Comm. Korean Math. Soc. **11** (1996), No. 3, pp. 707–723

CONVOLUTION PRODUCT AND GENERALIZED ANALYTIC FOURIER-FEYNMAN TRANSFORMS

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ABSTRACT. We first define the concept of the generalized analytic Fouri er-Feynman transforms of a class of functionals on function space induced by a generalized Brownian motion process and study of functionals which plays on important role in physical problem of the form

$$F(x) = \left\{ \int_0^T f(t, x(t))dt \right\}$$

where f is a complex-valued function on $[0, T] \times \mathbb{R}$. We next show that the generalized analytic Fourier-Feynman transform of the convolution product is a product of generalized analytic Fourier-Feynman transform of functionals on function space.

1. Introduction

In various Feynman integration theories, the integrand of the Feynman integral is a functional of the standard Wiener process. In [3,4] Cameron and Martin investigated various linear transformations of the standard Wiener measure. Since then many related papers have appeared in the literature. The concept of an L_1 analytic Fourier-Feynman transform was introduced by Brue in [1]. In [3] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In [10] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ which gave various relationships between L_1 and L_2 theories. In this paper we extend the ideas from the Wiener processes to

Received March 23, 1996. Revised June 10, 1996.

1991 AMS Subject Classification: 60.

Key words and phrases: analytic fuction space integral convolution product, generalized analytic Fourier-Feynman transform.

The present research has been conducted by the research grant of Dankook University in 1996.

more general stochastic processes. We note that the Wiener process is free of drift and is stationary in time. However, the stochastic process considered in this paper is a process subject to drift and is nonstationary in time. We first define the concept of the generalized analytic Fourier-Feynman transforms of a class of functionals on function space induced by a generalized Brownian motion process and study a class of functionals which plays an important role in physical problem of the form

$$F(x) = \exp\left\{ \int_0^T f(t, x(t))dt \right\}$$

where f is a complex-valued function on $[0,T] \times \mathbb{R}$ such that $f(t,\cdot)$ is in $L_p(\mathbb{R})$ for almost all $t \in [0,T]$. We next show that the generalized analytic Fourier-Feynman transform of the convolution product is a product of generalized analytic Fourier-Feynman transform of functionals on function space.

2. Definitions and preliminaries

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real valued stochastic process X on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $X(0,\omega)=0$ a.e. and for $0 \le t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(X(t_1,\omega), \cdots, X(t_n,\omega))$ is normal distributed with the density function

(2.1)

$$K(t,\vec{\eta}) = \left((2\pi)^n \prod_{j=1}^n (b(t_j) - b(t_{j-1}))\right)^{-1/2} \exp\left\{ -\frac{1}{2} \sum_{j=1}^n \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})} \right\}$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $a(\cdot)$ is a real valued continuous function with $a(0) = 0$, and $b(\cdot)$ is a strictly increasing continuously differentiable real valued function with $b(0) = 0$.

As explained in [15, p.18-20], X induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space.

We note that the generalized Brownian motion process X determined by $b(\cdot)$ is a Gaussian process with covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [15, p.187], the probability measure μ induced by X, taking a separable version, is supported by $C_b[0,T]$(=the Banach space of continuous functions x on $[0,T]$ with x(0)=0 under the sup norm). Hence $(C_b[0,T], \mathcal{B}(C_b[0,T]), \mu)$ is the function space induced by X where $\mathcal{B}(C_b[0,T])$ is the Borel σ-algebra of $C_b[0,T]$. Let W be a stochastic process on $(C_b[0,T], \mathcal{B}(C_b[0,T]), \mu)$ and D defined by $W(t,x) = x(t)$, $t \in D$, $x \in C_b[0,T]$. Then W is a generalized Brownian motion process whose the sample space is $C_b[0,T]$. We denote the function space integral of a $\mathcal{B}(C_b[0,T])$- measurable function F by

$$\int_{C_b[0,T]} F(x)d\mu(x)$$

whenever the integral exists.

A subset E of $C_b[0,T]$ is said to be scale invariant measurable [11] if $\rho E \in \mathcal{B}(C_b[0,T])$ for each $\rho > 0$. A scale-invariant measurable set N is said to be scale-invariant null if $\mu(\rho N) = 0$ for every $\rho > 0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.).

Next we give the definitions of the generalized analytic Feynman integral and the generalized analytic Fourier-Feynman transform.

Definition 2.1. Let $\mathbb{C}$, $\mathbb{C}_+$, and $\tilde{\mathbb{C}}_+$ denote respectively the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part. Let F be a complex-valued scale-invariant measurable function on $C_b[0,T]$ such that the function space integral

$$J(\lambda) = \int_{C_b[0,T]} F(\lambda^{-\frac{1}{2}}x)d\mu(x)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$, analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_b[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$E^{\mathrm{an}_\lambda}(F) = J^*(\lambda).$$

Let $q \neq 0$ be a real number and let F be a function such that $E^{\mathrm{an}_\lambda}(F)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$E^{\mathrm{anf}_q}(F) = \lim_{\lambda \to -iq} E^{\mathrm{an}_\lambda}(F)$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

DEFINITION 2.2.

i) Let F be a complex valued scale-invariant measurable functional on $C_b[0,T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_b[0,T]$, let

$$(2.2) \qquad (T_\lambda(F))(y) = \int_{C_b[0,T]}^{an_\lambda} F(x+y)d\mu(x)$$

ii) Given a number p with $1 \le p \le \infty$, p and p' will always be related by $1/p + 1/p' = 1$.

iii) Let $1 < p \le 2$ and let $\{H_n\}$ and H be scale-invariant measurable functionals such that for each $\rho > 0$,

$$(2.3) \qquad \lim_{n\to\infty} \int_{C_b[0,T]} |H_n(\rho y) - H(\rho y)|^{p'} d\mu(y) = 0.$$

Then we write

$$(2.4) \qquad \underset{n\to\infty}{\mathrm{l \cdot i \cdot m.}}(W_s^{p'})(H_n) \approx H$$

and we call H the scale invarinat limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ.

We finally ready to state the definition of the generalized L_p analytic Fourier-Feynman transform and our definition of the convolution product [cf, 5,10].

DEFINITION 2.3. Let $q \neq 0$ be real number. For $1 < p \leq 2$ we define the generalized L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F, by the formula

$$(T_q^{(p)}(F))(y) = \underset{\lambda \to -iq}{\mathrm{l \cdot i \cdot m.}}(W_s^{p'})(T_\lambda(F))(y) \tag{2.5}$$

whenever this limit exists for all $\lambda \in \mathbb{C}_+$. We define the generalized L_1-analytic Fourier-Feynamn transform $T_q^{(1)}(F)$ of F, by the formula

$$(T_q^{(1)}(F))(y) = \lim_{\lambda \to -iq}(T_\lambda(F))(y) \tag{2.6}$$

for s-a.e. y on $C_b[0,T]$. We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_q^{(p)}$ exists and if $F_1 \approx F_2$ then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_2) \approx T_q^{(p)}(F_1)$.

Now we give the definition of the convolution product for complex-valued measurable functionals on $C_b[0,T]$.

DEFINITION 2.4. Let F and G be complex-valued measurable functionals on $C_b[0,T]$. For $\lambda \in \tilde{\mathbb{C}}_+$, we define the convolution product of two functionals $F(x)$ and $G(x)$ to be

$$(F*G)_\lambda(y) = \begin{cases} \int_{C_b[0,T]}^{an_\lambda} F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})d\mu(x), & \lambda \in \mathbb{C}_+ \\ \int_{C_b[0,T]}^{anf_q} F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})d\mu(x), & \\ \qquad \lambda = -iq, \quad q \in \mathbb{R} - \{0\} & \end{cases} \tag{2.7}$$

if the integral in the right right hand side exists.

NOTATION.. When $\lambda = -iq$, we will denote $(F*G)_\lambda$ by $(F*G)_q$.

3. Generalized analytic Fourier-Feynman transforms

Let p be a real number with $1 \leq p \leq 2$ and let $r \in (2p/(2p-1), \infty]$. Let $L_{pr}([0,T] \times \mathbb{R})$ be the space of all complex-valued Lebesgue measurable functions f on $[0,T] \times \mathbb{R}$ such that $f(t,\cdot)$ is in $L_p(\mathbb{R})$ for almost all

$t \in [0,T]$ and as a function of t, $\|f(t,\cdot)\|_p$ is in $L_r([0,T])$. The second class $A \equiv A_{pr}$ of functionals is defined as follows : A functional $F(x)$ belong to A if

$$F(x) = \exp\left\{\int_0^T f(t,x(t))dt\right\} \tag{3.1}$$

where $f \in L_{pr}([0,T]\times\mathbb{R})$. Then $F(x)$ is defined s-a.e. and is scale-invariant measurable.

THEOREM 3.1. *Let $F \in A$ be given by (3.1) with $f \in L_{pr}([0,T]\times\mathbb{R})$. Then $T_\lambda(F)$ exists for all $\lambda \in \mathbb{C}_+$ and is given by*

(3.2)

$$\begin{aligned}&(T_\lambda(F))(y)\\ &= 1 + \sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbb{R}^n}\prod_{j=1}^{n}\Bigg[\left(\frac{\lambda}{2\pi(b(t_j)-b(t_{j-1}))}\right)^{\frac{1}{2}} f(t_j,u_j+y(t_j))\\ &\qquad\cdot\exp\left\{-\frac{\lambda((u_j-\lambda^{-\frac{1}{2}}a(t_j))-(u_{j-1}-\lambda^{-\frac{1}{2}}a(t_{j-1})))^2}{2(b(t_j)-b(t_{j-1}))}\right\}\Bigg]d\vec{u}d\vec{t}\end{aligned}$$

where $\Delta_n(T) = \{\vec{t} = (t_1,\cdots,t_n) \in [0,T]^n : 0 < t_1 < t_2 < \cdots < t_n \le T\}$, $\vec{u} = (u_1,\cdots,u_n)$, and $t_0 \equiv 0 \equiv u_0$.

PROOF. Note that

$$\begin{aligned}F(x) &= \exp\left\{\int_0^T f(t,x(t))dt\right\}\\ &= 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\left[\int_0^T f(t,x(t))dt\right]^n\\ &= 1 + \sum_{n=1}^{\infty}\int_{\Delta_n(T)}\prod_{j=1}^{n}\left[\int_0^T f(t_j,x(\tau_j))\right]d\vec{t}.\end{aligned} \tag{3.3}$$

Hence by using the Fubini theorem, the change of variable theorem, (2.2), and (3.3), we have for all $\lambda > 0$,

(3.4)

$$\begin{aligned}
&(T_\lambda(F))(y)\\
&= \int_{C_b[0,T]} F(\lambda^{-1/2}x + y)d\mu(x)\\
&=1 + \int_{C_b[0,T]} \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \prod_{j=1}^{n} \Big[f(t_j, \lambda^{-\frac{1}{2}} x(t_j) + y(t_j))\Big] d\vec{t} d\mu(x)\\
&=1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^n} \prod_{j=1}^{n} \Bigg[\left(\frac{\lambda}{2\pi(b(t_j) - b(t_{j-1}))}\right)^{\frac{1}{2}} f(t_j, u_j + y(t_j))\\
&\qquad \cdot \exp\left\{-\frac{\lambda((u_j - \lambda^{-\frac{1}{2}} a(t_j)) - (u_{j-1} - \lambda^{-\frac{1}{2}} a(t_{j-1})))^2}{2(b(t_j) - b(t_{j-1}))}\right\}\Bigg] d\vec{u}d\vec{t}.
\end{aligned}$$

Thus by analytic continuation in λ, we have that equation (3.2) holds throughout $\mathbb{C}_+$.

THEOREM 3.2. *Let $F \in A$ be as in Theorem 3.1. Then for all $1 \le p \le 2$, the Fourier-Feynman transform $T_q^{(p)}(F)$ exists for all $q \neq 0$ and is given by the formula*

(3.5)

$$\begin{aligned}
&(T_q^{(p)}(F))(y)\\
&=1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^n} \prod_{j=1}^{n} \Bigg[\left(\frac{-iq}{2\pi(b(t_j) - b(t_{j-1}))}\right)^{\frac{1}{2}} f(t_j, u_j + y(t_j))\\
&\qquad \cdot \exp\left\{\frac{iq((u_j - (\frac{i}{q})^{\frac{1}{2}} a(t_j)) - (u_{j-1} - (\frac{i}{q})^{\frac{1}{2}} a(t_{j-1})))^2}{2(b(t_j) - b(t_{j-1}))}\right\}\Bigg] d\vec{u}d\vec{t}.
\end{aligned}$$

where $\Delta_n(T)$ and $\vec{u}$ are as in Theorem 3.1.

PROOF. By [10] and [11], $T_q^{(p)}(F)$ exists for all real $q \neq 0$ and is scale-invariant measurable. Since $\Phi(z) = \exp\{z\}$ is an entire function of order 1 and $f \in L_{pr}([0,T] \times \mathbb{R})$, we can easily show that for all $q \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^n} |q|^{\frac{n}{2}} \prod_{j=1}^{n} \left[\left(\frac{1}{2\pi(b(t_j) - b(t_{j-1}))}\right)^{\frac{1}{2}} |f(t_j, u_j)|\right] d\vec{u}d\vec{t} < \infty.$$

Thus for all $y \in C_b[0,T]$ the series on the right hand side of equation (3.5) converges absolutely and hence uniformly in q on compact subsets of $\mathbb{R} - \{0\}$. Moreover the series converges in the $L_p(C_b[0,T])$ mean. Hence the representation (3.5) for $T_q^{(p)}(F)$ follows from equation (3.2).

4. Convolution products and transforms of convolutions

In this section for any $F, G \in A$, the first theorem gives a series representation for the convolution product of F and G and the second theorem gives a series representation for $T_\lambda(F * G)_\lambda$.

THEOREM 4.1. *Let $F, G \in A$ be given by*

(4.1)
$$F(x) = \exp\left\{\int_0^T f(t, x(t))dt\right\} \quad \text{and} \quad G(x) = \exp\left\{\int_0^T g(t, x(t))dt\right\}$$

where $f, g \in L_{pr}([0,T] \times \mathbb{R})$. Then for all $\lambda \in \mathbb{C}_+$, the convolution product $(F * G)_\lambda$ exists and is given by the formula

(4.2)
$$\begin{aligned}
&(F * G)_\lambda(y) \\
&= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^n} \prod_{j=1}^{n} \Bigg[\left(\frac{\lambda}{2\pi(b(t_j) - b(t_{j-1}))} \right)^{\frac{1}{2}} \\
&\qquad \cdot \exp\left\{ -\frac{\lambda((u_j - \lambda^{-\frac{1}{2}} a(t_j)) - (u_{j-1} - \lambda^{-\frac{1}{2}} a(t_{j-1})))^2}{2(b(t_j) - b(t_{j-1}))} \right\} \\
&\qquad \cdot \left\{ f\left(t_j, \frac{y(t_j) + u_j}{\sqrt{2}}\right) + g\left(t_j, \frac{y(t_j) - u_j}{\sqrt{2}}\right) \right\} \Bigg] d\vec{u} d\vec{t}.
\end{aligned}$$

PROOF. By using the Fubini theorem, the change of variable theorem, and (2.7), we have for all $\lambda > 0$,

$$\begin{aligned}
&(F * G)_\lambda(y) \\
&= \int_{C_b[0,T]} F\left(\frac{y + \lambda^{-\frac{1}{2}} x}{\sqrt{2}}\right) G\left(\frac{y - \lambda^{-\frac{1}{2}} x}{\sqrt{2}}\right) d\mu(x) \\
&= \int_{C_b[0,T]} \exp\left\{\int_0^T f\left(t, \frac{y(t) + \lambda^{-\frac{1}{2}} x(t)}{\sqrt{2}}\right) dt\right\} \\
&\qquad \exp\left\{\int_0^T g\left(t, \frac{y(t) - \lambda^{-\frac{1}{2}} x(t)}{\sqrt{2}}\right) dt\right\} d\mu(x) \\
&= \int_{C_b[0,T]} \exp\left\{\int_0^T \left[f\left(t, \frac{y(t) + \lambda^{-\frac{1}{2}} x(t)}{\sqrt{2}}\right) + g\left(t, \frac{y(t) - \lambda^{-\frac{1}{2}} x(t)}{\sqrt{2}}\right)\right] dt\right\} d\mu(x)
\end{aligned}$$

$$= \int_{C_b[0,T]} \Big\{ 1 + \sum_{n=1}^{\infty} \Big[\int_0^T \Big[f\Big(t, \frac{y(t) + \lambda^{-\frac{1}{2}} x(t)}{\sqrt{2}}\Big) + g\Big(t, \frac{y(t) - \lambda^{-\frac{1}{2}} x(t)}{\sqrt{2}}\Big)\Big] dt \Big]^n \Big\} d\mu(x)$$

$$= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{C_b[0,T]} \prod_{j=1}^{n} \Big[f\Big(t_j, \frac{y(t_j) + \lambda^{-\frac{1}{2}} x(t_j)}{\sqrt{2}}\Big) + g\Big(t_j, \frac{y(t_j) - \lambda^{-\frac{1}{2}} x(t_j)}{\sqrt{2}}\Big)\Big] d\mu(x) d\vec{t}$$

$$= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^n} \prod_{j=1}^{n} \Big[\Big(\frac{\lambda}{2\pi(b(t_j) - b(t_{j-1}))} \Big)^{\frac{1}{2}} \Big\{ f\Big(t_j, \frac{y(t_j) + u_j}{\sqrt{2}}\Big) + g\Big(t_j, \frac{y(t_j) - u_j}{\sqrt{2}}\Big)\Big\} \cdot \exp\Big\{ - \frac{\lambda((u_j - \lambda^{-\frac{1}{2}} a(t_j)) - (u_{j-1} - \lambda^{-\frac{1}{2}} a(t_{j-1})))^2}{2(b(t_j) - b(t_{j-1}))} \Big\}\Big] d\vec{u} d\vec{t}.$$

Hence by analytic continuation in λ, we have the equation (4.2) holds throughout $\mathbb{C}_+$.

Now by using Theorem 4.1 for all $F, G \in A$, we have a series representation for $T_\lambda(F * G)_\lambda$.

THEOREM 4.2. *Let f, g, F and G be as in Theorem 4.1. Then for all $\lambda \in \mathbb{C}_+$, the transform $T_\lambda(F * G)_\lambda$ of convolution product exists and is given by the formula*

(4.3)

$$(T_\lambda(F * G)_\lambda)(z) = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} \Big[\Big(\frac{\lambda}{2\pi(b(t_j) - b(t_{j-1}))} \Big) \cdot \{ f(t_j, v_j + \frac{(2-\sqrt{2})}{2} \lambda^{-\frac{1}{2}} a(t_j) + \frac{z(t_j)}{\sqrt{2}}) + g(t_j, r_j - \lambda^{-\frac{1}{2}} a(t_j) + \frac{z(t_j)}{\sqrt{2}})\} \cdot \exp\Big\{ - \frac{\lambda((v_j - \lambda^{-\frac{1}{2}} a(t_j)) - (v_{j-1} - \lambda^{-\frac{1}{2}} a(t_{j-1})))^2}{2(b(t_j) - b(t_{j-1}))} - \frac{\lambda((r_j - \lambda^{-\frac{1}{2}} a(t_j)) - (r_{j-1} - \lambda^{-\frac{1}{2}} a(t_{j-1})))^2}{2(b(t_j) - b(t_{j-1}))} \Big\}\Big] d\vec{r} d\vec{v} d\vec{t}.$$

PROOF. By using the Fubini theorem, the change of variable theorem, (3.5), and (4.2), we have for all $\lambda > 0$,

(4.4)

$$
\begin{aligned}
&(T_\lambda(F*G)_\lambda)(z)\\
&=\int_{C_b[0,T]}(F*G)_\lambda(\lambda^{-\frac{1}{2}}x+z)d\mu(x)\\
&=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbb{R}^n}\int_{C_b[0,T]}\prod_{j=1}^{n}\bigg[\Big(\frac{\lambda}{2\pi(b(t_j)-b(t_{j-1}))}\Big)^{\frac{1}{2}}\\
&\quad\cdot\exp\Big\{-\frac{\lambda((u_j-\lambda^{-\frac{1}{2}}a(t_j))-(u_{j-1}-\lambda^{-\frac{1}{2}}a(t_{j-1})))^2}{2(b(t_j)-b(t_{j-1}))}\Big\}\\
&\quad\cdot\Big\{f\Big(t_j,\frac{\lambda^{-\frac{1}{2}}x(t_j)+z(t_j)+u_j}{\sqrt{2}}\Big)+g\Big(t_j,\frac{\lambda^{-\frac{1}{2}}x(t_j)+z(t_j)-u_j}{\sqrt{2}}\Big)\Big\}d\mu(x)\bigg]d\vec{u}d\vec{t}\\
&=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbb{R}^{2n}}\prod_{j=1}^{n}\bigg[\Big(\frac{\lambda}{2\pi(b(t_j)-b(t_{j-1}))}\Big)\\
&\quad\cdot\exp\Big\{-\frac{\lambda((u_j-\lambda^{-\frac{1}{2}}a(t_j))-(u_{j-1}-\lambda^{-\frac{1}{2}}a(t_{j-1})))^2}{2(b(t_j)-b(t_{j-1}))}\\
&\qquad-\frac{\lambda((w_j-\lambda^{-\frac{1}{2}}a(t_j))-(w_{j-1}-\lambda^{-\frac{1}{2}}a(t_{j-1})))^2}{2(b(t_j)-b(t_{j-1}))}\Big\}\\
&\quad\cdot\Big\{f\Big(t_j,\frac{z(t_j)+w_j+u_j}{\sqrt{2}}\Big)+g\Big(t_j,\frac{z(t_j)+w_j-u_j}{\sqrt{2}}\Big)\Big\}\bigg]d\vec{w}d\vec{u}d\vec{t}.
\end{aligned}
$$

Now, in the last equation which appears in the above equation, let

$$v_j \equiv v_j(t) = \frac{w_j + u_j + (\sqrt{2}-2)\lambda^{-\frac{1}{2}}a(t_j)}{\sqrt{2}}$$

and

$$r_j \equiv r_j(t) = \frac{w_j - u_j + \sqrt{2}\lambda^{-\frac{1}{2}}a(t_j)}{\sqrt{2}}$$

then we have

$$v_j - \lambda^{-\frac{1}{2}}a(t_j) = \frac{(w_j - \lambda^{-\frac{1}{2}}a(t_j)) + (u_j - \lambda^{-\frac{1}{2}}a(t_j))}{\sqrt{2}}$$

and

$$r_j - \lambda^{-\frac{1}{2}}a(t_j) = \frac{(w_j - \lambda^{-\frac{1}{2}}a(t_j)) - (u_j - \lambda^{-\frac{1}{2}}a(t_j))}{\sqrt{2}}$$

and so the Jacobian of this transformation is one and

$$\begin{aligned}
&((u_j - \lambda^{-\frac{1}{2}}a(t_j)) - (u_{j-1} - \lambda^{-\frac{1}{2}}a(t_{j-1})))^2 \\
&\quad + ((w_j - \lambda^{-\frac{1}{2}}a(t_j)) - (w_{j-1} - \lambda^{-\frac{1}{2}}a(t_{j-1})))^2 \\
&= ((v_j - \lambda^{-\frac{1}{2}}a(t_j)) - (v_{j-1} - \lambda^{-\frac{1}{2}}a(t_{j-1})))^2 \\
&\quad + ((r_j - \lambda^{-\frac{1}{2}}a(t_j)) - (r_{j-1} - \lambda^{-\frac{1}{2}}a(t_{j-1})))^2
\end{aligned}$$

for $j = 1, 2, \cdots, n$. Substituting this expression in the last equation of (4.4), then we have for all $\lambda > 0$,

$$\begin{aligned}
&(T_\lambda(F * G)_\lambda)(z) \\
&= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} \Big[\Big(\frac{\lambda}{2\pi(b(t_j) - b(t_{j-1}))} \Big) \\
&\quad \cdot \{ f(t_j, v_j + \frac{(2-\sqrt{2})}{2}\lambda^{-\frac{1}{2}}a(t_j) + \frac{z(t_j)}{\sqrt{2}}) + g(t_j, r_j - \lambda^{-\frac{1}{2}}a(t_j) + \frac{z(t_j)}{\sqrt{2}}) \} \\
&\quad \cdot \exp\Big\{ -\frac{\lambda((v_j - \lambda^{-\frac{1}{2}}a(t_j)) - (v_{j-1} - \lambda^{-\frac{1}{2}}a(t_{j-1})))^2}{2(b(t_j) - b(t_{j-1}))} \\
&\qquad - \frac{\lambda((r_j - \lambda^{-\frac{1}{2}}a(t_j)) - (r_{j-1} - \lambda^{-\frac{1}{2}}a(t_{j-1})))^2}{2(b(t_j) - b(t_{j-1}))} \Big\} \Big] d\vec{r}d\vec{v}d\vec{t}.
\end{aligned}$$

By analytic continuation in λ, we have that equation (4.3) is holds throughout $\mathbb{C}_+$.

5. Generalized analytic Fourier-Feynman transforms of convolutions

In this section, we will show that there is a relationships involving convolution products and analytic function space integrals and then the generalized analytic Fourier-Feynman transform of the convolution product is the product of the transforms. The result of this paper is stated in the following three theorems.

THEOREM 5.1. *If $T_\lambda(F)$, $T_\lambda(G)$, and $T_\lambda(F * G)_\lambda$ exists for $\lambda > 0$, then we have*

$$(T_\lambda(F * G)_\lambda)(z) = (T_\lambda(F))(\frac{z}{\sqrt{2}})(T_\lambda(G))(\frac{z}{\sqrt{2}}). \tag{5.1}$$

PROOF. By using (2.2) and (2.7) we have for $\lambda > 0$,

$$\begin{aligned}
&(T_\lambda(F*G)_\lambda)(z)\\
=&\int_{C_b[0,T]}(F*G)(z+\lambda^{-\frac{1}{2}}y)d\mu(y)\\
=&\int_{C_b^2[0,T]}F\Big(\frac{z}{\sqrt{2}}+\frac{\lambda^{-\frac{1}{2}}(y+x)}{\sqrt{2}}\Big)G\Big(\frac{z}{\sqrt{2}}+\frac{\lambda^{-\frac{1}{2}}(y-x)}{\sqrt{2}}\Big)d\mu(y)d\mu(x).
\end{aligned}$$

Since two generalized Browian motion processes $v = \dfrac{y+x}{\sqrt{2}}$ and $w = \dfrac{y-x}{\sqrt{2}}$ are independent Gaussian random variables, by applying to the last equation which appears in the above equation, we have

$$\begin{aligned}
&(T_\lambda(F*G)_\lambda)(z)\\
=&\int_{C_b^2[0,T]}F\Big(\frac{z}{\sqrt{2}}+\frac{\lambda^{-\frac{1}{2}}v}{\sqrt{2}}\Big)G\Big(\frac{z}{\sqrt{2}}+\frac{\lambda^{-\frac{1}{2}}w}{\sqrt{2}}\Big)d\mu(v)d\mu(w)\\
=&\int_{C_b[0,T]}F\Big(\frac{z}{\sqrt{2}}+\frac{\lambda^{-\frac{1}{2}}v}{\sqrt{2}}\Big)d\mu(v)\int_{C_b[0,T]}G\Big(\frac{z}{\sqrt{2}}+\frac{\lambda^{-\frac{1}{2}}w}{\sqrt{2}}\Big)d\mu(w)\\
=&(T_\lambda(F))(\frac{z}{\sqrt{2}})(T_\lambda(G))(\frac{z}{\sqrt{2}}).
\end{aligned}$$

Hence we complete the proof.

By using Theorem 5.1, we have a relationships involving convolution products and analytic function space integrals.

THEOREM 5.2. *Let F and G be as in Theorem 4.1. Then we have for all $\lambda \in \mathbb{C}_+$,*

$$(T_\lambda(F*G)_\lambda)(z) = (T_\lambda(F))(\frac{z}{\sqrt{2}})(T_\lambda(G))(\frac{z}{\sqrt{2}}). \tag{5.2}$$

PROOF. By Theorem 5.1, equation (5.2) holds for all $\lambda \in \mathbb{C}_+$. Since $T_\lambda(F)$, $T_\lambda(G)$, and $T_\lambda(F*G)_\lambda$ have analytic extension throughout $\mathbb{C}_+$, we have the desired result.

In the following theorem, we give a convenient formula for evaluating the generalized analytic Fourier-Feynman transform of the convolution product.

THEOREM 5.3. *Let f, g, F and G be as in Theorem 4.1. Then for all real $q \neq 0$ and $1 \leq p \leq 2$, we have*

$$(5.3) \qquad (T_q^{(p)}(F * G)_q)(z) = (T_q^{(p)}(F))(\frac{z}{\sqrt{2}})(T_q^{(p)}(G))(\frac{z}{\sqrt{2}}).$$

PROOF. In view of the proof of Theorem 3.2, the series expansions for $(T_q^{(p)}(F))(\frac{z}{\sqrt{2}})$ and $(T_q^{(p)}(G))(\frac{z}{\sqrt{2}})$ both converges absolutely for all $z \in C_b[0,T]$ and hence the right hand side of (5.3) is a bounded continuous function of λ on $\tilde{\mathbb{C}}_+$ for all $z \in C_b[0,T]$. Thus by using (5.1), $T_q^{(p)}(F * G)_q$ exists and we have the desired equation (5.3) for all p and q.

COROLLARY 5.4. *Let f and F be as in Theorem 4.1. Then for all $q \neq 0$ and $1 \leq p \leq 2$, we have*

$$(T_q^{(p)}(F * F)_q)(z) = [(T_q^{(p)}(F))(\frac{z}{\sqrt{2}})]^2.$$

REMARK. Since $T_\lambda(F)$ and $T_\lambda(G)$ are easier to calculate than are $(F * G)_\lambda$ and $T_\lambda(F * G)_\lambda$, the formula (5.3) is very useful.

EXAMPLES. The following examples demonstrate that Theorem 5.3 above is useful to evaluate the transform of convolutions. By using equation (5.3), we need only transforms of the various functionals F on function space $C_b[0,T]$.

Let $v(t)$ be a real valued function on $[0,T]$ with $v \in L_2[0,T]$, $k \in \mathbb{C}$, and $x \in C_b[0,T]$. Then we have the followings:

1. If $F_1(x) = 1$, then $(T_\lambda(F_1))(z) = 1$.
2. For any $x \in C_b[0,T]$, let $F_2(x) = \int_0^T v(t)dx(t)$. Then by using (2.2), we have

$$(T_\lambda(F_2))(z) = \lambda^{-\frac{1}{2}} \int_0^T v(t)da(t) + \int_0^T v(t)dz(t).$$

In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener processes, then $a(t) \equiv 0$ and hence we have

$$(T_\lambda(F_2))(z) = \int_0^T v(t)dz(t).$$

3. For any $x \in C_b[0,T]$, let $F_3(x) = \int_0^T x^2(t)dt$. Then by using the change of variable theorem, the Fubini theorem, and (2.2), we have

$$(T_\lambda(F_3))(z) = \int_0^T \Big[\frac{b(t)}{\lambda} + 2\lambda^{-\frac{1}{2}} z(t)a(t) + z^2(t)\Big]dt.$$

In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener processes, then $a(t) \equiv 0$ and $b(t) = t$ and hence we have

$$(T_\lambda(F_3))(z) = \frac{T^2}{2\lambda} + \int_0^T z^2(t)dt.$$

4. For any $x \in C_b[0,T]$, let $F_4(x) = \big[\int_0^T v(t)dx(t)\big]^2$. Since $\int_0^T v(t)dx(t)$ is Gaussian with mean $\int_0^T v(t)da(t)$ and variance $\int_0^T v^2(t)db(t)$, by using this to the last equation which appears in the above equation, we have

$$\begin{aligned}&(T_\lambda(F_4))(z)\\ &= \Big[\int_0^T v(t)dz(t)\Big]^2 + 2\lambda^{-\frac{1}{2}}\int_0^T v(t)dz(t)\int_0^T v(t)da(t) + \frac{1}{\lambda}\int_0^T v^2(t)db(t).\end{aligned}$$

In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener processes, then $a(t) \equiv 0$ and $b(t) = t$ and hence we have

$$(T_\lambda(F_4))(z) = \Big[\int_0^T v(t)dz(t)\Big]^2 + \frac{\|v\|^2}{\lambda}.$$

5. For any $x \in C_b[0,T]$, let $F_5(x) = \int_0^T \exp\{x(t)\}dt$. Then by using the change of variable theorem, the Fubini theorem, and (2.2) we have

$$(T_\lambda(F_5))(z) = \int_0^T \exp\Big\{z(t) + \lambda^{-\frac{1}{2}}a(t) - \frac{b(t)}{2\lambda}\Big\}dt.$$

In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener processes, then $a(t) \equiv 0$ and $b(t) = t$ and hence we have

$$(T_\lambda(F_5))(z) = \int_0^T \exp\Big\{z(t) + \frac{t}{2\lambda}\Big\}dt.$$

6. For any $x \in C_b[0,T]$, let $F_6(x) = \exp\{k \int_0^T v(t)dx(t)\}$. Then by using the change of variable theorem, the Fubini theorem, and (2.2) we have

$$(T_\lambda(F_6))(z)$$
$$= \exp\left\{ k \int_0^T v(t)dz(t) + k\lambda^{-\frac{1}{2}} \int_0^T v(t)da(t) + \frac{k^2}{2\lambda}\left[\int_0^T v^2(t)db(t)\right]\right\}.$$

In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener processes, then $a(t) \equiv 0$ and $b(t) = t$ and hence we have

$$(T_\lambda(F_6))(z) = \exp\left\{ k \int_0^T v(t)dz(t) + \frac{k^2\|v\|^2}{2\lambda}\right\}.$$

7. For any $x \in C_b[0,T]$, let $F_7(x) = \exp\{k[\int_0^T v(t)dx(t)]^2\}$. Then by using the change of variable theorem, and the Fubini theorem, and (2.2) we have

$$(T_\lambda(F_7))(z) = \left(\frac{\lambda}{\lambda - 2k(\int_0^T v^2(t)db(t))}\right)^{\frac{1}{2}}$$
$$\exp\left\{\left(\frac{k}{\lambda - 2k(\int_0^T v^2(t)db(t))}\right)\cdot\left[\left(\lambda \int_0^T v(t)dz(t)\right)^2\right.\right.$$
$$\left.\left. + \left(2k\lambda^{\frac{1}{2}} \int_0^T v(t)da(t) \int_0^T v(t)dz(t) + \left(\int_0^T v(t)da(t)\right)^2\right)\right]\right\}.$$

provided $\mathrm{Re}(k/\lambda) < (\int_0^T v^2(t)db(t))^{-2}$. In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener processes, then $a(t) \equiv 0$ and $b(t) = t$ and hence we have

$$(T_\lambda(F_7))(z) = \left(\frac{\lambda}{\lambda - 2k\|v\|^2}\right)^{\frac{1}{2}} \exp\left\{\left(\frac{k\lambda}{\lambda - 2k\|v\|^2}\right)\left(\int_0^T v(t)dz(t)\right)^2\right\}.$$

8. For any $x \in C_b[0,T]$, let $F_8(x) = \exp\{k \int_0^T x(t)dt\}$. Since $\int_0^T x(t)dt$ is Gaussian with mean $\int_0^T a(t)dt$ and variance given by $M \equiv \int_0^T \int_0^T \min\{b(s), b(t)\}dsdt$. By using this to the last equation which appears in the above equation, we have

$$(T_\lambda(F_8))(z) = \exp\left\{ k \int_0^T z(t)dt + k\lambda^{-\frac{1}{2}} \int_0^T a(t)dt + \frac{k^2M^2}{2\lambda}\right\}.$$

In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener processes, then $a(t) \equiv 0$ and $b(t) = t$ and hence we have $M = \int_0^T \int_0^T \min\{s,t\} ds dt = \frac{1}{3}T^3$. Thus we have

$$(T_\lambda(F_8))(z) = \exp\left\{k\int_0^T z(t)dt + \frac{k^2T^3}{6\lambda}\right\}.$$

Now, by using the above examples, together with equation (5.3), we can find the transforms $T_q^{(p)}(F_j * F_k)_q$ of convolutions $(F_j * F_k)$ for various functionals F_j, F_k on function space $C_b[0,T]$ for $j, k = 1, 2, \cdots, 8$. For examples,

$$\begin{aligned}
&(T_q^{(p)}(F_2 * F_4)_q)(z) \\
&= \left(\int_0^T v(t)dz(t) + \left(\frac{i}{q}\right)^{\frac{1}{2}} \int_0^T v(t)da(t)\right) \\
&\quad \cdot \left(\left[\int_0^T v(t)dz(t)\right]^2 + 2\left(\frac{i}{q}\right)^{\frac{1}{2}} \int_0^T v(t)dz(t) \int_0^T v(t)da(t) \right. \\
&\qquad \left. + \frac{i\left[\int_0^T v^2(t)db(t)\right]^2}{q}\right),
\end{aligned}$$

$$\begin{aligned}
&(T_q^{(p)}(F_3 * F_5)_q)(z) \\
&= \left(\int_0^T \left[z^2(t) + 2\left(\frac{i}{q}\right)^{\frac{1}{2}} z(t)a(t) + \frac{ib(t)}{q}\right] dt\right) \\
&\quad \cdot \exp\left\{k\int_0^T v(t)dz(t) + k\left(\frac{i}{q}\right)^{\frac{1}{2}} \int_0^T v(t)da(t) + \frac{ik^2\left[\int_0^T v^2(t)db(t)\right]^2}{2q}\right\}.
\end{aligned}$$

Remark. In view of the above examples, in order to obtain the generalized analytic Fourier-Feynman transforms of convolution product of functionals on function space, we need only compute the transforms of the various functionals F_j on function space $C_b[0,T]$ for $j = 1, 2, \cdots, 8$.

Acknowledgement. I thank professor S. J. Kang for helpful discussion, professor D. M. Chung for communication of a manuscript before publication.

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