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ON GENERIC SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE

SEONG-BAEK LEE, SEUNG-GOOK HAN,
NAM-GIL KIM AND SEONG SOO AHN

ABSTRACT. The purpose of this paper is to compute the covariant derivative of a shape operator of a generic submanifold of a complex space form without using the Green-Stoke's theorem. In particular, we classify complete generic submanifolds of a complex number space C^m with parallel mean curvature vector satisfying a certain condition.

Introduction

One of typical natural submanifolds of a Kaehler manifold is the so-called generic submanifolds that are defined as follows : Let M be a submanifold of a Kaehler manifold $\tilde{M}$ with complex structure J. If each normal space is mapped into the tangent space under the action of J, M is called a generic submanifold of $\tilde{M}$. Real hypersurfaces of Riemannian manifolds are the most typical example of generic submanifolds. Compact submanifolds of Kaehler manifold have been studied by applying the Green-Stoke's theorem to compute the Simon's type (for example[8]).

In the present paper, we compute the covariant derivative of a shape operator of a generic submanifold of a complex space form without using the Green-Stoke's theorem. In particular, we classify complete generic submanifolds of a complex number space C^m with parallel mean curvature vector satisfying a certain condition.

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1. Generic submanifolds of a Kaehler manifold

Let $\tilde{M}$ be a real $2m$-dimensional Kaehler manifold with metric tensor $<,>$ and the complex structure J. Then, $J^2 = -I$ and $< JX, JY > = < X, Y >$, where I denotes the identity transformation of the tangent bundle and X and Y vector fields on $\tilde{M}$. Let $\tilde{\nabla}$ be the Riemannian connection compatible with $<,>$. Then, we get $\tilde{\nabla} J = 0$. Let M be an n-dimensional Riemannian manifold isomertrically immersed in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$. We then obtain the induced Levi-Civita connection on M. Then the equation of Gauss and Weingarten are respectively given by $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$, where h is the second fundamental form, A_ξ the shape operator associated to the normal vector field ξ satisfying $< h(X, Y)\xi > = < A_\xi X, Y >$ and D the connection in the normal bundle $T^\perp M$ of M. An n-dimensional submanifold M in a Kaehler manifold $\tilde{M}$ is called *generic* if $J(T_p^\perp M) \subset T_p M$ for each p in M, where $T_p M$ is the tangent space of M at p and $T_p^\perp M$ the normal space of M at p.

We now consider an n-dimensional generic submanifold M of a Kaehler manifold $\tilde{M}$. Let X be a vector field tangent to M and ξ a vector field normal to M. Then we may put

$$JX = pX - qX, \tag{1.1}$$

$$J\xi = t\xi, \tag{1.2}$$

where pX denotes the tangential part of JX, qX the normal part of JX and $t\xi$ a vector field defined by $< t\xi, X > = < qX, \xi >$. It follows from (1,1) and (1.2) that

$$p^2 = -I + tq, \quad qp = 0, \quad pt = 0, \quad qt = I \tag{1.3}$$

REMARK. Let M be a generic submanifold of a Kaehler manifold. We can easily find that $p^3 + p = 0$.

Differentiating (1.1) and (1.2) covariantly and making use of $\tilde{\nabla} J = 0$, we obtain

$$(\nabla_Y p)X = -A_{qX} Y + th(X, Y), \tag{1.4}$$

$$(1.5) \qquad (\nabla_Y q)X = h(Y, pX),$$

$$(1.6) \qquad (\nabla_Y t)\xi = -pA_\xi X,$$

$$(1.7) \qquad h(X, t\xi) = qA_\xi X,$$

where $(\nabla_Y p)X = \nabla_Y pX - p\nabla_Y X$, $(\nabla_Y q)X = D_Y qX - q\nabla_Y X$ and $(\nabla_Y t)\xi = \nabla_Y t\xi - tD_X\xi$ for all vector fields X and Y tangent to M and ξ normal to M.

We now assume that the ambient Kaehler manifold $\tilde{M}$ is a complex space form with constant holomorphic sectional curvature $4c$ and we shall denote it by $\tilde{M}(c)$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ is given by

$$\begin{aligned} < \tilde{R}(X,Y)Z, W > =& c\{< X, W >< Y, Z > - < Y, W >< X, Z > \\ &+ < JX, W >< JY, Z > - < JY, W >< JX, Z > \\ &-2 < JX, Y >< JZ, W >\}. \end{aligned}$$

It follows from (1.1) and (1.2) that the equations of Gauss, Codazzi and Ricci for M are respectively obtained

(1.8)

$$\begin{aligned} < R(X,Y)Z, W > = c\{&< X, W >< Y, Z > - < Y, W >< X, Z > \\ &+ < pX, W >< pY, Z > - < pY, W >< pX, Z > \\ &- 2 < pX, Y >< pZ, W >\} + < h(X, W), h(Y, Z) > \\ &- < h(X, Z), h(Y, W) >, \end{aligned}$$

(1.9)

$$\begin{aligned} (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) =& c\{- < pY, Z > qX \\ &+ < pX, Z > qY + 2 < pX, Y > qZ\}, \end{aligned}$$

$$(1.10) \qquad \begin{aligned} < R^\perp(X,Y)\xi, \eta > =& c\{< qX, \eta >< qY, \xi > \\ &- < qY, \eta >< qX, \xi >\} + < [A_\xi, A_\eta]X, Y >, \end{aligned}$$

where $\bar{\nabla}$ is the operator of covariant differentiation defined on the direct sum of the tangent bundle and cotangent bundle $TM \oplus T^{\perp}M$ given by $(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$, R and $R^{\perp}$ and the Riemann curvature tensor of M and that in the normal bundle respectively and $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$. Let H be the mean curvature vector field on M defined by $\frac{1}{n}\mathrm{Tr}h$, where $\mathrm{Tr}h$ means the trace of h.

2. Basic formulas

Let M be an n-dimensional generic submanifold of a real $2m$-dimensio nal complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature $4c$. A normal vector field ξ is said to be *parallel* if $D_X\xi = 0$ for any vector field X on M. We assume that the mean curvature vector field H is nonvanishing and parallel in the normal bundle. Let $\{e_1, e_2, \cdots, e_n, \xi_1, \xi_2, \cdots, \xi_{2m-n}\}$ be an orthonormal frame of $\tilde{M}(c)$ of M such that $e_1, e_2, \cdots, e_n$ are tangent to M and $\xi_1, \xi_2, \cdots, \xi_{2m-n}$ normal to M with $\xi_1 = H/||H||$. Let $'\Delta$ be the so-called restricted Laplacian operator (see[8] for detail). Let $A_{\xi_x} = A_x$. Throughout this paper the indices i, j and k run over the range $\{1, 2, \cdots, n\}$ and x, y, z, u belong to $\{1, 2, \cdots, 2_{m-n}\}$. Since ξ_1 is parallel, $'\Delta A_1$ is given by

(2.1)
$$('\Delta A_1)X = \sum_i [R(e_i, X), A_1]e_i + c\sum_i \nabla_{e_i}\{- < t\xi_1, e_i > pX + < t\xi_1, X > pe_i - 2 < pX, e_i > t\xi_1\}$$

By straightforward computation and making use of (1.4) - (1.7), we can obtain from (2.1)

$$(2.2)\quad \begin{aligned} <('\Delta A_1)X, Y> &= c(n+3) < A_1X, Y > -c(\mathrm{Tr}A_1) < X, Y > \\ &+ (\mathrm{Tr}A_1) < A_1X, A_1Y > - \sum_x \mathrm{Tr}(A_1A_x) < A_xX, Y > \\ &+ 3c(\mathrm{Tr}A_1) < t\xi_1, X >< t\xi_1, Y > -6c < A_1pX, pY > - \\ & c\sum_x\{3 < A_1X, t\xi_x >< Y, t\xi_x > + < A_xY, t\xi_x >< X, t\xi_1 > \\ &+ 2 < A_xX, t\xi_1 >< Y, t\xi_x > - < A_xY, t\xi_1 >< X, t\xi_x > \\ &+ < A_xX, t\xi_x >< Y, t\xi_1 >\}. \end{aligned}$$

We now define the property(*):

$$(*) \qquad A_\eta p = pA_\eta$$

for any vector field η normal to M, which is equivalent to $h(pX, Y) + h(X, pY) = 0$ for any vector fields X and Y on M.

Applying p to (*) and making use of (1.3), we get

$$A_\eta X = -pA_\eta pX + tqA_\eta X \tag{2.3}$$

for any vector field X tangent to M. If we put $X = t\zeta$ for some vector field ζ normal to M, then

$$A_\eta t\zeta = th(t\zeta, t\eta) \tag{2.4}$$

because of (1.3). Let $\{\xi_1, \xi_2, \cdots, \xi_{2m-n}\}$ be an orthonormal normal vectors at a point p of M. Then we may set (2.4) as

$$A_\eta t\zeta = \sum_x Q(\xi_x, \zeta, \eta) t\xi_x, \tag{2.5}$$

where $Q(\xi_x, \zeta, \eta) =< h(t\zeta, t\eta), \xi_x >$. If we put $Q_{xyz} = Q(\xi_x, \xi_y, \xi_z)$, then we can easily see that Q_{xyz} is symmetric with respect to x, y and z by means of (1.7). We now assume that the mean curvature vector field H is nonvanishing and parallel in the normal bundle and ξ_1 is chosen as $H/||H||$. We extend $\xi_1, \xi_2, \cdots, \xi_{2m-n}$ to differentiable orthonormal normal vector fields defined on a normal neighborhood O of p by parallel translation with respect to normal connection along geodesics in M and we denote them by the same notation as $\xi_1, \xi_2, \cdots, \xi_{2m-n}$. Then we have $(D_X\xi_X)(p) = 0$. From (2.5) we get

$$A_1 t\xi_y = \sum_x Q(\xi_x, \xi_y, \xi_1) t\xi_x. \tag{2.6}$$

It gives that

$$< h(X, t\xi_y), \xi_1 >= \sum_x Q(\xi_x, \xi_y, \xi_1) < t\xi_x, X >$$

for any vector field X tangent to M. Differentiating this equation covariantly and making use of (1.6) and $(*)$, we find

$$<(\tilde{\nabla}_Y h)(X, t\xi_y), \xi_1> - <h(X, pA_yY), \xi_1>$$
$$= \sum_x (YQ(\xi_x, \xi_y, \xi_1)) <t\xi_1, X> - \sum_x Q(\xi_x, \xi_y, \xi_1) <pA_xY, X> \text{ at } p.$$

By means of the equation of Codazzi, we get

$$\begin{aligned} 2c <pY, X> \delta_{y1} + 2\sum_x Q(\xi_x, \xi_y, \xi_1) <pA_x, X> \\ - <p(A_1A_y + A_yA_1)Y, X> \\ = \sum_x (YQ(\xi_x, \xi_y, \xi_1)) <t\xi_x, X> \\ - \sum_x (XQ(\xi_x, \xi_y, \xi_1) <t\xi_x, Y> \end{aligned} \tag{2.7}$$

If we put $X = t\xi_z$ and use (1.3), then we see that the right hand side vanishes. On the other hand, the fact that ξ_1 is parallel implies

$$(A_1A_y - A_yA_1)X = c\{<qX, \xi_1> t\xi_y - <qX, \xi_y> t\xi_1\} \tag{2.8}$$

for any vector field X on M. Considering (2.8), (2.7) yields

$$A_yA_1pY = c\delta_{y1}pY + \sum_x Q(\xi_x, \xi_y, \xi_1)A_xpY$$

for every vector field Y on M Applying p to the last equation and using (1.3), we obtain

(2.9)

$$A_yA_1Y = c\delta_{y1}(I - tq)Y + \sum_x Q(\xi_x, \xi_y, \xi_1)A_xY$$
$$- \sum_x \sum_z \{Q(\xi_x, \xi_y, \xi_1)Q(\xi_z, qY, \xi_x) - Q(\xi_x, qY, \xi_1)Q(\xi_z, \xi_x, \xi_y)\}t\xi_z.$$

Combining (2.8) and (2.9) and making use of the fact that $Q(\xi_x, \xi_y, \xi_z)$ is symmetric with respect to x, y and z, we have

$$\begin{aligned} \sum_x \{Q(\xi_x, \xi_w, \xi_1)Q(\xi_z, \xi_x, \xi_y) - Q(\xi_x, \xi_w, \xi_y)Q(\xi_z, \xi_x, \xi_1)\} \\ = c(-\delta_{yz}\delta_{1w} + \delta_{yw}\delta_{1z}). \end{aligned} \tag{2.10}$$

Together with (2.10), (2.9) implies that

$$A_1^2 = \sum_x Q(\xi_x, \xi_1, \xi_1)A_x + c(I - tq). \tag{2.11}$$

Putting (2.2) and (2.11) together and taking account of (2.5) and (2.6), we obtain

(2.12)

$$\begin{aligned} < ('\Delta A_1)X, Y >= c\{-2 < A_1X, Y > +2(\mathrm{Tr}A_1) < t\xi_1, X >< t\xi_1, Y > \\ +2\sum_x\sum_y Q(\xi_x, \xi_y, \xi_1) < t\xi_x, X >< t\xi_y, Y > \\ -\sum_x\sum_y Q(\xi_x, \xi_y, \xi_y)\{< t\xi_x, X >< t\xi_1, Y > + < t\xi_1, X >< t\xi_x, Y >\} \end{aligned}$$

for any vector fields X and Y on M. Putting $X = t\xi_1$ and $Y = t\xi_1$, we have

$$< ('\Delta A_1)t\xi_1, t\xi_1 >= 2c\{\mathrm{Tr}A_1 - \sum_x Q(\xi_x, \xi_x, \xi_1)\}. \tag{2.13}$$

If we denote $Q(\xi_1, \xi_1, \xi_1)$ by Q, then

$$(XQ)(p) =< (\nabla_X A_1)t\xi_1, t\xi_1 > (p) \tag{2.14}$$

for any vector field X on M because of (1.3) and (1.6). If we choose an orthonormal frame $\{e_1, \cdots, e_n\}$ satisfying $(\nabla_{e_i} e_j)(p) = 0$. Then we have

(2.15)

$$(\Delta Q)(p) =< ('\Delta A_1)t\xi_1, t\xi_1 > (p) - 2\sum_i < (\Delta_{e_i} A_1)t\xi_1, pA_1e_i > (p)$$

since A_1 is symmetric, where Δ denotes the Laplacian operator defined by $\sum_i \nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}} e_i$. Using the another form of equation of Codazzi, that is,

$$\begin{aligned} &(\nabla_X A_\eta)Y - (\nabla_Y A_\eta)X + A_{D_X\eta}Y - A_{D_Y\eta}X \\ &= \text{the tangential part of } \tilde{R}(X, Y)\eta \end{aligned}$$

for all vector fields X and Y on M and a normal vector field η on M, we can reduce (2.15) to

$$(\Delta Q)(p) =< ('\Delta A_1)t\xi_1, t\xi_1 > (p) - 2c\sum_i < pe_i, pA_1e_i > (p)$$

with the help of (1.3) and (1.4). Taking account of (1.3) and (2.13), we obtain

$$(\Delta Q)(p) = 0.$$

This equation holds for every point p in M. Thus Q is a harmonic function on M. It follows that $\mathrm{Tr}A_1^2$ is also harmonic.

We now define a tensor T by

$$T(X,Y) = (\nabla_X A_1)Y - c\{< pX, Y > t\xi_1 + < qY, \xi_1 > pX\}.$$

We then have

$$\|T\|^2(p) = \|\nabla A_1\|^2(p) - 4c^2(n-m) \tag{2.16}$$

because of (1.3). Putting $X = e_i$ and $Y = A_1e_i$ in (2.12) and summing up together, we obtain

$$\sum_{i=1} < ('\Delta A_1)e_i, A_1e_i > (p) = 4c^2(m-n) \tag{2.17}$$

with the help of (2.11). Putting (2.16) and (2.17) together, we have

$$\|T\|^2(p) = 0$$

for every point of p in M since $\frac{1}{2}\Delta\mathrm{Tr}A_1^2 =<' \Delta A_1, A_1 > + \|\Delta A_1\|^2$, that is,

$$(\nabla_X A_1)Y = c\{< pX, Y > t\xi_1 + < qY, \xi_1 > pX\} \tag{2.18}$$

for all vector fields X and Y on M.

NOTE. By considering (2.8), we see that Q and $\mathrm{Tr}A_1^2$ are constant along M.

PROPOSITION 2.1. *Let M be an n-dimensional generic submanifold of a complex space form $\tilde{M}(c)$ with nonvanishing parallel mean curvature vector H. If (*) is satisfied on M, then all the principal curvatures of H are constant.*

PROOF. If we defined a function $h_k = \mathrm{Tr}(A_H^k)$ for any integer $k \geq 1$, then h_k is constant by considering (1.3), (2.5) and (2.18). Thus, every principal curvature of H is constant. (Q. E. D.)

3. Generic submanifolds of a complex number space

In this section we assume that a generic submanifold M of a complex number space C^m satisfies the condition $(*)$ in section 2 and the mean curvature vector field H is nonvanishing and parallel in the normal bundle. Then we have from (2.18)

PROPOSITION 3.1. *Let M be an n-dimensional generic submanifold of a complex number space C^m with nonvanishing parallel mean curvature vector H. If $(*)$ is satisfied on M, then the weingarten map A_H associated to H is parallel.*

We now prove

THEOREM 3.2. *Let M be an n-dimensional generic submanifold of a complex number space C^m with nonvanishing parallel mean curvature vector H. If $(*)$ is satisfied on M, then M is either a minimal submanifold or a product submanifold $M_1 \times M_2 \times \cdots \times M_a$, where M_t $(t = 1, 2, \cdots, a)$ is a n_t-dimensional submanifold imbedded in C^{m_t} and M_t is contained in a hypersphere in C^{m_t}.*

PROOF. If $H = 0$, then M is minimal. Suppose that $H \neq 0$. By Proposition 3.1, A_H is parallel. According to Proposition 2.1, we see that every principal curvature of H is constant along M. Let $c_1, c_2, \cdots, c_a$ be mutually distinct principal curvatures of H and let $n_1, n_2, \cdots, n_a$ be their multiplicities. Since A_H is parallel, the distribution D_t defined by c_t is parallel and hence M is a product of submanifolds $M_1 \times M_2 \times \ldots \times M_a$ by de Rham decomposition Theorem, where M_t is the integral submanifold of D_t for each $t = 1, 2, \cdots, a$. Moreover, $A_x D_t \subset D_t$ for each x and t

since $[A_1, A_x] = 0$. The theorem of Moore [7] gives that $M = M_1 \times M_2 \times \cdots \times M_a$ is a product submanifold imbedded in $C^m = C^{m_1} \times C^{m_2} \times \cdots \times C^{m_a}$, $m_1 + m_2 + \cdots + m_a = m$. Let $\pi_t(H)$ be the component of H in the subspace C^{m_t}. Then $\pi_t(H)$ is the parallel mean curvature vector of M_t in C^{m_t} and it is a umbilical section of M_t. Thus, M_t lies in a hypersphere in C^{m_t} which is orthogonal to $\pi_t(H)$. Futhermore. M_t is minimal in the sphere. (Q. E. D.)

REMARK. ([5]) Let M be an n-dimensional complete generic submanifold of complex number space C^m with flat normal connection and parallel mean curvatrue vector. If $(*)$ is satisfied on M, then M is a product of spheres.

4. Submersions and immersions

Let $\tilde{\pi} : S^{2m+1} \rightarrow CP^m$ be the Riemannian submersion defined by the Hopf-fibration, where S^{2m+1} is the unit hypersphere and CP^m the complex projective space with constant holomorphic sectional curvature 4. Then we get

$$\hat{\nabla}_{X^*} Y^* = (\tilde{\nabla}_X Y)^* + < JX, Y > V, \tag{4.1}$$

$$\hat{\nabla}_{X^*} V = \hat{\nabla}_V X^* = -(JX)^*, \tag{4.2}$$

where V is the unit vertical vector field whose integral curves are great circles S^1 of S^{2m+1}, X^* denotes the horizontal lift of X on CP^m and $\hat{\nabla}$ the metric connection on S^{2m+1}. Then $< X, Y > (q) =< X^*, Y^* > (\bar{q})$, where $\tilde{\pi}\bar{q}$ =q. Let M be a generic submanifold of complex projective space CP^m. Denote by $\bar{M} = \tilde{\pi}^{-1}(M)$, the inverse image of M. Then $\bar{M}$ is a principal circle bundle over M with totally geodesic fibres. From this fact we have the following commutative diagram :

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\ \bar{i}\ } & S^{2m+1} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ M & \xrightarrow{\ i\ } & CP^m, \end{array}$$

where i and $\tilde{i}$ denote the immersions and $\pi = \tilde{\pi}|_{\bar{M}}$. Since V spans the vertical subspace of the submersion $\tilde{\pi} : S^{2m+1} \to CP^m$, we have the orthogonal decomposition

$$T_{\bar{p}}\bar{M} = (T_{\pi(\bar{p})}M)^* \oplus \quad \text{Span} \quad \{V\}. \tag{4.3}$$

By the Gauss equation and (1.1) we find

$$\hat{\nabla}_{X^*}Y^* = (\nabla_X Y)^* + h(X,Y)^* + < pX, Y > V \tag{4.4}$$

for X, Y tangent to M. Let ξ be a normal vector field of M in CP^m, (4.1) gives

$$\hat{\nabla}_{X^*}\xi^* = (\tilde{\nabla}_X \xi)^* + < JX, \xi > V,$$

which together with the Weingarten equation and (1.1) implies

$$\tilde{A}_{\xi^*}X^* = (A_\xi X)^* + < qX, \xi > V, \tag{4.5}$$

$$\hat{D}_X \xi^* = (D_X \xi)^* \tag{4.6}$$

where $\tilde{A}$ denotes the Weingarten map of $\bar{M}$ associated with ξ^* and $\hat{D}$ the normal connection defined in the normal bundle of $\bar{M}$. Let ∇' be the metric connection on $\bar{M}$.

Then we have

$$\hat{\nabla}_{X^*}Y^* = \nabla'_{X^*}Y^* + \tilde{h}(X^*, Y^*), \tag{4.7}$$

where $\tilde{h}$ denotes the second fundamental form of $\bar{M}$ in S^{2m+1}. Combining (4.1) and (4.7), we obtain

$$\nabla'_{X^*}Y^* = (\nabla_X Y)^* + < pX, Y > V, \tag{4.8}$$

$$\tilde{h}(X^*, Y^*) = h(X,Y)^*. \tag{4.9}$$

From (1.1), (4.2) and (4.7), we get

$$\tilde{h}(X^*, V) = (qX)^*, \tag{4.10}$$

$$\nabla'_{X^*}V = \nabla'_V X^* = -(pX)^*. \tag{4.11}$$

Since V is the unit vector field tangent to the totally geodesic fibres, we have

$$\tilde{h}(V, V) = 0. \tag{4.12}$$

It is well-known (for example, see[5])

LEMMA 4.1. *Let M be a submanifold of CP^m. Then the mean curvature vector of M is parallel in the normal bundle if and only if that of $\tilde{M}$ is parallel in the normal bundle.*

LEMMA 4.2. *Let M be a generic submanifold of CP^m with non vanishing parallel mean curvature vector field H. If $h(X,pY)+H(pX,Y)=0$ is satisfied on M, then $\tilde{A}_1$ is parallel, where $\tilde{A}_1$ is the weingarten map associated with ξ_1^* and $\xi_1 = H/||H||$.*

PROOF. Let's comput $(\nabla'_{X^*}\tilde{A}_1)Y^*$:

$$(\nabla'_{X^*}\tilde{A}_1)Y^* = \nabla'_{X^*}\tilde{A}_1 Y^* - \tilde{A}_1\nabla'_{X^*}Y^*. \tag{4.13}$$

By (4.8) we see that

$$\begin{aligned}\tilde{A}_1\nabla'_{X^*}Y^* =&\tilde{A}_1\{(\nabla_X Y)^* + < pX, Y > V\}\\ =&(A_1\nabla_X Y)^* + < q\nabla_X Y, \xi_1 > V + < PX, Y > \tilde{A}_1 V\end{aligned} \tag{4.14}$$

because of (4.5). Differentiating $\tilde{A}_1 Y^* = (A_1 Y)^* + < qY, \xi_1 > V$ covariantly, we get

$$\nabla'_{X^*}\tilde{A}_1 Y^* = \nabla'_{X^*}(A_1 Y)^* + X < qY, \xi_1 > V + < qY, \xi_1 > \nabla'_{X^*}V,$$

or, using (4.8) and (4.11),

$$\begin{aligned}\nabla'_{X^*}\tilde{A}_1 Y^* =&(\nabla_X A_1 Y)V + < pX, A_1 Y > V\\ &+ X < qY, \xi_1 > V - < qY, \xi_1 > (pX)^*.\end{aligned} \tag{4.15}$$

Substituting (4.14) and (4.15) into (4.13) and using (1.5), we obtain

$$\begin{aligned}(\nabla'_{X^*}\tilde{A}_1)Y^* =&((\nabla_X A_1)Y)^* + < pX, A_1 Y > V + < h(X,pY), \xi_1 > V\\ &- < qY, \xi_1 > (pX)^* - < pX, Y > \tilde{A}_1 V.\end{aligned}$$

We now suppose that $h(X,pY) + h(pX,Y) = 0$. Then the last equation can be reduced to

$$(\nabla'_{X^*}\tilde{A}_1)Y^* = ((\nabla_X A_1)Y)^* - < qY, \xi_1 > (pX)^* - < pX, Y > \tilde{A}_1 V. \tag{4.16}$$

On the other hand, we find by (4.12)

$$(4.17) \qquad <(\nabla'_{X^*}\tilde{A}_1)Y^*, V>=0.$$

We now consider :

$$\begin{aligned}<(\nabla'_{X^*}\tilde{A}_1)Y^*, Z^*> &= <(\nabla_X A_1)Y, Z> - <qY, \xi_1><pX, Z> \\ &\quad - <pX, Y><\tilde{A}_1 V, Z^*> \\ &= <(\nabla_X A_1)Y - <qY, \xi_1> pX \\ &\quad - <pX, Y> t\xi_1, Z> \quad (\text{By}(4.10)) \\ &= 0 \quad (\text{By } (2.18) \text{ with } c=1)\end{aligned}$$

and

$$\begin{aligned}&<(\nabla'_{Y^*}\tilde{A}_1)V, X^*> = Y^* <\tilde{A}_1 V, X> - <\tilde{A}_1 \nabla'_{Y^*} V, X^*> \\ &- <\tilde{A}_1 V, \nabla'_{Y^*} X^*> \\ &= Y^* <\tilde{h}(X^*, V), \xi_1^*> - <\tilde{h}(X^*, (pY)^*), \xi_1^*> - <\tilde{h}(V, \nabla'_{Y^*} X^*), \xi^*> \\ &= Y <qX, \xi_1> + <h(X, pY), \xi_1> + <q\nabla_Y X, \xi_1> \\ &= <h(Y, pX> + h(X, pY), \xi_1>=0.\end{aligned}$$

We also easily obtain $<(\nabla_V \tilde{A}_1)Y^*, X^*>=0$. Similarly, we can compute :

$$<(\nabla'_V \tilde{A}_1)Y^*, V>=0, \quad (\nabla'_V \tilde{A}_1)V=0.$$

Summing up these results, we find that $\tilde{A}_1$ is parallel.

(Q. E. D.)

By the same argument developed in the previous section, we see that the principal curvatures of ξ_1^* are constant and $\bar{M}$ is a product of submanifolds $\bar{M}_1 \times \bar{M}_2 \times \cdots \times \bar{M}_b$. Thus we have

THEOREM 4.3. *Let M be a generic submanifold of CP^m with non vanishing parallel mean curature vector field H. If $h(X, pY) + h(pX, Y) = 0$ is satisfied on M, then M is the projection of a product of submanifolds $\tilde{\pi}(\bar{M}_1 \times \bar{M}_2 \times \cdots \times \bar{M}_b)$, where $\tilde{\pi}$ is the natural projection defined by the Hopf -fibration $S^1 \to S^{2m+1} \to CP^m$.*

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Professors Seong-Baek Lee, Seung-Gook Han and Nam-Gil Kim
Chosun University
Kwangju 501-759, Korea

Professor Seong Soo Ahn
Dong Shin University
Naju, Korea 520-714