Comm. Korean Math. Soc. 11 (1996), No. 3, pp. 757–766

ON REAL HYPERSURFACES OF A COMPLEX SPACE FORM IN TERMS OF THE RICCI TENSOR

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ABSTRACT. The purpose of this paper is to study a real hypersurface of $M_n(c)$ where structure vector ξ is principal and satisfying $\nabla_\xi S = (\nabla S)\xi$ (section 2) and also satisfying $\nabla_\xi S = a(S\phi - \phi S)$ (section 3) where a is constant.

0. Introduction

A complex n-dimensional Kählerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denote by $M_n(c)$. A complete and simply connected complex space form is a complex projective space P_nC, a complex Euclidean space E_nC or a complex hyperbolic space H_nC, according as $c > 0$, $c = 0$ or $c < 0$. Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kählerian metric and complex structure J of $M_n(c)$. We denote by ∇, A, and S, the Levi-Civita connection with respect to g, the shape operator, and the Ricci tensor of type (1,1) on M respectively. R. Takagi [13] classified homogeneous real hypersurfaces of P_nC as six model spaces of type A_1, A_2, B, C, D, and E. T.E. Cecil and P.J. Ryan [2] extensively investigated real hypersurfaces of a complex projective space P_nC on which $\xi = -JN$ is principal curvature vector, where N is a local unit normal vector field. By making use of this notion and R. Takagi's classification, M. Kimura [8] proved the following.

Received July 3, 1995. Revised November 18, 1995.
1991 AMS Subject Classification: 53C15.
Key words and phrases: structure vector, principal, real hypersurface.
The present studies were supported by the Chosun University Research funds, 1994.

THEOREM A. *Let M be a connected real hypersurface of P_nC. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the followings;*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $P_{n-1}C$, where $0 < r < \frac{\pi}{2}$),*
(A_2) *a tube of radius r over a totally geodesic $P_kC(1 \le k \le n-2)$, where $0 < r < \frac{\pi}{2}$,*
(B) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,*
(C) *a tube of radius r over $P_1C \times P_{\frac{n-1}{2}}C$, where $0 < r < \frac{\pi}{4}$ and $n(\ge 5)$ is odd,*
(D) *a tube of radius r over a complex Grassmann $G_{2,5}(C)$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*
(E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

We note that the number of distinct principal curvatures of the above homogeneous real hypersurface is $2, 3, 5$ and that the structure vector field ξ is a principal curvature vector with principal curvature $\alpha = 2\cot 2r$ (for more details, see [14]).

On the other hand, real hypersurfaces of a complex hyperbolic space H_nC have also been investigated by J. Bernt [1], S. Montiel [10] and A. Romero [11] and so on. J. Bernt [1] classified real hypersurfaces with constant principal curvatures of H_nC under the condition that ξ is principal curvature vector. Namely he proved the following.

THEOREM B. *Let M be a connected real hypersurface of H_nC. Then M has constant principal curvatures and ξ is principal curvature vector if and only if M is locally congruent to one of the followings;*
(A_0) *a horosphere in H_nC,*
(A_1) *a tube over a complex hyperbolic hyperplane $P_{n-1}C$,*
(A_2) *a tube over a totally geodesic $P_kC(1 \le k \le n-2)$,*
(B) *a tube over a totally real hyperbolic space H_nR.*

The principal curvatures and their multiplicities of the above hypersurfaces are also given in [1]. U-H. Ki [4] proved

THEOREM C. *There does not exist a real hypersurface with the parallel Ricci tensor of complex space form $M_n(c), c \ne 0, n \ge 3$.*

From the above Theorem C, many authors investigated a real hypersurfaces of $M_n(c)$ under weaker conditions than the parallel Ricci tensor. Y.J. Suh [12] determined a real hypersurface of $M_n(c)$ whose structure vector ξ is principal and satisfying $g((\nabla_X S)Y, Z) = 0$, where X, Y, Z are vector fields which are orthogonal to ξ. S. Maeda [9] classified a real hypersurface of P_nC whose structure vector ξ is principal and satisfying $\nabla_\xi S = 0$. Recently, in [3] J.T. Cho and U-H. Ki investigated a real hypersurface of P_nC on which structure vector ξ is principal and the Ricci tensor is parallel with respect to a canonical connection. Also, in [5] it was proved that there does not exist a real hypersurface with harmonic Weyl tensor of complex space form, $c \neq 0, n \geq 3$.

In these circumstances, in the present paper, we investigate a real hypersurface of $M_n(c)$ whose structure vector ξ is principal and satisfying $\nabla_\xi S = (\nabla S)\xi$ (section 2) and in section 3, we study a real hypersurface of $M_n(c)$ whose structure vector ξ is principal and satisfying $\nabla_\xi S = a(S\phi - \phi S)$ where a is constant. All manifolds in this paper are assumed to be connected and of class C^∞.

The author would like to express his thanks to Professor U-H. Ki and Dr. J.T. Cho for their valuable suggestions and advices during the preparation of this paper.

1. Preliminaries

Let M be an orientable real hypersurface of $M_n(c)$ and N be a unit normal vector field on M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection in $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $M_n(c)$. An eigenvector(resp. eigenvalue) of the shape operator A is called a *principal curvature vector*(resp. *principal curvature*). For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi. \tag{1.1}$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$
(1.2) \qquad \begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y). \end{aligned}
$$

From (1.2), we get

$$
(1.3) \qquad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi).
$$

From the $\widetilde{\nabla} J = 0$ and (1.1), making use of Gauss and Weingarten formulas, we have

$$
(1.4) \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi
$$

$$
(1.5) \qquad \nabla_X \xi = \phi AX.
$$

Since the ambient space is of constant holomorphic sectional curvature c, we have the following equations of Gauss and Codazzi :

$$
(1.6) \qquad \begin{aligned} R(X, Y)Z =& \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned}
$$

$$
(1.7) \qquad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}
$$

Using (1.2), (1.3), (1.5) and (1.6), we get

$$
(1.8) \qquad SX = \frac{c}{4}\{(2n + 1)X - 3\eta(X)\xi\} + hAX - A^2 X,
$$

and further

$$
(1.9) \qquad \begin{aligned} (\nabla_X S)Y = & -\frac{3c}{4}\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} + dh(X)AY \\ & + h(\nabla_X A)Y - A(\nabla_X A)Y - (\nabla_X A)AY, \end{aligned}
$$

where $h = traceA$ and d denotes the exterior derivative.

Now we assume that the structure vector ξ is principal with corresponding principal curvature α, i.e., $A\xi = \alpha\xi$. Then it is seen in [7] and [8] that α is constant. Differentiating $A\xi = \alpha\xi$ and using (1.5), we obtain

$$(\nabla_X A)\xi = \alpha\phi AX - A\phi AX.$$

This equation and the equation (1.7) of Codazzi give rise to

$$2A\phi A = \frac{c}{2}\phi + \alpha(A\phi + \phi A). \tag{1.10}$$

Comparing (1.10) with the above equation, we have

$$(\nabla_X A)\xi = -\frac{c}{4}\phi X - \frac{\alpha}{2}(A\phi - \phi A)X. \tag{1.11}$$

By (1.7) and (1.11) we get

$$(\nabla_\xi A)X = -\frac{\alpha}{2}(A\phi - \phi A)X. \tag{1.12}$$

which implies

$$dh(\xi) = 0. \tag{1.13}$$

If X is a principal vector with corresponding principal curvature λ, then (1.10) gives us to

$$(2\lambda - \alpha)A\phi X = (\frac{c}{2} + \alpha\lambda)XY. \tag{1.14}$$

2. Real hypersurfaces satisfying $\nabla_\xi S = (\nabla S)\xi$

In this section, we prove

THEOREM 2.1. *There does not exist a real hypersurface of $M_n(c)$, $c \neq 0$ which satisfies $A\xi = \alpha\xi$ and $\nabla_\xi S = (\nabla S)\xi$,* PROOF. Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Assume that $A\xi = \alpha\xi$. Then from (1.9)and (1.13) we have

$$
\begin{aligned}
&(\nabla_\xi S)Y - (\nabla_Y S)\xi \\
&= \frac{3c}{4}\phi AY - \alpha dh(Y)\xi + (hI - A)\{(\nabla_\xi A)Y - (\nabla_Y A)\xi\} \\
&\quad - (\nabla_\xi A)AY + \alpha(\nabla_Y A)\xi.
\end{aligned} \tag{2.1}
$$

The equation (2.1) and the hypothesis, together with (1.7),(1.11) and (1.12), yield

$$
\begin{aligned}
&\frac{3c}{4}\phi AY - \alpha dh(Y)\xi + \frac{c}{4}(hI - A)\phi Y \\
&\quad + \frac{\alpha}{2}(A\phi - \phi A)AY + \alpha(\alpha\phi AY - A\phi AY) = 0.
\end{aligned} \tag{2.2}
$$

If we multiply (2.2) by ξ and use (1.3), We see that $\alpha dh(Y) = 0$ for any vector field Y on M. Thus from (1.10) and (2.2), we have

$$
3(c+\alpha^2)\phi AY - (c+\alpha^2)A\phi Y - 2\alpha\phi A^2 Y + c(h - \frac{\alpha}{2})\phi Y = 0. \tag{2.3}
$$

Let Y be any principal vector orthogonal to ξ and put $AY = \lambda Y$. Then we see that $2\lambda \neq \alpha$. In fact, suppose that there exists a point p of M such that $2\lambda(p) \neq \alpha$. Then it follows from (1.14) that $c + \alpha^2 = 0$, which together with (2.3) yields

$$
\alpha^2 h(p) = 0. \tag{2.4}
$$

If $\alpha = 0$, then $c = 0$ and this contradicts to $c \neq 0$. Therefore $\alpha \neq 0$ and $h(p) = 0$. Since $dh(X) = 0$ for any vector field X on M, then $h = o$ on M. Because $\lambda = \frac{\alpha}{2}$, we have $h = 2(n-1)\frac{\alpha}{2} + \alpha = n\alpha$ and hence this also contradicts.

Then (1.14) gives

$$
A\phi Y = \mu\phi Y, \quad \text{where} \quad \mu = \frac{\alpha\lambda + \frac{c}{2}}{2\lambda - \alpha}.
$$

Thus from (2.3), we obtain

$$2\alpha\lambda^2 - 3(c+\alpha^2)\lambda + (c+\alpha^2)\mu - c(h - \frac{\alpha}{2}) = 0. \tag{2.5}$$

If we substitute ϕY instead of Y into (2.3), we also have

$$2\alpha\mu^2 - 3(c+\alpha^2)\mu + (c+\alpha^2)\lambda - c(h - \frac{\alpha}{2}) = 0. \tag{2.6}$$

From (2.5) and (2.6), we get

$$\alpha(\lambda^2 - \mu^2) - 2(c+\alpha^2)(\lambda - \mu) = 0. \tag{2.7}$$

Now suppose there exists a point $p \in M$ such that $\lambda(p) = \mu(p)$. Then (2.5) gives

$$2\alpha\lambda^2(p) - 2(\alpha^2 + c)\lambda(p) - c(h(p) - \frac{\alpha}{2}) = 0. \tag{2.8}$$

On the other hand, (1.14) gives

$$\lambda^2(p) - \alpha\lambda(p) - \frac{c}{4} = 0. \tag{2.9}$$

The equations (2.8) and (2.9) yield

$$h(p) = \alpha - 2\lambda(p). \tag{2.10}$$

Let $e_1 = \xi, e_2, ..., e_{2n-1}$ be principal vectors at p and put $Ae_i = \lambda_i(p)e_i$, $\lambda_1(p) = \alpha, i = 1, 2, ..., 2n-1$. Then $h(p) = \alpha + \sum_{i=2} \lambda_i(p)$. Since it follows from (2.10) that $h(p) = \alpha - 2\lambda_i(p)$ for $i \geq 2$, then we see $\lambda_i(p) = 0$ for $i \geq 2$, that is, $\lambda(p) = 0$ in (2.10).

Thus (2.9) gives rise to $c = 0$ and this contradicts. Since $\lambda \neq \mu$ on M, from (2.7) we get

$$\alpha(\lambda + \mu) = 2(\alpha^2 + c), \tag{2.11}$$

which shows that $\alpha \neq 0$ on M. Moreover from (1.14) and (2.11) we have

$$\lambda\mu = \alpha^2 + \frac{5}{4}c. \tag{2.12}$$

Hence we see from (2.11) and (2.12) that λ and μ are two distinct solutions of

$$t^2 - \frac{2(c+\alpha^2)}{\alpha}t + (\alpha^2 + \frac{5}{4}c) = 0$$

and α is not a solution of it. This implies that there exist three distinct constant curvatures α, λ and μ.

Also, from (2.5) and (2.6) we obtain

$$c(h - 2\alpha) = \alpha(\lambda^2 + \mu^2) - (\alpha^2 + c)(\lambda + \mu). \tag{2.13}$$

From (2.13), taking account of (2.11) and (2.12), we obtain $h = 2\frac{c+\alpha^2}{\alpha}$. But, from (2.11) and (2.12), we also see that $h = \alpha + 2(n-1)\frac{c+\alpha^2}{\alpha}$. Therefore we have

$$\alpha^2 = -\frac{2n-4}{2n-3}c, \tag{2.14}$$

Since $\alpha \neq 0$, then $n \neq 2$. Therefore (2.14) shows that $c < 0$. But according to J.Bernt's work [1], we see that $\lambda\mu = 1$, and from (2.12) that $\alpha^2 = 1 - \frac{5}{4}c$, which together with (2.14) yield $c > 0$. That is impossible. At last we have proved Theorem1. (Q.E.D)

3. Real hypersurfaces satisfying $\nabla_\xi S = a(S\phi - \phi S)$

In the present section, we determine real hypersurfaces of $M_n(c)$, $c \neq 0$, which satisfy $A\xi = \alpha\xi$ and $\nabla_\xi S = a(S\phi - \phi S)$ (a : constant).

Assume that $A\xi = \alpha\xi$. Then from (1.9) and (1.12) we get

$$(\nabla_\xi S)Y = -\frac{\alpha}{2}h(A\phi - \phi A)Y + \frac{\alpha}{2}(A^2\phi - \phi A^2)Y, \tag{3.1}$$

for any vector field Y on M. On the other hand, from (1.8), we get

$$(S\phi - \phi S)Y = h(A\phi - \phi A)Y - (A^2\phi - \phi A^2)Y. \tag{3.2}$$

Therefore from (3.1) and (3.2), taking account of Theorem 3.1 in [6], we have

THEOREM 3.1. *Let M be a real hypersurface of $P_n(c)$, $n \geq 3$. Suppose that M satisfies $A\xi = \alpha\xi$ and $\nabla_\xi S = a(S\phi - \phi S)$ ($a \neq -\frac{\alpha}{2}$, constant). If $\alpha \neq 0$ and the multiplicities of principal curvatures except α are not equal to 1, then M is locally congruent to a tube of radius r over one of the following Kählerian submanifolds:*

(A_1) *a hyperplane $P_{n-1}C$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{8}$,*
(A_2) *a totally geodesic $P_kC(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{8}$,*
(B) *a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = n-2$ and $n \neq 3$,*
(C) *$P_1C \times P_{\frac{n-1}{2}}C$, where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = \frac{1}{n-2}$ and $n(\geq 5)$ is odd,*
(D) *a complex Grassmann $G_{2,5}(C)$, where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = \frac{3}{5}$ and $n = 9$,*
(E) *a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = \frac{5}{9}$ and $n = 15$.*

THEOREM 3.2. *Let M be a real hypersurface of H_nC, $n \geq 3$. Suppose that M satisfies $A\xi = \alpha\xi$ and $\nabla_\xi S = a(S\phi - \phi S)$ ($a \neq -\frac{\alpha}{2}$, constant). If $\alpha \neq 0$. then M is locally congruent to one of the types (A_0), (A_1) or (A_2) in Theorem B.*

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