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ON THE CONTINUITY OF THE MAP INDUCED BY SCALAR-INPUT CONTROL SYSTEM

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ABSTRACT. In the control system
$\dot{x}(t) = f(t, x(t)) + g(t, x(t))\dot{u}(t), \; x(0) = \bar{x}, \; t \in [0, T]$,
this paper shows that the map from u with $L^1(m)$−topology to x_u with $L^1(\mu)$−topology is Lipschitz continuous where f is C^1, μ is the Stieltjes measure derived from the function g which is not smooth in the variable t and x_u is the solution of the above system corresponding to u under the assumption that $\dot{u}$ is bounded.

1. Introduction

Let f, g be the maps from $I\!R \times I\!R^n$ into $I\!R^n$. Consider the control system

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))\dot{u}(t), \quad x(0) = \bar{x} \in I\!R^n, \quad t \in [0, T], \quad (1.1)$$

where $\cdot = \frac{d}{dt}$, $u(t) \in I\!R$ and f is a C^1 function. For a smooth control u, denote by x_u the corresponding solution of (1.1) if it exists. If g is a C^2 function, A. Bressan[4] proved that the map $\sigma : U \to V$ defined by $\sigma(u) = x_u$ is Lipschitz continuous with respect to L^1−norm, where for some $k, k' \in I\!R$,

$$U = \{u : [0, T] \to I\!R \mid u \text{ is } C^1 \text{ and } |u(t)| \leq k \text{ for any } t \in [0, T]\}$$

and

$$V = \{x : [0, T] \to I\!R^n \mid x \text{ is } C^1 \text{ and } |x(t)| \leq k' \text{ for any } t \in [0, T]\},$$

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that is, there exists $L > 0$ such that for any $\tau \in [0, T]$ and any $u, v \in U$,

$$|x_u(\tau) - x_v(\tau)| + \int_0^T |x_u(t) - x_v(t)|dt$$
$$\leq L\left[|u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^T |u(t) - v(t)|dt\right]. \quad (1.2)$$

As long as u is just measurable, the corresponding solution of (1.1) should be interpreted as a distribution which is not unique.[7] Due to (1.2), A. Bressan in [4] defined a unique generalized solution of (1.1) corresponding to a Lebesgue integrable function u by taking a sequence $\{u_n\}$ of C^1 functions which converges to u in $L^1(m)$, where m is the Lebesgue measure and in [3] described a maximum principle. However, if g is not differentiable in the variable t, then the transformation which was adopted in [4] to prove inequality (1.2) is not valid and ineqaulity (1.2) does not hold. (The counterexample is provided in §3.) For each $\beta > 0$, define the sets

$$U_\beta = \{u : [0, T] \to I\!R \mid u \text{ is } C^1 \text{ and } |\dot{u}(t)| \leq \beta \text{ for any } t \in [0, T]\}$$

and

$$X = \{x : [0, T] \to I\!R^n \mid x \text{ is absolutely continuous }\}.$$

Let μ be the Stieltjes measure defined by (1.4). Put $B_n(\alpha) = \{x \in I\!R^n : |x| \leq \alpha\}$. Now, we fix $\beta > 1$. This paper proves that when g is not differentiable in the variable t, the map σ from U_β with $L^1(m)$–topology into X with $L^1(\mu)$–topology is Lipschitz continuous under the assumptions :

(A1) *There exists $K > 0$ such that for any $u \in U_\beta$ and $t \in [0, T]$,*

$$|x_u(t)| \leq K.$$

(A2) *For every x, the function $t \to g(t, x)$ is right continuous; for every t, the function $x \to g(t, x)$ is C^1; there exists $N_1 > 1$ such that the operator norm $\|D_x g(t, x)\| \leq N_1$ for any $(t, x) \in [0, T] \times B_n(K)$; and there exists a nondecreasing and right continuous function ϕ on $[0, T]$ such that*

$$|g(t,x) - g(s,x)| \leq \phi(t) - \phi(s) \tag{1.3}$$

for any $t \geq s$ and any $x \in B_n(K)$, where $D_x g(t,x)$ represents the Jacobian matrix of g with components $\dfrac{\partial g_i}{\partial x_j}$.

Let μ be the Stieltjes measure defined by

$$\mu((a,b]) = b - a + \phi(b) - \phi(a). \tag{1.4}$$

2. Continuity of σ

Under assumptions (A1) and (A2), the map $\sigma : u \mapsto x_u$ is well-defined.[1] For each $u \in U_\beta$, the solution x_u of (1.1) is the fixed point of the map $x \mapsto \Phi(u,x)$, where

$$\Phi(u,x)(t) = \bar{x} + \int_0^t (f(s,x(s)) + g(s,x(s))\dot{u}(s))\,ds, \tag{2.1}$$

that is,

$$x_u(t) = \bar{x} + \int_0^t (f(s,x_u(s)) + g(s,x_u(s))\dot{u}(s))\,ds.$$

By the smoothness of f and (1.3), there exists $N_2 > 1$ such that

$$|f(t,x)| \leq N_2, \qquad \text{the operator norm } \|D_x f(t,x)\| \leq N_2$$

$$\text{and} \quad |g(t,x)| \leq N_2 \quad \text{for any } (t,x) \in [0,T] \times B_n(K).$$

Let $M = \max(N_2 + \beta N_1, \beta N_1 N_2)$. Now, we fix $\tau \in [0,T]$. To investigate the continuity of σ, define a norm $\|\cdot\|_{U_\beta}$ on U_β by

$$\|u\|_{U_\beta} = |u(0)| + |u(\tau)| + \int_0^T |u(t)|d\mu \tag{2.2}$$

and a norm $\|\cdot\|_X$ on X by

$$\|x\|_X = \frac{e^{-4MT}}{4M}|x(\tau)| + \int_0^T e^{-4Ms}|x(s)|ds. \tag{2.3}$$

LEMMA 1. *Let h be a continuous real-valued function on $[0,T]$ and let $v \in U_\beta$. Then Stieltjes integral $\int_0^T h(s)dg(s,x_v(s))$ exists and satisfies*

$$|\int_0^T h(s)dg(s,x_v(s))| \leq 2\beta N_1 N_2 \int_0^T h(s)|d\mu, \tag{2.4}$$

where $g(s,x) = (g_1(s,x),\cdots,g_n(s,x))$ and $\int_0^T h(s)dg(s,x_v(s)) = (\int_0^T h(s)dg_1(s,x_v(s)),\cdots,\int_0^T h(s)dg_n(s,x_v(s)))$.

PROOF. Define $F(\tau,t) = h(\tau)g(t,x_v(t))$ on $[0,T]\times[0,T]$. Then there exists $\bar{M} > 0$ such that

$$|F(\tau,t_1) - F(\tau,t_2)| \leq \bar{M}|\phi(t_1) - \phi(t_2) + t_1 - t_2|$$

and

$$\begin{aligned}|F(\tau_1,t_1) - F(\tau_1,t_2) - F(\tau_2,t_1) + F(\tau_2,t_2)|\\ \leq \bar{M}|h(\tau_1) - h(\tau_2)||\phi(t_1) - \phi(t_2) + t_1 - t_2|\end{aligned}$$

for any $t_1,t_2,\tau_1,\tau_2 \in [0,T]$. Hence the Perron integral $\int_0^T DF(\tau,t)$ exists and it is equal to the Stieltjes integral $\int_0^T h(s)dg(s,x_v(s))$. [6] In the same way, $\bar{F}(\tau,t) = |h(\tau)|(\phi(t)+t)$ is Perron integrable on $[0,T]$. For any $\varepsilon > 0$, there exists a partition $0 = \alpha_0 \leq \tau_1 \leq \alpha_1 \leq \tau_2 \leq \cdots \alpha_n = T$ of $[0,T]$ fulfilling

$$\left|\int_0^T DF(\tau,t) - \sum_{i=1}^n [F(\tau_i,\alpha_i) - F(\tau_i,\alpha_{i-1})]\right| < \varepsilon$$

and

$$\left|\int_0^T D\bar{F}(\tau_i,\alpha_i) - \sum_{i=1}^n [\bar{F}(\tau_i,\alpha_i) - \bar{F}(\tau_i,\alpha_{i-1})]\right| < \frac{\varepsilon}{2\beta N_1 N_2}.$$

On the other hand,

$$
\begin{aligned}
&|\sum_{i=1}^{n}[F(\tau_i,\alpha_i)-F(\tau_i,\alpha_{i-1})]| \\
&\qquad = |\sum_{i=1}^{n} h(\tau_i)[g(\alpha_i, x_v(\alpha_i)) - g(\alpha_{i-1}, x_v(\alpha_{i-1}))]| \\
&\qquad = |\sum_{i=1}^{n} h(\tau_i)[g(\alpha_i, x_v(\alpha_i)) - g(\alpha_{i-1}, x_v(\alpha_i)) \\
&\qquad\qquad + g(\alpha_{i-1}, x_v(\alpha_i)) - g(\alpha_{i-1}, x_v(\alpha_{i-1}))]| \\
&\qquad \le \sum_{i=1}^{n} |h(\tau_i)|[N_1 \int_{\alpha_{i-1}}^{\alpha_i} |f(s,x_v(s)) + g(s,x_v(s))\dot v(s)|ds \\
&\qquad\quad + \phi(\alpha_i) - \phi(\alpha_{i-1})] \\
&\qquad \le 2\beta N_1 N_2 \sum_{i=1}^{n}[\bar F(\tau_i,\alpha_i) - \bar F(\tau_i,\alpha_{i-1})] \\
&\qquad \le 2\beta N_1 N_2 \int_0^T D\bar F(\tau_i,\alpha_i) + \varepsilon
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, $|\int_0^T h(s)dg(s,x_v(s))| \le 2\beta N_1 N_2 \int_0^T |h(s)|d\mu$.

Examples 2 in §3 shows that the constant C_β in the following lemma depends on the size of $\dot u$ and $\dot v$.

LEMMA 2. *Under assumptions (A1) and (A2), there exists a constant $C_\beta > 0$ such that for any $u, v \in U_\beta$,*

(1) $\|\Phi(u,x_u) - \Phi(u,x_v)\|_X \le \frac{1}{4}\|x_u - x_v\|_X$,

(2) $\|\Phi(u,x_v) - \Phi(v,x_v)\|_X \le C_\beta\|u-v\|_{U_\beta}$.

PROOF. Let $u, v \in U_\beta$.

(1)

$$\begin{aligned}
&\|\Phi(u,x_u)-\Phi(u,x_v)\|_X \\
&\quad = \|\int_0^t \big(f(s,x_u(s))-f(s,x_v(s)) \\
&\qquad + (g(s,x_u(s))-g(s,x_v(s)))\dot{u}(s)\big)ds\|_X \\
&\quad \le \frac{e^{-4MT}}{4M}\int_0^\tau (N_2+\beta N_1)|x_u(s)-x_v(s)|ds \\
&\qquad + \int_0^T\int_0^t e^{-4Mt}(N_2+\beta N_1)|x_u(s)-x_v(s)|dsdt \\
&\quad = \frac{e^{-4MT}}{4M}\int_0^\tau (N_2+\beta N_1)|x_u(s)-x_v(s)|ds \\
&\qquad + \int_0^T (N_2+\beta N_1)|x_u(s)-x_v(s)|\left(\frac{e^{-4Ms}}{4M}-\frac{e^{-4MT}}{4M}\right)ds \\
&\quad \le \frac{1}{4}\int_0^T e^{-4Ms}|x_u(s)-x_v(s)|ds+\frac{e^{-4MT}}{16M}|x_u(\tau)-x_v(\tau)| \\
&\quad = \frac{1}{4}\|x_u-x_v\|_X.
\end{aligned}$$

(2) By virtue of (2.4),

$$\begin{aligned}
&\|\Phi(u,x_v)-\Phi(v,x_v)\|_X \\
&\quad = \|\int_0^t g(s,x_v(s))(\dot{u}(s)-\dot{v}(s))ds\|_X \\
&\quad = \frac{e^{-4MT}}{4M}|g(\tau,x_v(\tau))(u(\tau)-v(\tau))-g(0,x_v(0))(u(0)-v(0)) \\
&\qquad - \int_0^\tau (u(s)-v(s))dg(s,x_v(s))| \\
&\qquad + \int_0^T e^{-4Mt}|g(t,x_v(t))(u(t)-v(t))-g(0,x_v(0))(u(0)-v(0))
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^t (u(s) - v(s)) dg(s, x_v(s))|dt \\
&\leq \frac{e^{-4MT}}{4M} \big(N_2 |u(\tau) - v(\tau)| + N_2 |u(0) - v(0)| \\
&+ \int_0^\tau 2\beta N_1 N_2 |u(s) - v(s)| d\mu\big) \\
&+ \int_0^T e^{-4Mt} (N_2 |u(t) - v(t)| + N_2 |u(0) - v(0)| \\
&+ \int_0^t 2\beta N_1 N_2 |u(s) - v(s)| d\mu) dt \\
&\leq e^{-4MT} \big(|u(\tau) - v(\tau)| + |u(0) - v(0)| \\
&+ \int_0^\tau \frac{e^{-4MT}}{4M} 2\beta N_1 N_2 |u(s) - v(s)| d\mu\big) \\
&+ \int_0^T N_2 |u(t) - v(t)| dt + N_2 T |u(0) - v(0)| \\
&+ \int_0^T 2\beta N_1 N_2 |u(s) - v(s)| \left(\frac{e^{-4Ms}}{4M} - \frac{e^{-4MT}}{4M} \right) d\mu \\
&\leq (1 + N_2 T) \|u - v\|_{U_\beta}
\end{aligned}
$$

Depending on Lemma 2, we can prove the following main theorem:

THEOREM 1. *Under assumptions (A1) and (A2), there exists a constant $D_\beta > 0$ such that for any $\tau \in [0, T]$ and any $u, v \in U_\beta$,*

$$
\begin{aligned}
&|x_u(\tau) - x_v(\tau)| + \int_0^T |x_u(t) - x_v(t)| dt \\
&\leq D_\beta \left[|u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^T |u(t) - v(t)| d\mu \right] \quad (2.5)
\end{aligned}
$$

PROOF. By Lemma 2,

$$
\frac{3}{4} \|x_u - x_v\|_X \leq C_\beta \|u - v\|_{U_\beta}
$$

and (2.5) holds with $D_\beta = \dfrac{16M(1+N_2T)e^{4MT}}{3}$.

REMARK. It remains as an open question if the constant D_β in the above theorem is independent of the size of $\dot{u}, \dot{v}$. If it is true, then we can define the generalized solution x_u of (1.1) corresponding to an integrable function u by taking a sequence $\{u_n\}$ of C^1 functions converging to u in $L^1(\mu)$ so that x_{u_n} converges to x_u in $L^1(m)$.

3. Examples

EXAMPLE 1. This example shows that the input-output map σ is not continuous with respect to $L^1(m)$–topology without the assumption of smoothness of g in the variable t, but the map σ from U_β with $L^1(m)$–topology into X with $L^1(\mu)$–topology is continuous. In (1.1), put $T = 2, f \equiv 0$. For each $n \in \mathbb{N}$, let

$$a_n = 1 - 1/n, \qquad b_n = (a_n + a_{n+1})/2, \qquad k_n = n^{10} \qquad m_n = 1/k_n,$$

$$\ell_{1,n} = b_n, \qquad \ell_{2,n} = \ell_{1,n} + m_n, \qquad \ell_{3,n} = \ell_{2,n} + m_n.$$

Let $g(t)$ be a function from $[0,2]$ into $\mathbb{R}$ such that g vanishes on $[1,2]$ and is defined on each interval $[a_n, a_{n+1})$ by

$$g(t) = \begin{cases} 0 & \text{on } [a_n, \ell_{1,n}) \\ n(t - \ell_{1,n}) & \text{on } [\ell_{1,n}, \ell_{2,n}) \\ -n(t - \ell_{3,n}) & \text{on } [\ell_{2,n}, \ell_{3,n}) \\ 0 & \text{on } [\ell_{3,n}, a_{n+1}) \end{cases}.$$

Then $g(t)$ is absolutely continuous and we can take the function $\phi(t)$ in (1.3) as the abolutely continuous function such that $\phi(0) = 0, \phi(t)$ is constant on $[1,2]$ and on each interval $[a_n, a_{n+1})$,

$$\dot{\phi}(t) = \begin{cases} 0 & \text{on } [a_n, \ell_{1,n}) \cup [\ell_{3,n}, a_{n+1}) \\ n & \text{on } [\ell_{1,n}, \ell_{3,n}). \end{cases}$$

To define the sequence $\{u_n\}$ of C^1 functions which does not satisfy (1.2) with $u = u_n$ and $v \equiv 0$, we introduce the points in the interval (a_n, a_{n+1});

$$\alpha_{1,n} = b_n - m_n - 2{m_n}^2, \qquad \alpha_{2,n} = b_n - 2{m_n}^2,$$

$$\alpha_{3,n} = b_n, \quad \text{and} \quad \alpha_{4,n} = \ell_{2,n}.$$

For each $n \in I\!N$, define functions $v_n : [0,2] \to I\!R$ such that each v_n vanishes on $[0, a_n) \cup (a_{n+1}, 2]$ and on $[a_n, a_{n+1}]$

$$v_n(t) = \begin{cases} 0 & \text{on } [a_n, \alpha_{1,n}) \\ 2k_n(t-\alpha_{1,n})^{2k_n-1} & \text{on } [\alpha_{1,n}, \alpha_{2,n}) \\ -2k_n^3 {m_n}^{2k_n-1}\left(t - \dfrac{\alpha_{2,n}+\alpha_{3,n}}{2}\right) & \text{on } [\alpha_{2,n}, \alpha_{3,n}) \\ 2k_n(t-\alpha_{4,n})^{2k_n-1} & \text{on } [\alpha_{3,n}, \alpha_{4,n}) \\ 0 & \text{on } [\alpha_{4,n}, a_{n+1}]. \end{cases}$$

Each v_n is continuous on $[0,2]$. Let u_n be the solution of the initial value problem

$$\dot{u}_n(t) = v_n(t), \qquad u_n(0) = 0$$

Then each u_n is a C^1 function, $|\dot{u}_n(t)| < 1$ for any $t \in [0,2]$ and

$$\begin{aligned} \int_0^2 |u_n(t)|dt &= \int_{\alpha_{1,n}}^{\alpha_{2,n}} u_n(t)dt + \int_{\alpha_{2,n}}^{\alpha_{3,n}} u_n(t)dt + \int_{\alpha_{3,n}}^{\alpha_{4,n}} u_n(t)dt \\ &\le \frac{2{m_n}^{2k_n+1}}{2k_n+1} + 2{m_n}^{2k_n+2} + {m_n}^{2k_n+4} \\ &= {m_n}^{2k_n+1} O(n^{-10}). \end{aligned} \tag{3.1}$$

Let $x_n(t)$ be the solution of the initial value problem

$$\dot{x}(t) = g(t)\dot{u}_n(t), \qquad x(0) = 0.$$

Then

$$\begin{aligned} |x_n(\alpha_{4,n})| &= \left| \int_{\alpha_{3,n}}^{\alpha_{4,n}} n(t-\alpha_{3,n}) 2k_n (t-\alpha_{4,n})^{2k_n-1} dt \right| \\ &= \left| -\frac{2{m_n}^{2k_n+1} k_n n}{4{k_n}^2 + 2k_n} \right| \\ &= {m_n}^{2k_n+1} O(n^{-9}). \end{aligned} \tag{3.2}$$

Since x_n is nonincreasing, for any $L > 0$ there exists $n \in \mathbb{N}$ such that (1.2) does not hold. But

$$\int_0^2 |u_n(t)|d\mu \geq \int_{\alpha_{1,n}}^{\alpha_{2,n}} u_n(t)dt + \int_{\alpha_{1,n}}^{\alpha_{2,n}} u_n(t)d\phi(t) \geq \frac{{m_n}^{2k_n+1}}{2k_n+1}(1+n) \tag{3.3}$$

and

$$\int_0^2 |x_n(t)|dt \leq \frac{{m_n}^{2k_n+1}}{4{k_n}^2 + 2k_n} \cdot 2k_n \cdot 2. \tag{3.4}$$

From (3.3) and (3.4), there exists a constant satisfying (2.5) for $\tau \equiv 0$.

EXAMPLE 2. This example shows that the constant C_β in Lemma 2 depends on the size of $\dot{u}$ and $\dot{v}$. Consider the control system

$$\dot{x} = g(x)\dot{u}(t), \qquad t \in [0,1], \qquad x(0) = 1,$$

where $g(x) = x$. For each $n \in \mathbb{N}$, define the functions $u_n(t)$ and $v_n(t)$ on $[0,1]$ such that both of them are periodic functions with period $1/n$ and on $[0,1/n]$

$$v_n(t) = \begin{cases} nt & t \in [0,1/2n) \\ -n(t-1/n) & t \in [1/2n,1/n], \end{cases}$$

$$u_n(t) = \begin{cases} -nt & t \in [0,1/4n) \\ 3n(t-1/4n) - 1/4 & t \in [1/4n,3/4n) \\ -5n(t-1/n) & t \in [3/4n,1/n]. \end{cases}$$

Then u_n and v_n are differentiable almost everywhere, uniformly bounded and satisfy that $v_n \geq u_n$ on $[0,1/2n]$ and $v_n \leq u_n$ on $[1/2n,1/n]$. Denote $x_n = x_{v_n}$. Now, let us compare

$$\int_0^1 \left| \int_0^t g(x_n(s))(\dot{v}_n(s) - \dot{u}_n(s))ds \right| dt$$

$$\left(= \int_0^1 |\Phi(v_n,x_n)(s) - \Phi(v_n,x_n)(s)|ds\right)$$

and

$$\int_0^1 |u_n(t) - v_n(t)|dt.$$

To compute the first one, simplify the integrand by

$$\begin{aligned}\int_0^t g(x_n(s))(\dot v_n(s) - \dot u_n(s))ds &= g(x_n(t))(v_n(t) - u_n(t)) \\ &\quad - \int_0^t (v_n(s) - u_n(s))D_x g(x_n(s))\dot x_n(s)ds \\ &= x_n(t)(v_n(t) - u_n(t)) - \int_0^t (v_n(s) - u_n(s))x_n(s)\dot v_n(s)ds.\end{aligned}$$

Thus

$$\begin{aligned}&\int_0^1 |\int_0^t g(x_n(s))(\dot v_n(s) - \dot u_n(s))ds|dt \\ &\quad \geq \int_0^1 \int_0^t n|v_n(s) - u_n(s)|dsdt - \int_0^1 e^{1/2}|v_n(t) - u_n(t)|dt \quad (3.5)\end{aligned}$$

On the other hand

$$\begin{aligned}&\int_0^1 \int_0^t |v_n(s) - u_n(s)|dsdt \\ &\quad = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_0^t |v_n(s) - u_n(s)|dsdt \\ &\quad = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} ((i-1)A + \int_{\frac{i-1}{n}}^t |v_n(s) - u_n(s)|ds)dt \\ &\quad = \frac{A}{n}\frac{n(n-1)}{2} + n\int_0^{\frac{1}{n}} \int_0^t |v_n(s) - u_n(s)|dsdt \\ &\quad = \frac{3(n-1)}{16n} + \frac{5}{32n}\end{aligned}$$

where $A = \int_0^{\frac{1}{n}} |v_n(s) - u_n(s)|ds = \frac{3}{8n}$. From (3.5),

$$\int_0^1 |\int_0^t g(x_n(s))(\dot v_n(s) - \dot u_n(s))ds|dt \geq \frac{3(n-1)}{16} + \frac{5}{32} - e^{1/2}\frac{3}{8}.$$

Since $\int_0^1 |v_n(s) - u_n(s)|ds = \frac{3}{8}$, for any $L > 0$, there exists $n \in \mathbb{N}$ such that

$$\int_0^1 \left| \int_0^t g(x_n(s))(\dot{v}_n(s) - \dot{u}_n(s))ds \right| dt > L \int_0^1 |v_n(s) - u_n(s)|ds.$$

Approximating u_n and v_n by C^1 functions, we can conclude that the constant C_β in Lemma 2.2 depends on β.

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