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ON M-IDEAL PROPERTIES OF CERTAIN SPACES OF COMPACT OPERATORS

CHONG-MAN CHO AND BEOM SOOL KIM

ABSTRACT. It is proved that $K(c_0, Y)$ is an M-ideal in $L(c_0, Y)$ if Y is a closed subspace of c_0. And a new direct proof of the fact that $K(L_1[0,1], \ell_1)$ is not an M-ideal in $L(L_1[0,1], \ell_1)$ is given.

1. Introduction and Preliminary

A closed subspace J of a Banach space X is called an *L-summand* if there exists a closed subspace J' of X so that X is an algebraic direct sum of J and J', and if $j \in J$ and $j' \in J'$ then

$$\|j + j'\| = \|j\| + \|j'\|.$$

A closed subspace J of a Banach space X is called an *M-ideal* in X if $J^{\perp} = \{x^* \in X^* : x^*(j) = 0 \text{ for all } j \in J\}$, the anihilator of J in X^*, is an L-summand in X^*.

Since the notion of an M-ideal in a Banach space was introduced by Alfsen and Effros [1], many authors have studied the problem determining those Banach spaces X and Y for which $K(X, Y)$, the space of compact linear operators from X to Y, is an M-ideal in $L(X, Y)$, the space of bounded linear operators from X to Y [3, 7, 8, 9, 11, 14, 15, 16]. It is well known that if X is a Hilbert space, ℓ_p $(1 < p < \infty)$ or c_0, then $K(X)(=K(X,X))$ is an M-ideal in $L(X)(=L(X,X))$ [4, 7, 15] while $K(\ell_1)$ and $K(\ell_\infty)$ are not M-ideals in the corresponding space of operators [15]. Also several authors proved that $K(\ell_p, \ell_q)$ for $1 < p \le q < \infty$ is an M-ideal in $L(\ell_p, \ell_q)$ [6, 11, 14] and $K(X, c_0)$ is an M-ideal in $L(X, c_0)$

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for every Banach space X [14, 15]. However, in the case of $K(c_0, X)$ in $L(c_0, X)$ the situation is different. It is known that $K(c_0, \ell_\infty)$ is not an M-ideal in $L(c_0, \ell_\infty)$ [14] and as we will see presently $K(c_0, Y)$ is an M-ideal in $L(c_0, Y)$ for some Banach space Y.

The aim of this article is to formulate a necessary and sufficient condition for $K(c_0, Y)$ to be an M-ideal in $L(c_0, Y)$ for a Banach space Y (Theorem 2.1) and prove that $K(c_0, Y)$ is an M-ideal in $L(c_0, Y)$ if Y is a closed subspace of c_0 (Corollary 2.2). We also give a new proof of the result of Cho [2] stating that $K(L_1[0,1], \ell_1)$ is not an M-ideal in $L(L_1[0,1], \ell_1)$ (Theorem 2.6).

Alfsen and Effros [1], and Lima [10] characterized an M-ideal by various intersection properties of balls. In particular, Lima [10, Theorem 6.17] proved the following useful theorem.

THEOREM 1.1. *A closed subspace J of a Banach space X is an M-ideal in X if and only if for all $x \in X$ with $\|x\| \le 1$, for all $y_1, y_2, y_3 \in J$ with $\|y_j\| \le 1$ ($j = 1, 2, 3$), and all $\varepsilon > 0$, there exists $y \in J$ such that*

$$\|x + y_i - y\| < 1 + \varepsilon \quad (i = 1, 2, 3).$$

In 1993, Kalton and Werner [9] established a necessary and sufficient condition for $K(X, Y)$ to be an M-ideal in $L(X, Y)$ for Banach spaces X and Y. More specifically, they proved the following theorem.

THEOREM 1.2 [9]. *Suppose that X is a Banach space such that there exists a sequence $\{K_n\}_{n=1}^{\infty}$ in $K(X)$ satisfying*

(i) $K_n \to I_X$ *strongly,*
(ii) $K_n^* \to I_{X^*}$ *strongly,*
(iii) $\|I_X - 2K_n\| \to 1$,

where I_X and I_{X^*} denote the identity maps on X and X^*, respectively. If Y is a Banach space, then $K(X, Y)$ is an M-ideal in $L(X, Y)$ if and only if every $T \in L(X, Y)$ with $\|T\| \le 1$ has property (M).

According to Kalton and Werner [9] a continuous linear operator T with $\|T\| \le 1$ from a Banach space X to a Banach space Y is said to have property (M) if

$$\limsup_{n\to\infty} \|y + Tx_n\| \le \limsup_{n\to\infty} \|x + x_n\|$$

for all $x \in X$, $y \in Y$ with $\|y\| \leq \|x\|$ and all weakly null sequences $\{x_n\}_{n=1}^{\infty}$ in X.

Dualizing property (M), we say that a contractive operator $T : X \to Y$ has property (M*) if

$$\limsup_{n\to\infty} \|x^* + T^* y_n^*\| \leq \limsup_{n\to\infty} \|y^* + y_n^*\|$$

for all $x^* \in X^*$, $y^* \in Y^*$ with $\|x^*\| \leq \|y^*\|$ and all weak* null sequence $\{y_n^*\}_{n=1}^{\infty}$ in Y^*.

2. Results

Kalton and Werner [9, Corollary 2.4] proved that for $1 < p < \infty$ and for a Banach space Y every $T \in L(\ell_p, Y)$ with $\|T\| \leq 1$ has property (M) if and only if for every $y \in Y$ and every $T \in L(\ell_p, Y)$ the inequality

$$\limsup_{n\to\infty} \|y + Te_n\| \leq \left(\|y\|^p + \|T\|^p\right)^{1/p}$$

holds, where $\{e_n\}_{n=1}^{\infty}$ is the unit vector basis for ℓ_p.

In the case of $L(c_0, Y)$, we have an analogous result which is formulated in the following theorem. It is not hard to see that the Kalton-Werner's proof in $L(\ell_p, Y)$ case [9, Corollary 2.4] adapted to our case $L(c_0, Y)$ still works. However, for the completeness we include the proof in the theorem.

THEOREM 2.1. *Suppose that Y is a Banach space. Then $K(c_0, Y)$ is an M-ideal in $L(c_0, Y)$ if and only if for every $y \in Y$ and every $T \in L(c_0, Y)$ the inequality*

$$(*) \qquad \limsup_{n\to\infty} \|y + Te_n\| \leq \max\{\|y\|, \|T\|\}$$

holds, where $\{e_n\}_{n=1}^{\infty}$ is the unit vector basis of c_0.

PROOF. In view of Theorem 1.2 it suffices to show that every contractive operator in $L(c_0, Y)$ has property (M) if and only if every $T \in L(c_0, Y)$ satisfies the inequality $(*)$.

Suppose that every contraction in $L(c_0, Y)$ has property (M). Let $x \in c_0$, $y \in Y$ with $\|x\| = \|y\|$ and let $T \in L(c_0, Y)$. Without loss of generality we may assume that $\|T\| = 1$ in $(*)$. Since T has property (M) and $e_n \to 0$ weakly, we have

$$\begin{aligned}\limsup_{n\to\infty} \|y + Te_n\| &\le \limsup_{n\to\infty} \|x + e_n\| \\ &= \max\{\|x\|, 1\} \\ &= \max\{\|x\|, \|T\|\}.\end{aligned}$$

Conversely, suppose that inequality $(*)$ holds for every operator in $L(c_0, Y)$. If $T \in L(c_0, Y)$ with $\|T\| \le 1$ does not have property (M), then there exists a weakly null sequence $\{x_n\}_{n=1}^{\infty}$ in c_0, and $x \in c_0$, $y \in Y$ with $\|y\| \le \|x\|$ such that

$$\limsup_{n\to\infty} \|y + Tx_n\| > \limsup_{n\to\infty} \|x + x_n\|.$$

By passing to subsequences and by multiplying $\{x_n\}_{n=1}^{\infty}$ by a constant we may assume that $\|x_n\| \to 1$ and

$$\alpha = \lim_{n\to\infty} \|y + Tx_n\| > \lim_{n\to\infty} \|x + x_n\| = \max\{\|x\|, 1\}.$$

Since $x_n \to 0$ weakly, by the gliding hump argument for every $\varepsilon > 0$ we can choose a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and an isomorphism $\Phi : c_0 \to \overline{\text{span}\,\{x_{n_k}\}_{k=1}^{\infty}}$ such that $\Phi(e_k) = x_{n_k}$ for all k and

$$(1-\varepsilon)\|x\| \le \|\Phi(x)\| \le (1+\varepsilon)\|x\|$$

for all $x \in c_0$.

Thus we have

$$\begin{aligned}\alpha &= \lim_{k\to\infty} \|y + Tx_{n_k}\| \\ &= \lim_{k\to\infty} \|y + (T\Phi)e_n\| \\ &\le \max\{\|y\|, 1+\varepsilon\} \\ &\le \max\{\|x\|, 1+\varepsilon\}.\end{aligned}$$

Since $\max\{\|x\|, 1\} < \alpha$ and $\varepsilon > 0$ is arbitrary, we arrive at a contradiction.

As easy and interesting applications of the above theorem we have the following corollaries.

COROLLARY 2.2. *If Y is a closed subspace of c_0, then $K(c_0, Y)$ is an M-ideal in $L(c_0, Y)$.*

PROOF. Let $y \in Y$ and $T \in L(c_0, Y)$. Since $Te_n \to 0$ weakly in $Y \subseteq c_0$, we have

$$\limsup_{n\to\infty} \|y + Te_n\| = \max\left\{\|y\|, \limsup_{n\to\infty} \|Te_n\|\right\}$$
$$\leq \max\{\|y\|, \|T\|\}.$$

By Theorem 2.1, $K(c_0, Y)$ is an M-ideal in $L(c_0, Y)$.

The following corollary is a new simplified proof of the result of Saatkamp stating that $K(c_0, \ell_\infty)$ is not an M-ideal in $L(c_0, \ell_\infty)$ [14]. In his proof, Saatkamp used matrix representations of operators and some other facts. But our new proof is a direct consequence of Theorem 2.1.

COROLLARY 2.3. *$K(c_0, \ell_\infty)$ is not an M-ideal in $L(c_0, \ell_\infty)$.*

PROOF. Let $T : c_0 \to \ell_\infty$ be the canonical embedding. If $\{e_n\}_{n=1}^\infty$ is the unit vector basis of c_0 and $y = (1, 1, 1, \cdots)$, then

$$2 = \limsup_{n\to\infty} \|y + Te_n\| > \max\{\|y\|, \|T\|\} = 1.$$

Therefore, by Theorem 2.1 $K(c_0, \ell_\infty)$ is not an M-ideal in $L(c_0, \ell_\infty)$.

Recently, Cho [2] proved that $K(L_1[0,1], \ell_1)$ is not an M-ideal in $L(L_1[0,1], \ell_1)$ using a property (M*) version of Theorem 1.2. The rest of this section is devoted to a new direct proof of the result of Cho.

LEMMA 2.4. *There exists a norm one projection P on $L_1 (= L_1[0,1])$ with the range $P(L_1)$ isometric to ℓ_1.*

PROOF. We consider the partition $\{I_n : n \in \mathbb{N}\}$ of the interval $[0,1)$, where

$$I_n = \left[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n}\right) \quad (n \geq 1).$$

For every f in L_1 we write $f = \sum_{n=1}^{\infty} f\chi_{I_n}$ and define a projection $P : L_1 \to L_1$ by

$$Pf = \sum_{n=1}^{\infty} P(f\chi_{I_n}) \text{ and } P(f\chi_{I_n}) = \left(\frac{1}{m(I_n)} \int_{I_n} f\right) \chi_{I_n},$$

where χ_{I_n} is the characteristic function of I_n and $m(I_n)$ is the Lebesgue measure of I_n. Since $P(\chi_{I_n}) = \chi_{I_n}$ for each n and $\|Pf\| \leq \|f\|$ for all $f \in L_1$, P is a norm one projection.

Next to see that $P(L_1)$ is isometric to ℓ_1 we define a linear map ψ from $P(L_1)$ to ℓ_1 by linearly extending the map $\frac{1}{m(I_n)}\chi_{I_n} \mapsto e_n$, where $\{e_n\}_{n=1}^{\infty}$ is the unit vector basis of ℓ_1. If $f \in L_1$, then $Pf = \sum_{n=1}^{\infty} P(f\chi_{I_n})$ and so $\psi(Pf) = \left\{\int_{I_n} f\right\}_{n=1}^{\infty}$. Since $\{P(f\chi_{I_n})\}_{n=1}^{\infty}$ is a sequence in L_1 with disjoint supports, we have

$$\begin{aligned} \|Pf\|_{L_1} &= \sum_{n=1}^{\infty} \|P(f\chi_{I_n})\|_{L_1} \\ &= \sum_{n=1}^{\infty} \left|\int_{I_n} f\right| \\ &= \left\|\left\{\int_{I_n} f\right\}_{n=1}^{\infty}\right\|_{\ell_1} \\ &= \|\psi(Pf)\|_{\ell_1}. \end{aligned}$$

Hence ψ is an isometry. If $\{a_n\}_{n=1}^{\infty} \in \ell_1$, then $f = \sum_{n=1}^{\infty} \frac{a_n}{m(I_n)}\chi_{I_n} \in L_1$ and $\psi(Pf) = \psi f = \{a_n\}_{n=1}^{\infty}$. Therefore, ψ is surjective and $P(L_1)$ is isometric to ℓ_1.

LEMMA 2.5. *There exists a norm one projection τ on $L(L_1, \ell_1)$ whose range is isometric to $L(P(L_1), \ell_1)$ ($\cong L(\ell_1)$).*

PROOF. Let P and ψ be mappings constructed above. For each $T : L_1 \to \ell_1$, let $\tau(T) = TP$. Then τ is a projection on $L(L_1, \ell_1)$. Since $\|\tau(T)\| \leq \|T\|$ for all $T \in L(L_1, \ell_1)$, $\tau(\psi P) = \psi P$ and $\|\psi P\| = \|P\| = 1$, $\|\tau\| = 1$

Let ϕ be the linear map from the range $\tau(L(L_1, \ell_1))$ of τ to $L(P(L_1), \ell_1)$ defined by $\phi(TP) = TP|_{P(L_1)}$ for $T \in L(L_1, \ell_1)$. Since P is a norm one projection, ϕ is an isometry. If $S \in L(P(L_1), \ell_1)$, then $SP \in \tau(L(L_1, \ell_1))$ and $\phi(SP) = SP|_{P(L_1)} = S$. Hence ϕ is surjective. Therefore, the range $\tau(L(L_1, \ell_1))$ of τ is isometric to $L(P(L_1), \ell_1)$.

THEOREM 2.6. *$K(L_1, \ell_1)$ is not an M-ideal in $L(L_1, \ell_1)$.*

PROOF. We will use Lima's characterization of an M-ideal in Theorem 1.1. Assume that $K(L_1, \ell_1)$ is an M-ideal in $L(L_1, \ell_1)$. Let $S_i \in K(P(L_1), \ell_1)$ with $\|S_i\| \leq 1$ $(i = 1, 2, 3)$, $T \in L(P(L_1), \ell_1)$ with $\|T\| \leq 1$ and $\epsilon > 0$. Then $\widetilde{S}_i = S_i P \in K(L_1, \ell_1)$, $\widetilde{T} = TP \in L(L_1, \ell_1)$, $\|\widetilde{S}_i\| \leq 1$ $(i = 1, 2, 3)$ and $\|\widetilde{T}\| \leq 1$. By the assumption, there is $\widetilde{S} \in K(L_1, \ell_1)$ such that

$$\|\widetilde{S}_i + \widetilde{T} - \widetilde{S}\| < 1 + \epsilon \qquad (i = 1, 2, 3).$$

Then $S = \widetilde{S}P|_{P(L_1)} \in K(P(L_1), \ell_1)$ and

$$\begin{aligned} \|S_i + T - S\| &= \|(\widetilde{S}_i + \widetilde{T} - \widetilde{S})P|_{P(L_1)}\| \\ &\leq \|\widetilde{S}_i + \widetilde{T} - \widetilde{S}\| < 1 + \epsilon \qquad (i = 1, 2, 3). \end{aligned}$$

Hence $K(P(L_1), \ell_1)$ is an M-ideal in $L(P(L_1), \ell_1)$ and so $K(\ell_1)$ is an M-ideal in $L(\ell_1)$. This contradicts to the fact that $K(\ell_1)$ is not an M-ideal in $L(\ell_1)$ [15]. Therefore, $K(L_1, \ell_1)$ is not an M-ideal in $L(L_1, \ell_1)$.

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Department of Mathematics
Hanyang University
Seoul 133-791, Korea