{1}{2}\sum\{R_\nabla(E_a, s)E_a + R_\nabla(JE_a, s)JE_a\} \\
&= \frac{1}{2}\rho_\nabla(s).
\end{aligned}$$

Thus we have

THEOREM 3.6. *Let $(M, g_M, \mathcal{F})$ be as in Theorem 3.5 . Then on $\Gamma(Q)$ we have*

$$2(\mathcal{D}_b\bar{\mathcal{D}}_b + \bar{\mathcal{D}}_b\mathcal{D}_b) = \frac{1}{2}\nabla_\Gamma^*\nabla_\Gamma + \frac{1}{2}\rho_\nabla + \mathcal{K},$$

where ρ_∇ is the transversal Ricci operator on $\Gamma(Q)$.

THEOREM 3.7. *Let $(M, g_M, \mathcal{F})$ be an isoparametric Kähler foliation with bundle-like metric g_M. Suppose that the mean curvature form k satisfies $\delta k = 0$, then*

a) *If ρ_∇ is nonnegative and positive at some point of M , then every normal section $s \in Ker\ \mathcal{D}_b \cap Ker\ \bar{\mathcal{D}}_b$ vanishes, and*
b) *If ρ_∇ is non-negative , then every $s \in Ker\ \mathcal{D}_b \cap Ker\ \bar{\mathcal{D}}_b$ is parallel.*

By means of (3.11) and Theorem 3.7, we get

COROLLARY 3.8. *Let $(M, g_M, \mathcal{F})$ be a harmonic kähler foliation with bundle like metric g_M. Then if the transversal Ricci curvature is non-negative and positive at some point of M, then there are no nontrivial basic harmonic 1-forms.*

4. Vanishing theorems on Kähler spin foliations

Let $(M, g_M, \mathcal{F})$ be an isoparametric Kähler spin foliation. In this case there exists a principal *Spin* $(2n)$-bundle, $P_{Spin}(Q) \to M$, with a *Spin* $(2n)$-equivalent map, $\xi : P_{Spin}(Q) \to P_{So}(Q)$, to the bundle of (oriented) transversal orthonormal frames on M. The *foliated spinor bundle*, S, is then defined to be the vector bundle associated to the unitary representation τ of *Spin (2n)* given by the unique inreducible complex representation of $Cl(2n)$, i.e., $S = P_{Spin}(Q) \otimes_\tau \mathbf{C}^{2^n}$. This bundle is naturally a bundle of modules over $\mathbf{C}l(Q)$ and carries a cannonical connection induced from the lift of the Riemannian connection on $P_{So}(Q)$ ([4]). Since $\mathcal{F}$ is Kähler foliation, this bundle S is naturally holomorphic and its connection is hermitian. To compute the term $\mathcal{R}$ and $\bar{\mathcal{R}}$ in (3.12) we need to know the curvature tensor R^S of S. This is given in terms of the Riemannian curvature tensor on Q by the formula ([5])

$$R^S(X,Y)s = \frac{1}{4}\sum_{\alpha,\beta} g_Q(R_\nabla(X,Y)E_\alpha, E_\beta)E_\alpha E_\beta s \tag{4.1}$$

for all $X, Y \in \Gamma(Q)$ and all $s \in S$, where $\{E_\alpha\}$ is any real orthonormal basis of $\Gamma(Q)$. Choosing a basis $\{E_a, JE_a\}$, we can reexpress (4.1) as

(4.2)
$$R^S(X,Y) = \sum\{g_Q(R_\nabla(X,Y)\epsilon_a\bar{\epsilon_b})\bar{\epsilon_a}\epsilon_b + g_Q(R_\nabla(X,Y)\bar{\epsilon_a},\epsilon_b)\epsilon_a\bar{\epsilon_b}\}$$
$$= 2\sum g_Q(R_\nabla(X,Y)\epsilon_a,\bar{\epsilon_b})\bar{\epsilon_a}\epsilon_b + \sum g_Q(R_\nabla(X,Y)\epsilon_a,\bar{\epsilon_a}),$$

where we have used the fact : $\epsilon_a\bar{\epsilon_b} + \bar{\epsilon_b}\epsilon_a = -\delta_{ab}$. It follows that from (3.12) and (4.2),

$$\mathcal{R} = \sum \epsilon_a\bar{\epsilon_b}R^S(\bar{\epsilon_a},\epsilon_b) = \sum g_Q(R_\nabla(\bar{\epsilon_a},\epsilon_b)\epsilon_c,\bar{\epsilon_c})\epsilon_a\bar{\epsilon_b}. \tag{4.3}$$

Here we have used the Bianchi identity and the curvature properties on Kähler foliation. Similarily, we have

$$\bar{\mathcal{R}} = \sum g_Q(R_\nabla(\epsilon_a,\bar{\epsilon_b})\bar{\epsilon_c},\epsilon_c)\bar{\epsilon_a}\epsilon_b. \tag{4.4}$$

Therefore we have

$$\mathcal{R}^E = \mathcal{R} + \bar{\mathcal{R}} = \sum g_Q(R_\nabla(\epsilon_a,\bar{\epsilon_b})\epsilon_b,\bar{\epsilon_a}) = \frac{1}{8}\sigma_\nabla, \tag{4.5}$$

where σ_∇ is the scalar curvature on Q. Thus we have

THEROEM 4.1. *Let $(M, g_M, \mathcal{F})$ be an isoparametric Kähler spin foliation. Then on the foliated spinor bundle S, we have*

$$2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}) = \frac{1}{2}\nabla_T^*\nabla_T + \frac{1}{8}\sigma_\nabla + \mathcal{K}.$$

By means of Theorem 3.5 and Theorem 4.1, we have

THEOREM 4.2. *Let $(M, g_M, \mathcal{F})$ be an isoparametric Kähler spin foliation. Suppose that the mean curvature k satisfies $\delta k = 0$. If $\sigma_\nabla \geqslant 0$ and > 0 at some point, then every $s \in Ker\ \mathcal{D}_b \cap Ker\ \bar{\mathcal{D}}_b$ vanishes, and if $\sigma_\nabla \geqslant 0$, then every $s \in Ker\ \mathcal{D}_b \cap ker\ \bar{\mathcal{D}}_b$ is parallel.*

REMARK. *To understand $\mathcal{D}$ and $\bar{\mathcal{D}}$ in (3.5), we now introduce the transversal Dirac operator $\mathcal{D}_{tr}$ on $\Gamma(E)$:*

$$\mathcal{D}_{tr} = \sum \{E_a \nabla_{E_a} + (JE_a)\nabla_{JE_a}\} - \frac{1}{2}k.$$

Then they are related as follows ([6]) :

$$\mathcal{D}_{tr} = 2(\mathcal{D} + \bar{\mathcal{D}}).$$

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