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SOME RESULTS ON METRIC FIXED POINT THEORY AND OPEN PROBLEMS

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ABSTRACT. In this paper we give some sharp expressions of the weakly convergent sequence coefficient $WCS(X)$ of a Banach space X. They are used to prove fixed point theorems for involution mappings T from a weakly compact convex subset C of a Banach space X with $WCS(X) > 1$ into itself which T^2 are both of asymptotically nonexpansive type and weakly asymptotically regular on C. We also show that if X satisfies the semi-Opial property, then every nonexpansive mapping $T : C \to C$ has a fixed point. Further, some questions for asymtotically nonexpansive mappings are raised.

1. Introduction

Let X be a Banach space and $C \subseteq X$. A mapping $T : C \to X$ is said to be *nonexpansive* if for each $x, y \in C$, $\|T(x) - T(y)\| \leq \|x - y\|$. It was the 1965 discovery of a fundamental fixed point theorem for the class of nonexpansive mappings that provided the foundations for much of the subsequent metric fixed point theory. The central result of [32] asserts that if X is reflexive, and if K is a bounded closed convex subset of X which possesses a geometrical property called ' normal structure' [6], then every nonexpansive $T : K \to K$ has a fixed point, a fact also proved (at the same time) by Browder [7] and Göhde [18] under the somewhat stronger assumption that X is uniformly convex. For another rich fixed point theory for mappings of this class, see Goebel-Kirk [17].

On the other hand, if T is merely assumed to be *k-lipschitzian*, that is, for some fixed $k \geq 0$, $\|T(x) - T(y)\| \leq k\|x - y\|$ for each $x, y \in C$, then

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no comparable theory exists if the Lipschitz constant $k > 1$. Indeed, it is known that for any $k > 1$ there exists a k-lipschitzian self-mapping of the unit ball B of the infinite dimensional Hilbert space ℓ_2 which has no fixed point (see [32]). It has been known for some time, however, that such mappings T will always have fixed points if k is sufficiently near 1 and if appropriate constraints are placed on the iterates of T. One of the first results of this type is due to Goebel [14] who proved that if K is a weakly compact convex subset of a *strictly convex* Banach space, and if K has normal structure, then a mapping $T : K \to K$ will always have a fixed point if T^2 is nonexpansive and if T is k-lipschitzian for $k < 2$. We know that it remains true without the strict convexity assumption (see [31]). Note that if $K = [0,1] \subseteq \mathbb{R}$ and if $T : K \to K$ is defined by

$$T(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{7}{16}; \\ 2(x - \frac{7}{16}) & \text{if } \frac{7}{16} \leq x \leq \frac{9}{16}; \\ \frac{1}{4} & \text{if } \frac{9}{16} \leq x \leq 1, \end{cases}$$

then $T : K \to K$ is not nonexpansive but $T^2 = 0$.

A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* [15] if for each $n \in \mathbb{N}$, there exists a real number $k(n)$ such that

$$\|T^n x - T^n y\| \leq k(n)\|x - y\| \qquad \text{for all } x, y \in C$$

and $\lim_{n\to\infty} k(n) = 1$, where $\mathbb{N}$ denotes the set of natural numbers. We say that a mapping $T : C \to C$ is said to be *asymptotically nonexpansive type* (simply, a.n.t.) on C [33] if, for each $x \in C$,

$$\limsup_{n\to\infty}\Big(\sup\big\{[\|T^n x - T^n y\| - \|x - y\|] : y \in C\big\}\Big) \leq 0.$$

In section 2 of this paper, we give some sharp expressions of the weakly convergent sequence coefficient $WCS(X)$ of a Banach sapce X. In section 3, using a characterization of $WCS(X)$ we present a fixed point theorem for an involution map T from a weakly compact convex subset C of a Banach space X with $WCS(X) > 1$ into itself which T^2 is both of asymptotically nonexpansive type and weakly asymptotically regular on C. Finally, in section 4, we show that if X satisfies the semi-Opial property, then every nonexpansive mapping $T : C \to C$ has a fixed point. Further, we give some questions for asymptotically nonexpansive mappings. For variant open questions for nonexpansive mappings, see also Kirk [34].

2. Geometrical coefficients of X

Let X be a non-Schur Banach space and $A \subset X$.

$$\begin{aligned}\operatorname{diam}(A) &= \sup_{x,y\in A} \|x-y\|,\\ r_A(A) &= \inf_{x\in A}\big(\sup_{y\in A}\|x-y\|\big) \quad \text{“the } \textit{self- Chebyshev radius} \text{ of } A\text{”}.\end{aligned}$$

For each $x \in X$ and each sequence $\{x_n\} \in X$, we set

$$\begin{aligned}r(x,\{x_n\}) &= \limsup_{n\to\infty}\|x_n - x\|\\ A(\{x_n\}) &= \lim_{n\to\infty}\big(\sup\{\|x_i - x_j\| : i,j \ge n\}\big)\\ &\qquad \text{“the } \textit{asymptotic diameter} \text{ of } \{x_n\}\text{”}\\ r(\{x_n\}) &= \inf\big\{r(x,\{x_n\}) : x \in \overline{co}(\{x_n\})\big\}\\ &\qquad \text{“the } \textit{Chebyshev radius} \text{ of } \{x_n\} \text{ relative to } \overline{co}(\{x_n\})\text{”}\\ D(\{x_n\}) &= \limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - x_m\|,\end{aligned}$$

where $\overline{co}(A)$ denotes the closed convex hull of A.

First, let us introduce some geometrical coefficients introduced by Bynum [9].

(1) “the *normal structure coefficient* of X”

$$N(X) = \inf\left\{\frac{\operatorname{diam}(A)}{r_A(A)} : A \subset X \text{ bounded closed convex with } \operatorname{diam}(A) > 0\right\}.$$

(2) “the *bounded sequence coefficient* of X”

$$BS(X) = \sup\Big\{M : \text{ for any bounded sequence } \{x_n\} \text{ in } X, \ \exists\ y \in \overline{co}(\{x_n\}) \text{ such that } M\cdot\limsup_{n\to\infty}\|x_n - y\| \le A(\{x_n\})\Big\}.$$

(3) "the *weakly convegent sequence coefficient* of X"

$$WCS(X) = \sup\Big\{M : \text{ for each weakly convergent sequence } \{x_n\}, \exists\ y \in \overline{co}(\{x_n\}) \text{ such that } M \cdot \sup_{n\to\infty} \|x_n - y\| \le A(\{x_n\})\Big\},$$

$$A(X) = \inf\Big\{\frac{A(\{x_n\})}{r(\{x_n\})} : \ \{x_n\} \text{ bounded nonconvergent sequence in } X\Big\}.$$

REMARK 2.1. (a) $N(X) = BS(X) = A(X)$ for any Banach space (see Lim [36]).

$$WCS(X) = \inf\Big\{\frac{A(\{x_n\})}{r(\{x_n\})} : \ \{x_n\} \text{ converges weakly (not strongly)}\Big\}$$

$$W(X) = \inf\Big\{\frac{A(\{x_n\})}{r_X(\{x_n\})} : \ \{x_n\} \text{ converges weakly (not strongly)}\Big\},$$

where $W(X)$ is a geometrical constant first introduced by Webb-Zhao [44] and $r_X(\{x_n\}) = \inf_{x\in X} r(x, \{x_n\})$.

(b) $1 \le N(X) = BS(X) \le WCS(X) \le W(X) \le 2$. Let $X = l_2$-direct sum of $\mathbb{R}^n_\infty$, $n \ge 1$, where $\mathbb{R}^n_\infty$ is the space $\mathbb{R}^n$ with the maximum norm. Then X is separable, reflexive and $1 = N(X) = BS(X) < WCS(X) = \sqrt{2}$ (see Baillon [3]).

We say that X has the *uniform normal structure* (UNS) if $N(X) > 1$ and it has the *weak uniform normal structure* (WUNS) if $WCS(X) > 1$.

(c) $[\ WCS(X) > 1 \Rightarrow\ (\text{WNS})\ ]$, where (WNS) means that any weakly compact subset of X has normal structure, i.e., any convex subset K of C containing more than one point must contain a point $z \in K$ which has the property:

$$\sup_{y\in K} \|z - y\| < \operatorname{diam}(K).$$

DEFINITION 2.1. (a) X with a Schauder basis $\{e_n\}$ has the *Gossez-Lami Dozo Property* (GLD) [19] if for each $c > 0$, $\exists\ r = r(c) > 0$ such that

$$(\text{GLD})\ [\ \forall x \in X,\ \forall n \in \mathbb{N},\ \|P_n(x)\| = 1, \text{ and } \|(I - P_n)(x)\| \ge c \Longrightarrow \|x\| \ge 1 + r\ ],$$

where $P_n(x) = \sum_{i=1}^{n} x_i e_i$ and $(I - P_n)(x) = x - P_n(x) = \sum_{i=n+1}^{\infty} x_i e_i$ for all $x = \sum_{i=1}^{\infty} x_i e_i \in X$.

(b) X has the *generalized Gossez-Lami Dozo property* (GGLD) [21] if for every weakly null sequence $\{x_n\}$ s.t. $\lim_{n\to\infty} \|x_n\| = 1$, we have $D(\{x_n\}) > 1$.

(c) X has the *Tingley property* (T) [43] if for every weakly null (and not constant) sequence $\{x_n\}$,

$$\text{(T)} \qquad \sup_{m\in\mathbb{N}} (\limsup_{n\to\infty} \|x_n - x_m\| > \liminf_{n\to\infty} \|x_n\|.$$

Note that the following implications hold:

$$\begin{array}{l} \text{(GLD)} \overset{[19]}{\Longrightarrow} \text{(WNS)} \overset{[32]}{\Longrightarrow} \text{(FPP:N)} \\ \quad\Downarrow \\ \text{(GGLD)} \overset{[21]}{\Longrightarrow} \text{(T)} \overset{[43]}{\Longrightarrow} \text{(WNS)} \overset{[32]}{\Longrightarrow} \text{(FPP:N)}, \end{array}$$

where (FPP:N) means that for every weakly compact convex subset C of X, every nonexpansive map $T : C \to C$ has a fixed point.

Recently, Zhang [47] established the following sharp expression of the weakly convergent sequence coefficient of X, $WCS(X)$.

$$(*) \quad WCS(X) = \sup\Big\{M : x_n \rightharpoonup u \Rightarrow M \cdot \limsup_{n\to\infty} \|x_n - u\| \le A(\{x_n\})\Big\},$$

where "$\rightharpoonup$" means the weak convergence. The idea of the proof in Zhang [47] is as follows: For each $x \in X$, define

$$r(x) = \limsup_{n\to\infty} \|x_n - x\|.$$

Then, the functional $r(x)$ is weakly lower semicontinuous and by using the separability of $\overline{co}\{x_n\}$ the property $(*)$ is easily obtained.

REMARK 2.2. (a) $D(\{x_n\}) \leq A(\{x_n\})$ and
(b) $D(\{x_n\}) \neq A(\{x_n\})$ in general.

Consider the James' quasi-reflexive space J consisting of all real sequences $x := \{x_n\} = \sum_{n=1}^{\infty} x_n e_n$ for which $\lim_{n\to\infty} x_n = 0$ and $\|x\|_J < \infty$, where

$$\|x\|_J = \sup\left\{ \left[(x_{p_1} - x_{p_2})^2 + \cdots + (x_{p_{m-1}} - x_{p_m})^2 + (x_{p_m} - x_{p_1})^2\right]^{\frac{1}{2}} \right\}$$

and the supremum is taken over all choices of m and $p_1 < p_2 < \cdots < p_m$. Then J is a Banach space with the norm $\|\cdot\|_J$ and the sequence $\{e_n\}$ given by $e_n = (0, ..., 0, 1, 0, ...)$ where the 1 is in the nth position, is a Schauder basis for J.

Take $x_n = e_n - e_{n+1}$ for each $n \in \mathbb{N}$. Then,
(i) $\|x_n\|_J = \sqrt{6}$, $x_n \in J$,
(ii) $D(\{x_n\}) = 2\sqrt{3} < A(\{x_n\}) = 2\sqrt{5}$.

LEMMA 2.1. *If* $z_n = y_n/\|y_n\|$, $\alpha = \lim_{n\to\infty} \|y_n\| \neq 0$, *then*

$$D(\{z_n\}) = \frac{1}{\alpha} D(\{y_n\}).$$

LEMMA 2.2. *Let* $M > 0$. *Then the following statements are equivalent:*

(a) $M \cdot \limsup_n \|x_n - x\| \leqslant A(\{x_n\})$ *for any* $x_n \rightharpoonup x$ *(not strongly convergent).*
(b) $M \cdot \limsup_n \|x'_n - x'\| \leqslant D(\{x'_n\})$ *for any* $x'_n \rightharpoonup x'$ *(not strongly convergent).*

As a direct consequence of Lemma 2.2 and $(*)$, we give some sharp expressions of $WCS(X)$ which improves the results due to Zhang [47].

THEOREM 2.1.

$$WCS(X) = \sup\Big\{M : x_n \rightharpoonup u \Rightarrow M \cdot \limsup_{n\to\infty} \|x_n - u\| \leqslant D(\{x_n\})\Big\}$$
$$= \inf\Big\{\{\frac{D(\{x_n\})}{r(u, \{x_n\})} :$$
$$\{x_n\} \text{ weakly (not strongly) converges tou}\Big\}$$
$$= \inf\Big\{D(\{x_n\}) : \{x_n\} \subset S(X) \text{ and } x_n \rightharpoonup 0\Big\},$$

where $S(X)$ *denotes the unit sphere of* X, *i.e.,* $S(X) = \{x \in X : \|x\| = 1\}$.

Jiménez-Melado [21] has defined a geometrical coefficient $\beta(X)$ for a Banach space X, i.e.,

$$\beta(X) := \inf\Big\{D(\{x_n\}) : x_n \rightharpoonup 0 \text{ and } \lim_{n\to\infty} \|x_n\| = 1\Big\}$$

and he has shown that if $\beta(X) > 1$ then X has property (T). As a direct consequence of Theorem 2.1, we obtain the following result due to Benavides-Acedo-Xu [5].

COROLLARY 2.1. $WCS(X) = \beta(X)$.

3. Fixed point theorems

Recall that the *modulus of convexity* of X is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}.$$

The *characteristic of convexity* of X is the number $\varepsilon_0(X) = \sup\{\varepsilon : \delta(\varepsilon) = 0\}$. It is easy to see that X is *uniformly convex* iff $\varepsilon_0(X) = 0$; *uniformly nonsquare* iff $\varepsilon_0(X) < 2$; and *strictly convex* iff $\delta(2) = 1$. We say that $T : H \to H$ satisfies *Goebel's Lipschitz condition* if

$$\|T(x) - T(y)\| \leq k\|x - y\|$$

for all $x, y \in H$, where k satisfies the condition

$$\frac{k}{2}(1 - \delta(\frac{2}{k})) < 1.$$

Note that this condition always holds if $k < 2$.

Recall that $F \subseteq K \subseteq X$, then F is said to be a *1-local retract* of K if every family $\{B_i : i \in I\}$ of closed balls centered at points of F has the property:

$$(\cap_{i \in I} B_i) \cap K \neq \emptyset \implies (\cap_{i \in I} B_i) \cap F \neq \emptyset.$$

This concept is due to Khamsi [26,27] who used it to prove the existence of common fixed points for commuting families of nonexpansive mappings in more general context. He proved in [27] that F is a 1-local retract of K if and only if F is a nonexpansive retract of $F \cup \{x\}$, for every $x \in K$, where $A \subseteq X$ means a *nonexpansive retract* of X if there exists a nonexpansive map $r : X \to A$ such that $r_A = I$. It is easy to see that a 1-local retract of a convex set is metrically convex, and a 1-local retract of a closed set must itself be closed. It is well-known that if F is a nonexpansive retract of K, then it is a 1-local retract of K but not conversely.

The following, which is less immediate, basically follows the argument of Goebel [14]. For more detail proof, see [31].

LEMMA 3.1. *Let H be a nonempty subset of a Banach space X, and suppose H is a 1-local retract of $\overline{co}(H)$. Suppose $T : H \to H$ satisfies Gobel's Lipschitz condition, and $T^2 = I$. Then T has a fixed point.*

Let $\mathbb{R}_+$ be the set of nonnegative real numbers and let Φ be the family of continuous functions $\phi : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfying the following properties:
(i) $\phi(1,1,1) = k < 2$,
(ii) for $s \geq 0,\ t \geq 0,$ the inequality $s \leq \phi(t, 2t, s)$ implies that $s \leq ht$ for some $h \in [k, 2)$ (See [2], [10]).

Recall that a mapping $T : K \to K$ is called an *involution* if $T^2 = I$, where I denotes the identity map. With mimicking the proof of Lemma 3.1, we also have the following:

THEOREM 3.1. *Let X be a Banach space, let H be a nonempty 1-local retract of $\overline{co}(H)$. If $T : H \to H$ is an involution map satisfying*

$$\|Tx - Ty\| \leq \phi(\|x - y\|, \|x - Tx\|, \|y - Ty\|)$$

for all $x, y \in H$ and some $\phi \in \Phi$, then T has a fixed point in H.

Let $\{n_\alpha\}$ be an ultra subnet on $\mathbb{N}$. Let C be a weakly compact subset of a Banach space X and let $T : C \to C$ be a mapping such that for each $x \in C$,

$$T^{n_\alpha}(x) \rightharpoonup S(x).$$

It is easy to show that if $T : C \to C$ is a mapping of a.n.t. then $S : C \to C$ is nonexpansive. Obviously, $Fix(T) \subseteq Fix(S)$, where $Fix(T)$ denotes the set of all fixed points of T. Furthermore, if X has weak normal structure, by classical fixed point theorem of Kirk, $Fix(S) \neq \emptyset$. Now we will present a sufficient condition for which $Fix(S) \subseteq Fix(T)$. For the proof of the following lemma, see the lemma 3.1 of [29].

LEMMA 3.2. *Let C be a weakly compact convex subset of a Banach space X with $WCS(X) > 1$. Let $T : C \to C$ be a continuous mapping of a.n.t. and weakly asymptotic regular on C. Then $Fix(T) = Fix(S) \neq \emptyset$. Further it is a nonexpansive retract of C.*

Combined with Theorem 3.1, this yields the following result.

THEOREM 3.2. *Let C be a weakly compact convex subset of a Banach space X with $WCS(X) > 1$. Let $T : C \to C$ be a mapping such that T^2 is both a.n.t. and weakly asymptotic regular on C. If $T : C \to C$ is an involution map satisfying*

$$\|Tx - Ty\| \leq \phi(\|x - y\|, \|x - Tx\|, \|y - Ty\|)$$

for all $x, y \in C$ and some $\phi \in \Phi$, then T has a fixed point in C.

PROOF. By Lemma 3.2, $H := Fix(T^2)$ is a nonempty nonexpansive retract of C. It is obvious that $T : H \to H$ and that all assumptions of Theorem 3.1 are fulfilled. Therefore $Fix(T) \neq \emptyset$. □

Here we give an example of a mapping which is k-lipschitzian involution but not of a.n.t.

EXAMPLE. Let $X = \mathbb{R}$, $C = [-\frac{1}{k}, 1]$, where $1 < k < 2$. Define a mapping $T : C \to C$ by

$$T(x) = \begin{cases} -\frac{1}{k}x & \text{if } 0 \le x \le 1; \\ -kx & \text{if } -\frac{1}{k} \le x \le 0. \end{cases}$$

Then it is obvious that T is a uniformly k-lipschitzian involution mapping but it is not of a.n.t. Indeed, for $x = 0$,

$$\begin{aligned} &\limsup_{n\to\infty} \sup\{|T^n(y)| - |y| : y \in [-\frac{1}{k}, 1]\} \\ &= \sup\{|T(y)| - |y| : y \in [-\frac{1}{k}, 1]\} \\ &= \sup\{(k-1)|y| : -\frac{1}{k} \le y \le 0\} \\ &= (k-1)(\frac{1}{k}) = 1 - \frac{1}{k} > 0. \end{aligned}$$

4. Some questions

Recall that a Banach space X has the semi-Opial property (semi-O) [35] if for any bounded nonconstant sequence $\{x_n\}$ with $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x$ and

(semi-O) $$\lim_{k\to\infty} \|x - x_{n_k}\| < \operatorname{diam}\{x_n\}.$$

The following spaces have the semi-Opial property.

(1) X is reflexive and it has Opial's property [40], i.e., for any weakly null sequence $\{x_n\}$,

(O) $$\liminf_{n\to\infty} \|x_n\| < \liminf_{n\to\infty} \|x + x_n\| \quad \text{for every } x(\ne O) \in X.$$

(2) X has (UNS).

(3) X is nearly uniformly convex (NUC) [20], i.e., for $\forall \epsilon > 0,\ \exists\ \delta > 0$ such that

$$\text{(NUC)} \qquad [\ \|x_n\| \leq 1,\ \text{sep}(\{x_n\}) > \epsilon \Rightarrow \text{co}(\{x_n\} \cap B_{1-\delta}(O) \neq \emptyset\ ],$$

where $B_r(O) = \{x \in X : \|x\| \leq 1\}$ for $r > 0$.

Note that X is (NUC) iff it is reflexive and has a (UKK) norm.

Let $\text{sep}(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m\}$. A Banach space X is said to have *Kadec-Klee* (KK) norm if for every sequence $\{x_n\}$ in X the following implication holds:

$$\text{(KK)} \qquad [\ \|x_n\| \leq 1,\ \text{sep}(\{x_n\}) > 0 \text{ and } x_n \rightharpoonup x \Rightarrow \|x\| < 1\ ].$$

In other words, (weak convergence)$\Rightarrow$ (norm convergence) on the unit sphere of X, i.e., $\{x \in X : \|x\| = 1\}$. The norm of X is said to be (UKK) (*uniformly Kadec-Klee*) if $\forall \epsilon > 0,\ \exists\ \delta > 0$ such that

$$\text{(UKK)} \qquad [\ \|x_n\| \leq 1,\ \text{sep}(\{x_n\}) \geq \epsilon \text{ and } x_n \rightharpoonup x \Rightarrow \|x\| \leq 1 - \delta\ ],$$

where $\text{sep}(\{x_n\}) := \inf\{\|x_n - x_m\| : n \neq m\} \geq \varepsilon$ "ε-separate sequence". The norm of X is said to be (WUKK) (*weakly uniform Kadec-Klee*) [11] if $\exists\ \varepsilon,\ \delta > 0$ such that

$$\text{(WUKK)} \quad [\ \|x_n\| \leq 1,\ \text{sep}(\{x_n\}) \geq \epsilon \text{ and } x_n \rightharpoonup x \Rightarrow \|x\| \leq 1 - \delta\ ],$$

(4) (Baillon-Schöneberg [4]) $X = X_\beta$, where $1 < \beta < 2$, $X_\beta = (\ell^2, \|\ \|_\beta)$,

$$\|x\|_\beta = \max(\|x\|_2, \beta\|x\|_\infty)$$

for $x \in \ell^2$. Note that

(i) X_β is (UNS) if $1 < \beta < \sqrt{2}$.
(ii) X_β is (ANS) if $1 < \beta < 2$.
(iii) $X_{\sqrt{2}}$ fails to have (NS).

(5) X is the James quasi-reflexive space.

The following proposition gives an interesting result concerning minimal sets:

PROPOSITION ([1]). *Let $\alpha : K \to \mathbb{R}_+$ be a lower semicontinuous convex function. Assume that $\alpha(Tx) \leq \alpha(x)$ for all $x \in K$. Then α is a constant function.*

LEMMA 4.1 (GOEBEL [16]-KARLOVITZ [24,25]). *Let T be nonexpansive. Let K be a weakly compact convex subset which is minimal invariant under T. Let $\{x_n\}$ be a sequence of approximate fixed points, i.e., $x_n \in K$ and $\|Tx_n - x_n\| \to 0$. Then for each $x \in K$,*

$$\lim_{n\to\infty} \|x - x_n\| = diam(K).$$

QUESTION (I). *Does Geobel-Karlovitz lemma hold for any asymptotically nonexpansive mapping T ?*

To compete the proof of Lemma, we define $\alpha(x) = \limsup_{n\to\infty} \|x_n - x\|$ for $x \in K$. Since $\{x_n\}$ is a sequence of approximate fixed points for T, it follows that $\alpha(Tx) \leq \alpha(x)$ for all $x \in K$. By above proposition, α must be a constant function.

THEOREM 4.2. *(semi-O) $\Rightarrow$ (FPP:N).*

PROOF. For fixed a $\lambda \in (0,1)$, we set

$$S_\lambda := \lambda I + (1-\lambda)T,$$

where I is the identity operator of X. Then it is obvious that $S_\lambda : C \to C$ is nonexpansive with the same fixed point set of T. Moreover, it is well-known that S_λ is asymptotically regular on C (see [13]). Then, by Zorn's lemma there exists a nonempty weakly compact convex subset K of C which is invariant under S_λ. Suppose that $\operatorname{diam}(K) > 0$. Let $x_o \in K$. Taking $x_n := S_\lambda^n x_o$ in Lemma 4.1, for each $x \in K$,

$$\lim_{n\to\infty} \|x - x_n\| = \operatorname{diam}(K).$$

On the other hand, by the semi-Opial property of X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $w\text{-}\lim_{k\to\infty} x_{n_k} = x$ and

$$\lim_{k\to\infty} \|x_{n_k} - x\| < \operatorname{diam}(\{x_n\}) \leq \operatorname{diam}(K),$$

which gives a contradiction. Hence, $\operatorname{diam}(K) = 0$ and so S_λ has a fixed point in K. □

QUESTION (II). *Does Theorem 4.2 hold for any asymptotically nonexpansive mapping T ?*

We say that a Banach space X has the *uniform Opial's property* (UO) [39] if for any $c > 0$, there exists $r > 0$ such that

$$\text{(UO)} \qquad 1 + r \leq \liminf_{n \to \infty} \|x + x_n\|$$

for every $x \in X$ with $\|x\| \geq c$ and any sequence $\{x_n\}$ with $x_n \rightharpoonup O$, $\liminf_n \|x_n\| \geq 1$.

We say that X has the *local uniform Opial's property* (LUO) [39] if for any $c > 0$ and for any weakly null sequence $\{x_n\}$ in X with $\liminf_n \|x_n\| \geq 1$, there exists $r > 0$ such that

$$\text{(LUO)} \qquad 1 + r \leq \liminf_{n \to \infty} \|x + x_n\| \quad \text{for all } x \in X \text{ with } \|x\| \geq c.$$

THEOREM 4.3 ([45]). *If X has (UO), then (FPP) holds for any continuous mapping of a.n.t.*

Note that the following implications hold:

$$\text{(UC)+(O)} \overset{[7,45]}{\Longrightarrow} \text{(UO)} \Rightarrow \text{(LUO)} \Rightarrow$$
$$\text{(O)} \overset{\text{(R)}}{\Longrightarrow} \text{(semi-O)} \overset{\text{(Theorem 4.2)}}{\Longrightarrow} \text{(FPP:N)},$$

where (R) means the reflexivity of X.

QUESTION (III). *Let X have the property (O). Does (FPP) hold for any asymptotically nonexpansive mapping T ?*

For every $x, y \in X$ and nonnegative real number λ, we set

$$M_\lambda(x, y) = \left\{ z \in X : \max\{\|z - x\|,\ \|z - y\|\} \leq \frac{1}{2}(1 + \lambda)\|x - y\| \right\}.$$

If A is a bounded subset of X, we define $|A| = \sup\{\|z\| : z \in A\}$. For any sequence $\{x_n\}$ in X and any nonnegative real number λ, we set

$$A_\lambda(\{x_n\}) = \limsup_{n \to \infty} (\limsup_{m \to \infty} |M_\lambda(x_n, x_m)|).$$

A Banach space X is called *orthogonally convex* (OC) [22,23] if for any weakly null sequence $\{x_n\}$ with $D(\{x_n\}) > 0$,

$$\text{(OC)} \qquad \exists\ \lambda > 0 \text{ such that } A_\lambda(\{x_n\}) < D(\{x_n\}).$$

Note also that the following implications hold:

$$\text{(OC)} \overset{[22]}{\Longrightarrow} \text{(FPP:N)} \overset{[32]}{\Longleftarrow} \text{(WNS)}.$$

The following spaces have (OC) property:

(1) X is a Banach space with the Schur property, and so is ℓ^1.
(2) (UC) $\Longrightarrow$ (OC).
(3) c_o and c.
(4) The James quasi-reflexive space is (OC).
(5) (Dulst [11]) The space $VD = (\ell^2, \|\ \cdot\|)$ is (OC), where

$$\|x\| = \max\{\frac{1}{3}\|x\|_2,\ \sup_{n\geq 2}|x(1) + x(n) + x(n+1)|\}$$

for all $x = \sum_{n=1}^{\infty} x(n)e_n \in \ell^2$.

QUESTION (IV). *What is the relation between $WCS(X) > 1$ and (OC) ?*

QUESTION (V). *Let X=(OC). Does (FPP) hold for any asymptotically nonexpansive mapping T ?*

Finally recall a generalization of uniformly convex Banach spaces which is due to Sullivan [42]. Let $k \geq 1$ be an integer. Then a Banach space X is said to be k-(UR) (*k-uniformly rotund*) if given any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that if $\{x_1, \cdots, x_{k+1}\} \subset B_X$, the closed unit ball of X, satisfies $V(x_1, \cdots, x_{k+1}) \geq \epsilon$, then $\|(\sum_{i=1}^{k+1} x_i)/(k+1)\| \leq 1 - \delta(\epsilon)$. Here $V(x_1, \cdots, x_{k+1})$ is the volume enclosed by the set $\{x_1, \cdots, x_{k+1}\}$, i.e.,

$$V(x_1, \cdots x_{k+1}) = \sup\left\{\begin{vmatrix} 1 & \dots & 1 \\ f_1(x_1) & \dots & f_1(x_{k+1}) \\ \vdots & & \vdots \\ f_k(x_1) & \dots & f_k(x_{k+1}) \end{vmatrix}\right\},$$

where the supremum is taken over all $f_1, \cdots, f_k \in B_{X^*}$. The modulus of k-uniform rotundity of X is the function $\delta_X^{(k)}(\cdot)$ defined by

$$\delta_X^{(k)}(\epsilon) = \inf\left\{1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| : x_i \in B_X,\ V(x_1, \cdots, x_{k+1}) \geq \epsilon\right\}.$$

Then it is seen that X is k-(UR) if and only if $\delta_X^{(k)}(\epsilon) > 0$ for $\epsilon > 0$.

It is now known that the following implications hold:

$$\begin{array}{l}
(\mathrm{NUC}) \overset{[46]}{\Leftarrow} \cdots \Leftarrow k-(\mathrm{UR}) \cdots \Leftarrow 2-(\mathrm{UR}) \Leftarrow 1-(\mathrm{UR}) \Longleftrightarrow (\mathrm{UC}) \\
\Downarrow \\
(\mathrm{UKK}) \Rightarrow (\mathrm{WUKK}) \overset{[12]}{\Longrightarrow} (\mathrm{WNS}) \overset{[32]}{\Longrightarrow} (\mathrm{FPP{:}N}) \\
\Downarrow \\
(\mathrm{KK}).
\end{array}$$

Recall that a Banach space X is said to satisfy *Lim's condition* (L) [37] if there exists a function $\delta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties:

(a) $\delta(r, s)$ is continuous and strictly increasing in each variable,

(b) If $x_n \rightharpoonup 0$ and $\lim \|x_n\| = s > 0$, then

$$\lim \|y - x_n\| = \delta(\|y\|, s) \quad \text{for every } y \in X.$$

Khamsi [28] showed that a Banach space X having a weakly continuous duality map (or more generally, Lim's condition (L)) satisfies the uniform Opial condition. For more details, see [28]. Here, we have the following implications:

$$(\mathrm{J{=}WSC}) \Rightarrow (\mathrm{L}) \overset{[28]}{\Longrightarrow} (\mathrm{UO}) \Rightarrow (\mathrm{LUO}) \Rightarrow (\mathrm{O}),$$

$$k\text{-(UR)+(O)} \overset{[39]}{\Longrightarrow} \text{(LUO)},$$

$$\text{(UC)+(O)} \overset{[8,45]}{\Longrightarrow} \text{(UO)} \Longrightarrow \text{(LUO)} \Longrightarrow \text{(O)} \overset{(R)}{\Longrightarrow} \text{(semi-O)} \Longrightarrow \text{(FPP:N)}.$$

Recently, Xu [45] raised the following question.

QUESTION (VI). *For $k > 1$, k-(UR)+(O) $\Longrightarrow$ (UO) ?*

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