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ON THE ITERATION OF HOLOMORPHIC MAPPINGS IN $\mathbb{C}^2$

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ABSTRACT. Let F be a germ of analytic transformation from $(\mathbb{C}^2, O)$ to $(\mathbb{C}^2, O)$. Let a, b denote the eigenvalues of $DF(O)$. O is called a semi-attractive fixed point if $|a| = 1,\ 0 < |b| < 1$ (or $|b| = 1,\ 0 < |a| < 1$). O is called a super-attractive fixed point if $a = 0,\ b = 0$. We discuss such a mapping from the point of view of dynamical systems.

1. Introduction

We will consider a germ of analytic transformation F from $(\mathbb{C}^2, O)$ to $(\mathbb{C}^2, O)$, i.e., a holomorphic map defined in a neighborhood of the origin in $\mathbb{C}^2$ which leaves the origin $O = (0,0)$ of $\mathbb{C}^2$ fixed. Let a, b denote the eigenvalues of $DF(O)$. O is called a *semi-attractive* fixed point if $|a| = 1,\ 0 < |b| < 1$ (or $|b| = 1,\ 0 < |a| < 1$). O is said to be *super-attractive* if both of the eigenvalues of the Jacobian matrix at the origin, $DF(O)$, are zero. We discuss such a mapping from the point of view of dynamical systems. That is, we will be mainly concerned with the behaviour of the points in the vicinity of the fixed point O under the iterates $\{ F,\ F^{\circ 2} = F \circ F,\ \cdots,\ F^{\circ n},\ \cdots \}$. For the case of semi-attractive transformations, the dynamics of analogous mapping in one complex variable may be written with a convergent power series in x as

$$F(x) = x(1 + a_1 x + a_2 x^2 + \cdots)$$

and has been studied by Fatou and Leau. Their theory is quite complete (see [1], [5]). Voronin[8] showed that a map of the form $x \mapsto x(1 + x^k +$

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$\cdots$) is formally conjugate to $x \mapsto x(1 + x^k + \beta x^{2k})$, and the number $\beta \in \mathbb{C}$ is the only invariant in terms of formal power series. Ueda[6,7] studied the analytic transformation $(\mathbb{C}^2, O)$ with eigenvalues $\{1, b\}$ at O, such that $0 < |b| < 1$. He calls a classification $\{(1, b)_k\}$, for k integer, $1 \leq k \leq \infty$, on these transformations. Ueda concentrated his work on the case $(1, b)_1$. In Sections 2 and 3, it will be treated for *semi-attractive* transformations of type $(a, b)_k$ where $a^p = 1$, $0 < |b| < 1$.

In general, it is not possible to find an analytic change of coordinates around the super-attractive fixed point which transforms the dynamical system into a "simple normal form", for example, $(x, y) \mapsto (x^2, y^2)$. In Section 4, we describe a class of dynamical systems which can be "normalized" by an analytic change of coordinates into the simplest normal form.

2. Reduced forms of semi-attractive transformations

Let us consider a semi-attractive germ F of transformation of $(\mathbb{C}^2, O)$ with eigenvales a, b where $a^p = 1$, $0 < |b| < 1$. Let $E_1 \oplus E_2$ be the Jordan decomposition of $\mathbb{C}^2$ in characteristic subspaces. Here E_1 is associated to the eigenvalue a and E_2 to the eigenvalue b. There exists an analytic stable submanifold X attracted by O and tangent to E_2 (see [4] for the proof). Then a coordinate system (x, y) can be chosen in such a way that X is $\{x = 0\}$ and the matrix $DF(O)$ is triangular. With respect to this coordinate system, F has the form

$$(2.1) \qquad \begin{cases} x_1 = a a_1(y) x + a_2(y) x^2 + \cdots \\ y_1 = by + xh(x, y) \end{cases}$$

where $\{a_j(\cdot)\}$, $j = 2, \cdots$ and $h(\cdot, \cdot)$ are respectively germs of holomorphic functions from $(\mathbb{C}^1, 0)$ to $\mathbb{C}$, $(\mathbb{C}^1, 0)$ to $\mathbb{C}^1$, with $h(0,0) = 0$.

PROPOSITION 2.1. *Let F be a semi-attractive germ of transformation of $(\mathbb{C}^2, O)$ with eigenvalues a, b such that $a^p = 1$, $0 < |b| < 1$. For every integer m, there exists coordinates (x, y) in which the transformation has the form*

$$(2.2) \qquad \begin{cases} x_1 = ax + a_2 x^2 + \cdots + a_m x^m + a_{m+1}(y) x^{m+1} + \cdots \\ y_1 = by + xh(x, y) \end{cases}$$

as in (2.1), but with $a_1(y) = 1$, $a_2, \cdots a_m$ constants.

PROOF. We start with

$$\begin{cases} x_1 = aa_1(y)x + a_2(y)x^2 + \cdots \\ y_1 = by + xh(x,y) \end{cases}$$

and we proceed inductively on m.
1) Reduction to $a_1(y) \equiv 1$. We use the coordinate system

$$\begin{cases} X = u(y)x \\ Y = y \end{cases} \qquad \begin{cases} x = X/u(Y) \\ y = Y \end{cases}$$

where $u(y)$ is a germ of analytic function from $(\mathbb{C}^1,\ 0)$ to $\mathbb{C}$ such that $u(0) = 1$, to be chosen.
We want

$$\begin{aligned} X_1 &= u(y_1)x_1 = u(by + xh(x,y))[aa_1(y)x + a_2(y)x^2 + \cdots] \\ &= u(bY + \cdots)[a_1(y)a \cdot X/u(Y) + \cdots] \\ &= \frac{a_1(Y)u(bY)}{u(Y)} aX + O(X^2) = aX + O(X^2). \end{aligned}$$

So we have to choose u such that

$$\begin{aligned} u(Y) &= a_1(Y)u(bY) \\ u(bY) &= a_1(bY)u(b^2Y) \\ &\cdots \\ u(b^nY) &= a_1(b^nY)u(b^{n+1}Y). \end{aligned}$$

This gives for u the unique solution

$$u(Y) = \prod_{n=0}^{\infty} a_1(b^nY).$$

The infinite product is convergent in a neighborhood of 0 since $a_1(0) = 1$ and $|b| < 1$.

2) Suppose that for $m \geqslant 2$, with some coordinates (x, y), F takes the form

$$\begin{cases} x_1 = ax + a_2x^2 + \cdots + a_{m-1}x^{m-1} + a_m(y)x^m + \cdots \\ y_1 = by + xh(x,y) \end{cases}$$

with the a_j's constant for $1 \leqslant j \leqslant m-1$. We then use a coordinate transformation

$$\begin{cases} X = x + v(y)x^m \\ Y = y \end{cases} \quad \text{or} \quad \begin{cases} x = X - v(Y)X^m + \cdots \\ y = Y \end{cases}$$

with $v(y)$ a holomorphic function in a neighborhood of 0 in $\mathbb{C}$ such that $v(0) = 0$, v to be chosen. We get

$$\begin{aligned} X_1 &= x_1 + v(y_1)x_1^m \\ &= ax + a_2x^2 + \cdots + a_{m-1}x^{m-1} + a_m(y)x^m + v(bY)x^m + O(x^{m+1}) \\ &= aX - av(Y)X^m + a_2X^2 + \cdots + a_{m-1}X^{m-1} + a_m(y)X^m \\ &\quad + v(bY)X^m + O(X^{m+1}). \end{aligned}$$

So we need that

$$\begin{aligned} av(Y) - v(bY) &= a_m(y) - a_m(0) \\ av(bY) - v(b^2Y) &= a_m(by) - a_m(0) \\ &\cdots \\ av(b^nY) - v(b^{n+1}Y) &= a_m(b^ny) - a_m(0). \end{aligned}$$

The unique solution is then

$$v(y) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} a^{p-n}\{a_m(b^ny) - a_m(0)\} & \text{if} \quad p > 0 \\ \displaystyle\sum_{n=0}^{\infty} \{a_m(b^ny) - a_m(0)\} & \text{if} \quad p = 0. \end{cases}$$

The series converges in a neighborhood of 0 since $0 < |b| < 1$ and $a_m(y) - a_m(0) = 0$ for $y = 0$. □

Let us write again $F(x,y) = (x_1, y_1)$ as

$$(2.3)\qquad \begin{cases} x_1 = ax(1 + a_n x^n + a_{n+1}(y)x^{n+1} + \cdots), \ a_n \neq 0 \\ y_1 = by + xh(x,y). \end{cases}$$

PROPOSITION 2.2. *Let F be a semi-attractive germ of transformation of $(\mathbb{C}^2, O)$ with eigenvalues a, b such that $a^p = 1$, $0 < |b| < 1$. Then the transformation can be written in some coordinates (x,y)*

$$(2.4)\qquad \begin{cases} x_1 = ax(1 + a_{kp}x^{kp} + a_{kp+1}(y)x^{kp+1} + \cdots) \\ y_1 = by + xh(x,y) \end{cases}$$

for some positive integer k.

PROOF. Assume then that the transformation is written in the form (2.3). Consider the following holomorphic change of coordinates.

$$\begin{cases} X = x(1 - \alpha x^n) \\ Y = y \end{cases} \qquad\qquad \begin{cases} x = X(1 + \alpha X^n) + O(X^{n+2}) \\ y = Y. \end{cases}$$

We get

$$\begin{aligned} X_1 &= x_1(1 - \alpha x_1^n) \\ &= ax(1 + a_n x^n + a_{n+1}(y)x^{n+1} + \cdots) \\ &\qquad (1 - \alpha a^n x^n (1 + a_n x^n + a_{n+1}(y)x^{n+1} + \cdots)^n) \\ &= aX(1 + \alpha X^n)(1 + a_n X^n (1 + \alpha X^n)^n) \\ &\qquad (1 - \alpha a^n X^n (1 + \alpha X^n)^n (1 + a_n X^n (1 + \alpha X^n)^n)) + O(X^{n+2}) \\ &= aX(1 + \alpha X^n)(1 + a_n X^n)(1 - \alpha a^n X^n) + O(X^{n+2}) \\ &= aX(1 + (\alpha(1 - a^n) + a_n)X^n) + O(X^{n+2}) \\ &= aX(1 + a_n' X^n) + O(X^{n+2}). \end{aligned}$$

So we can solve for α if $n \neq kp$ for some positive integer k. We repeat this process inductively. □

PROPOSITION 2.3. *Let F be a semi-attractive germ of transformation of $(\mathbb{C}^2, O)$ of the form (2.4). Then the transformation can be written in some coordinates (x, y)*

$$\begin{cases} x_1 = ax(1 + x^{kp} + Cx^{2kp} + a_{2kp+1}(y)x^{2kp+1} + \cdots) \\ y_1 = by + xh(x, y) \end{cases}$$

with C a constant.

PROOF. We can suppose that F is in the form by Proposition 2.2

$$\begin{cases} x_1 = ax(1 + a_{kp}x^{kp} + a_{kp+1}x^{kp+1} + \cdots) \\ y_1 = by + xh(x, y) \end{cases}$$

with $a_{kp} \neq 0$. By a linear change of coordinates one can assume $a_{kp} = 1$. Now we use the coordinate transformation

$$\begin{cases} X = x(1 + c_n x^n) \\ Y = y \end{cases} \quad \text{or} \quad \begin{cases} x = X(1 - c_n X^n + O(X^{2n})) \\ y = Y. \end{cases}$$

Then we have

$$\begin{aligned} X_1 &= x_1(1 + c_n x_1^n) \\ &= ax(1 + a_{kp}x^{kp} + \cdots)(1 + c_n a^n x^n (1 + a_{kp}x^{kp} + \cdots)^n) \\ &= aX(1 - c_n X^n + O(X^{2n}))(1 + a_{kp}X^{kp}(1 - c_n X^n + O(X^{2n}))^{kp} + \cdots) \\ &\quad (1 + c_n a^n X^n (1 - c_n X^n + O(X^{2n}))^n (1 + a_{kp}X^{kp}(1 - c_n X^n + O(X^{2n}))^{kp} \\ &\quad + \cdots + a_{kp+n}X^{kp+n}(1 - c_n X^n + O(X^{2n}))^{kp+n} + \cdots)^n) \\ &= aX(1 + \sum_{i=kp}^{kp+n-1} a_i X^i + (a_{kp+n} - (kp - na^n)a_{kp}c_n)X^{kp+n} + O(X^{kp+n+1})). \end{aligned}$$

By taking $c_n = a_{kp+n}/a_{kp}(kp - na^n)$, $n \neq kp$, we have the desired result. □

3. Existence of attracting domains

In this section, we want to investigate the existence of attracting domains at $O = (0,0)$ in a neighborhood of O. As the partial derivative $|\frac{\partial}{\partial x_1} F_1^{\circ n}(O)| = 1$ in some coordinate system, the family $\{F^{\circ n}\}$ cannot converge to O in a neighborhood of O. So by attracting domains in a neighborhood of O, we mean open domains D with $O \in \partial D$ such that $x_n = F^{\circ n}(x)$ converge to O for $x \in D$. For the case of a semi-attractive invertible germ of $(\mathbb{C}^2,\ O)$ with $a = 1$, Ueda [6,7] showed the existence of attracting domains. The following theorem can be considered as a generalization of it. We will use a Fatou's method simplified here by using the reduced form for F which gives easily the Abel-Fatou invariant functions.

THEOREM 3.1. *Let F be a semi-attractive germ of transformation of $(\mathbb{C}^2, O)$ with engenvalues a, b such that $a^p = 1$, $0 < |b| < 1$. There exists an attracting domain with kp petals for some positive integer k.*

PROOF. Suppose that F is in the form by Proposition 2.3.

$$\begin{cases} x_1 = ax(1 + a_{kp}x^{kp} + a_{2kp}x^{2kp} + a_{2kp+1}x^{2kp+1} + \cdots) \\ y \ = by + xh(x,y) \end{cases}$$

with $a_{kp} \neq 0$. By a linear change of coordinate, we may assume $a_{kp} = -\frac{1}{kp}$.

Let R and ρ be positive constants to be adjusted later. The half complex-plane P_R and the subset $V_{R,\rho}$ of $\mathbb{C}^2$ is defined by

$$\begin{aligned} P_R &= \{X \in \mathbb{C} : \mathrm{Re} X \geqslant R\} \\ V_{R,\rho} &= \{(X,y) \in \mathbb{C}^2 : X \in P_R,\ |y| < \rho\}. \end{aligned} \tag{3.1}$$

Let D_R and $U_{R,\rho}$ be the images of P_R and $V_{R,\rho}$ by the inversion $z = \frac{1}{X}$. Then we have

$$\begin{aligned} D_R &= \{z \in \mathbb{C} : |z - \frac{1}{2R}| < \frac{1}{2R}\} \\ U_{R,\rho} &= \{(z,y) \in \mathbb{C}^2 : z \in D_R,\ |y| < \rho\}. \end{aligned}$$

There are kp branches of $z^{\frac{1}{kp}}$ in D_R. Let $\{\Delta_{Rj}\}_{0\leqslant j\leqslant kp-1}$ be the images of D_R by these determinations. We will show that, for R big enough and ρ small enough, the domains

$$(3.2)\qquad W_{R,\rho,j}=\{(x,y)\in\mathbb{C}^2 : x\in\Delta_{Rj},\ |y|<\rho\},\ 0\leqslant j\leqslant kp-1$$

are attracting domains.

Raising the relation

$$x_1=ax(1-\frac{1}{kp}x^{kp}+a_{2kp}x^{2kp}+\cdots)$$

to the power kp, we get

$$\begin{aligned}x_1^{kp}&=a^{kp}x^{kp}(1-\frac{1}{kp}x^{kp}+a_{2kp}x^{2kp}+\cdots)^{kp}\\&=x^{kp}(1-x^{kp}+c_{2kp}x^{2kp}+c_{2kp+1}(y)x^{2kp+1}+\cdots)\\y_1&=by+xh(x,y).\end{aligned}$$

We then restrict (x,y) to a $W_{R,\rho,j}$ for fixed R,ρ,j, and we make the transformations

$$(z=x^{kp},\ y=y)\quad\text{from}\quad W_{R,\rho,j}\quad\text{to}\quad U_{R,\rho}$$

and

$$(X=\frac{1}{z},\ y=y)\quad\text{from}\quad U_{R,\rho}\quad\text{to}\quad V_{R,\rho}.$$

For R big enough and ρ small enough, the transformation F is defined in $V_{R,\rho}$, where we get

$$\begin{aligned}X_1&=\frac{X}{1-x^{kp}+c_{2kp}x^{2kp}+c_{2kp+1}(y)x^{2kp+1}+\cdots}\\&=X(1+\frac{1}{X}+c\frac{1}{X^2}+O_y(\frac{1}{|X|^{2+\frac{1}{kp}}})).\end{aligned}$$

Therefore F becomes

$$\begin{cases}X_1=X+1+c\dfrac{1}{X}+O_y(\dfrac{1}{|X|^{1+\frac{1}{kp}}})\\[2ex] y_1=by+xh(x,y)=by+O_y(\dfrac{1}{|X|^{\frac{1}{kp}}}).\end{cases}$$

Here the notation $O_y(\frac{1}{|X|^\alpha})$ represents a holomorphic function of (X, y) in $V_{R,\rho}$ which is bounded by $\frac{K}{|X|^\alpha}$ for some constant K.

Let K be a constant such that

$$(3.3) \qquad \begin{cases} |X_1 - X - 1| \leqslant \dfrac{K}{|X|} \leqslant \dfrac{K}{R} \\ |y_1 - by| \leqslant \dfrac{K}{|X|^{\frac{1}{kp}}} \leqslant \dfrac{K}{R^{\frac{1}{kp}}} \end{cases}$$

in $V_{R,\rho}$.

Let R be a sufficiently large number such that

$$\frac{K}{R} < \frac{1}{2} \qquad \text{and} \qquad \frac{K}{R^{\frac{1}{kp}}} < (1 - |b|)\rho.$$

This condition implies $\mathrm{Re}X_1 \geqslant \mathrm{Re}X + \frac{1}{2}$ and $|y_1| \leqslant |y| \leqslant \rho$. Thus $V_{R,\rho}$ is mapped to itself.

In order to prove that $W_{R,\rho,j}$ is attracted by 0, it is enough to show that $V_{R,\rho}$ is attracted by $(\infty, 0)$. We see inductively that

$$\mathrm{Re}X_n \geqslant R + \frac{n}{2}.$$

Let C be a constant big enough to have $C \geqslant \frac{2K}{1-|b|}$ and $\rho \leqslant \frac{C}{R^{\frac{1}{kp}}}$. We prove by induction that if R is big enough, we have

$$|y_n| \leqslant \frac{C}{(R + \frac{n}{2})^{\frac{1}{kp}}}.$$

The inequality

$$(3.4) \qquad (\frac{R + \frac{1}{2}}{R})^{\frac{1}{kp}} \leqslant \frac{C}{bC + K}$$

holds because we see that $\dfrac{C}{|b|C+K} \geqslant \dfrac{2}{1+|b|} > 1$ from $C \geqslant \dfrac{2K}{1-|b|}$. So that (3.4) is true if R is big enough. Since

$$|y_{n+1}| \leqslant |b||y_n| + \frac{K}{|X_n|^{\frac{1}{kp}}} \leqslant \frac{bC+K}{(R+\frac{n}{2})^{\frac{1}{kp}}},$$

the inequality $|y_{n+1}| \leqslant \dfrac{C}{(R+\frac{n+1}{2})^{\frac{1}{kp}}}$ will be satisfied by (3.4).

We have now kp disjoint domains attracted by O. Each of them is positively invariant by F since $V_{R,\rho}$ is positively invariant. Furthermore since $x_{n+1} \sim x_n$ when $n \to \infty$, we have always the same branch of $x^{\frac{1}{kp}}$. Let D be the attracting domain of O. Then we want to prove if $\zeta \in D$, for n big enough, $\zeta_n = (x_n, y_n)$ is in one of the $W_{R,\rho,j}$'s, or equivalently that (x_n^{kp}, y_n) is in $U_{R,\rho}$, or that $(\dfrac{1}{x_n^{kp}}, y_n)$ is in $V_{R,\rho}$. But $y_n \to 0$ and we have

$$\frac{1}{x_1^{kp}} = \frac{1}{x^{kp}} + 1 + cx^{kp} + O_y(|x|^{1+\frac{1}{kp}}),$$

so $\mathrm{Re}\frac{1}{x_n^{kp}} \to \infty$ when $x_n \to 0$. So ζ belongs to the union of the increasing sequence of open sets

$$D_j = \bigcup_{n=0}^{\infty} F^{\circ -n}(W_{R,\rho,j}). \quad \square$$

4. Super-attractive fixed point

In this section, we consider the local behaviour of holomorphic mappings with a super-attractive fixed points. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex holomorphic mapping. A point $p \in \mathbb{C}$ is called a super-attractive fixed point if $f(p) = p$ and $f'(p) = 0$. If f is not a constant function, the classical Böttcher's theorem asserts that f is holomorphically conjugate to the map $z \mapsto z^k$ for some integer $k > 1$ in a neighborhood of p (see [5] for the proof).

Let us consider a complex 2-dimensional dynamical system $F : \mathbb{C}^2 \to \mathbb{C}^2$. Assume F is holomorphic in a neighborhood of the origin, $O = (0,0)$, and that the origin is a fixed point of F, i.e., $F(O) = O$. Futhermore, we assume both of the eigenvalues of the Jacobian matrix at the origin, $DF(O)$, are zero. Such a fixed point is said to be super-attractive. Hubbard and Papadopol [3] studied the case of super-attractive fixed points for homogeneous polynomial maps and their perturbations.

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be holomorphic in a neighborhood of the origin, $O = (0,0)$. Suppose that the origin is a fixed point of F, i.e., $F(O) = O$. Let

$$F(x,y) = (f_1(x,y), f_2(x,y))$$

We assume that the x-axis, $\{(x,0)\}$, and the y-axis, $\{(0,y)\}$ are invariant under F, i.e.,

$$f_2(x,0) = 0 \qquad \text{and} \qquad f_1(0,y) = 0$$

holds for all x and y near the origin. We assume

$$f_1(x,0) = x^k + \text{h.o.t.}, \qquad f_2(0,y) = y^p + \text{h.o.t.}$$

where $k, p \geqslant 2$. Moreover, we assume $\det(DF) = 0$ along the x-axis and the y-axis.

Under the assumptions above, we can apply the Bötcher's theorem to normalize the mapping on the x-axis and the y-axis respectively. We can rewrite the mapping F in the form

$$f_1(x,y) = x^k(1 + yg_1(x,y))$$
$$f_2(x,y) = y^p(1 + xg_2(x,y))$$

in a neighborhood of the origin, where $g_1(x,y)$ and $g_2(x,y)$ are holomorphic in the neighborhood of the origin. Let $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ denote the "normal form" mapping $\Psi(x,y) = (x^k, y^p)$.

THEOREM 4.1. *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be holomorphic mapping defined near the origin. Suppose F is of the form*

$$F(x,y) = (x^k(1 + yg_1(x,y)), y^p(1 + xg_2(x,y)))$$

where $k,\ p \geqslant 2$, and $g_1(x,y)$ and $g_2(x,y)$ are holomorphic near the origin. Then there exists a holomorphic change of coordinates $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ around the origin with

$$\Phi(0,0) = (0,0), \qquad D\Phi(O) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

such that

$$\Phi \circ F = \Psi \circ \Phi$$

holds in a neighborhood of the origin, where $\Psi(x,y) = (x^k, y^p)$.

PROOF. We will prove Theorem 4.1 for the case $k = 2,\ p = 2$. The same method holds for higher degrees of k and p.
Since

$$DF(O) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

there exists a neighborhood $U \subset \mathbb{C}^2$ of the origin, satisfying closure $(F(U)) \subset U$ and that for any $(x,y) \in U$, $\lim_{n\to\infty} F^{\circ n}(x,y) = O$ holds. Moreover, we can assume

$$|yg_1(x,y)| < \frac{1}{2}, \qquad |xg_2(x,y)| < \frac{1}{2}$$

for all $(x,y) \in U$. We shall denote the components of $F^{\circ n}$ as

$$F^{\circ n}(x,y) = (F_1^{\circ n}(x,y), F_2^{\circ n}(x,y)) = (x_n, y_n).$$

First, let us construct the first component Φ_1 of Φ. Let $\varphi_0(x,y) = x$ and define $\varphi_n(x,y) : U \to \mathbb{C}$ by

$$\varphi_n(x,y) = (F_1^{\circ n}(x,y))^{\frac{1}{2^n}}$$

for $n = 1, 2, \cdots$. Here, we choose the branch of the right hand side satisfying $\dfrac{\partial \varphi_n}{\partial x}(O) = 1$. As F maps the y-axis into itself, φ_n is holomorphic in the neighborhood. Let us verify that φ_n converges uniformly in U. We see

$$\frac{\varphi_{n+1}(x,y)}{\varphi_n(x,y)} = \frac{(F_1^{\circ(n+1)}(x,y))^{\frac{1}{2^{n+1}}}}{(F_1^{\circ n}(x,y))^{\frac{1}{2^n}}} = \left(\frac{(f_1(F^{\circ n}(x,y)))^{\frac{1}{2}}}{F_1^{\circ n}(x,y)}\right)^{\frac{1}{2^n}}$$
$$= \left(\frac{(f_1(x_n,y_n))^{\frac{1}{2}}}{x_n}\right)^{\frac{1}{2^n}} = \left(\frac{(x_n^2(1+y_n g_1(x_n,y_n)))^{\frac{1}{2}}}{x_n}\right)^{\frac{1}{2^n}}$$
$$= (1+y_n g_1(x_n,y_n))^{\frac{1}{2^{n+1}}}.$$

As $|yg_1(x,y)| < \dfrac{1}{2}$ holds in the neighborhood U,

$$\varphi_{n+1}(x,y) = x\prod_{j=0}^{n}(1+y_j g_1(x_j,y_j))^{\frac{1}{2^{j+1}}}$$

is uniformly convergent in U, where $(x_0,y_0)=(x,y)$. Hence, by setting

$$\lim_{n\to\infty}\varphi_n = \Phi_1,$$

Φ_1 is holomorphic in U and satisfies the function equation

$$\Phi_1 \circ F = \Phi_1^2.$$

Similarly, the second component Φ_2 can be defined. Therefore by setting

$$\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y)),$$

the function equation

$$\Phi\circ F = \Psi\circ\Phi$$

holds near the origin. $\square$

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