Comm. Korean Math. Soc. **11** (1996), No. 3, pp. 665–672

SOME PROPERTIES OF THE SET OF SCHWARZIANS OF CONFORMAL FUNCTIONS

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ABSTRACT. Let U denote the set of all Schwarzian derivatives S_f of conformal function f in the unit disk $\mathbf{D}$. We show that if S_f is a local extreme point of U, then f cannot omit an open set. We also show that if $S_f \in U$ is an extreme point of the closed convex hull $\overline{co}U$ of U, then f cannot omit a set of positive area. The proof of this uses Nguyen's theorem.

1. Introduction

In this paper, $E = E(\mathbf{D})$ will denote the Banach space of holomorphic functions ψ in the unit disk $\mathbf{D} = \{z : |z| < 1\}$, equipped with the norm

$$(1) \qquad ||\psi|| = ||\psi||_{\mathbf{D}} = \sup_{z\in\mathbf{D}} |\psi(z)|(1 - |z|^2)^2.$$

We define Banach space E by

$$E = \{\psi : \psi : \mathbf{D} \to \mathbf{C} \text{ holomorphic, } ||\psi|| < \infty\}.$$

Next for each function f which is meromorphic and locally univalent in $\mathbf{D}$ we let S_f denote the Schwarzian derivative of f. At finite points of $\mathbf{D}$ which are not poles of f, $S_f(z)$ is given by

$$S_f(z) = (f''(z)/f'(z))' - (1/2)(f''(z)/f'(z))^2$$

and it is holomorphic in $\mathbf{D}$. Direct computation gives the transformation rule

$$(2) \qquad S_{f\circ g}(z) = S_f(g(z))g'(z)^2 + S_g(z).$$

Received December 28, 1995. Revised June 21, 1996.
1991 AMS Subject Classification: 30C35, 30C99.
Key words and phrases: univalent function, extreme point, Schwarzian derivative, Lipschitz function.

If T is a Möbius transformation, we have $S_T = 0$, and so $S_{f\circ T}(z) = S_f(T(z))T'(z)^2$.

Let U denote the set of all Schwarzian derivatives S_f of conformal function f from $\mathbf{D}$ into the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Here, and in the rest of paper, conformal means holomorphic and univalent. It turns out that $U \subseteq E$; in fact U is a closed set in E [9,p.115]. Furthermore, U is contained in the closure of the ball $B(0,6) = \{\psi \in E : ||\psi|| < 6\}$ and U contains the closure of $B(0,2)$ ([7],[10]). The set U has been of some interest due to its connection with the Bers model

$$Q = \{S_f \in U : f \text{ has quasiconformal extention to } \hat{\mathbf{C}}\}$$

of the universal Teichmüller space. It was shown by [1] that Q is open, and the relationship between U and Q was clarified by [3], who showed that $Q = \text{int}(U)$.

It was for a long time an open question, due to Bers, whether U is equal to the closure of Q in E. This was disproved by [4], who showed by an example that $U - \overline{Q} \neq \phi$. Recently, Thurston[14] proved that in fact, U has isolated points.

In [12], we know that the omitted set of a conformal map f from $\mathbf{D}$ into $\mathbf{C}$ has zero area if S_f is an isolated point of U. The proof uses [11],which is also used in the proof of theorem 4.1 of this paper.

2. Extreme points and local extreme points

An extreme point of set $A \subset E$ is a $\psi \in A$ such that if ψ has a convex decomposition $\psi = t\psi_1 + (1-t)\psi_2$ with $0 < t < 1$ and $\psi_1,\ \psi_2 \in A$, then $\psi_1 = \psi_2$; i.e., the decomposition is trivial. The set of extreme points of A is denote $e(A)$.

A local extreme point of a set $A \subseteq E$ is a $\psi \in A$ such that there exists a $\delta > 0$ such that $\psi \in e(\{\varphi \in A : ||\varphi - \psi|| \leq \delta\})$.

We will denote the set of local extreme points of A by $le(A)$. Clearly we have $e(A) \subseteq le(A)$, with equality whenever A is convex. But $le(A)$ may contain other points besides extreme points; an isolated point is always a local extreme point, for instance.

We now consider local extreme points of U:

PROPOSITION 2.1. *Let $f, g : \mathbf{D} \to \hat{\mathbf{C}}$ be conformal into, with $f(\mathbf{D}) \subseteq g(\mathbf{D})$. If $S_f \in le(U)$, then $S_g \in le(U)$.*

PROOF. By assumption $f = g \circ T$ where $T : \mathbf{D} \to \mathbf{D}$ is a conformal automorphism [15,p.39]. Suppose $S_g \notin le(U)$. Then $S_g = tS_{g_1} + (1-t)S_{g_2}$ with $S_{g_1}, S_{g_2} \in U, S_{g_1} \neq S_{g_2}, 0 < t < 1, ||S_g - S_{g_1}|| \leq \delta$ and $||S_g - S_{g_2}|| \leq \delta$. We have

$$\begin{aligned} S_f = S_{g\circ T} &= S_g(T)T'^2 + S_T \\ &= [tS_{g_1}(T) + (1-t)S_{g_2}(T)]T'^2 + S_T \\ &= t[S_{g_1}(T)T'^2 + S_T] + (1-t)[S_{g_2}(T)T'^2 + S_T] \\ &= tS_{g_1\circ T} + (1-t)S_{g_2\circ T}. \end{aligned}$$

It is clear that $S_{g_1\circ T}, S_{g_2\circ T} \in U$, and $S_{g_1\circ T} \neq S_{g_2\circ T}$. Furthermore

$$\begin{aligned} ||S_{g\circ T} - S_{g_1\circ T}|| &= ||S_g(T)T'^2 + S_T - S_{g_1}(T)T'^2 - S_T|| \\ &= ||(S_g(T) - S_{g_1}(T))T'^2|| \\ &= \sup_{z\in\mathbf{D}} |S_g(T(z)) - S_{g_1}(T(z))||T'(z)|^2(1-|z|^2)^2 \\ &\leq \sup_{z\in\mathbf{D}} |S_g(T(z)) - S_{g_1}(T(z))|(1-|T(z)|^2)^2 \\ &\leq \sup_{z\in\mathbf{D}} |S_g(z) - S_{g_1}(z)|(1-|z|^2)^2 \\ &= ||S_g - S_{g_1}|| \leq \delta \end{aligned}$$

by the Schwarz-Pick lemma [1,p.3]. By the similar method we have $||S_{g\circ T} - S_{g_2\circ T}|| \leq \delta$. Consequently $S_f \notin le(U)$. Thus we are finished □

The above theorem is also valid for extreme points of U, of course; the proof is just a subset of the above proof.

THEOREM 2.2. *If $S_f \in le(U)$, then f cannot omit a nonempty open set.*

PROOF. Suppose f omits a nonempty open set. Then it will in particular omit some closed disk D_o, say. Let g be a Möbius transformation mapping $\mathbf{D}$ onto $\hat{\mathbf{C}} - D_o$. Clearly $f(\mathbf{D}) \subseteq g(\mathbf{D})$, so proposition 2.1 would imply that $0 = S_g \in le(U)$, which is false. We can see this by considering the functions $f_p(z) = [(1+z)/(1-z)]^p$ which are univalent for $0 < p \leq 2$. Since $S_{f_p}(z) = 2(1-p^2)(1-z^2)^{-2}$, it follows that $0 \notin le(U)$. □

3. The hyperbolic metric

Now we give a brief introduction to the hyperbolic metric. We refer the reader [8] and [16] for further details. Let $\Omega \subset \hat{\mathbf{C}}$ be a simply connected region. A simply connected region Ω is called hyperbolic if the complement of Ω in $\hat{\mathbf{C}}$ contains at least three points. By the Uniformization theorem ([2],p.142],[15,p.9]) there exists a holomorphic universal covering projection g of $\mathbf{D}$ onto Ω. Since Ω is simply connected, then g is just a conformal function of $\mathbf{D}$ onto Ω. The collection of all holomorphic universal covering projections of $\mathbf{D}$ onto Ω consists of the functions $g \circ T$, where $T \in \mathrm{Aut}(\mathbf{D})$, the group of conformal automorphisms of $\mathbf{D}$. The hyperbolic metric on $\mathbf{D}$ is defined by

$$\lambda_{\mathbf{D}}(z)|dz| = (1-|z|^2)^{-1}|dz|.$$

The density $\lambda_\Omega(w)$ of the hyperbolic metric $\lambda_\Omega(w)|dw|$ on a hyperbolic region Ω is determined by

$$\lambda_\Omega(g(z))|g'(z)| = \lambda_{\mathbf{D}}(z) = (1-|z|^2)^{-1}, \tag{3}$$

where $w = g(z)$ is any holomorphic universal covering projection of $\mathbf{D}$ onto Ω. The density of the hyperbolic metric is independent of the choice of the holomorphic universal covering projection g since

$$|T'(z)|(1-|T(z)|^2)^{-1} = (1-|z|^2)^{-1}, \qquad z \in \mathbf{D}$$

for any $T \in \mathrm{Aut}(\mathbf{D})$ [1,p.3]. Using the density λ_Ω, we can define the following norm

$$||h||_\Omega = \sup_{w\in\Omega} |h(w)|\lambda_\Omega(w)^{-2}, \tag{4}$$

which is analogous to the norm $||\psi|| = ||\psi||_{\mathbf{D}}$ from (1).

We shall need the following theorem due to [3].

THEOREM 3.1 (GEHRING' THEOREM). *If $f : \Omega \to \hat{\mathbf{C}}$ is conformal into, then $||S_f||_\Omega \leq 12$.*

In the following, $m(A)$ will always denote the Lebesgue planar measure of a set A. We shall also need the theorem of [11]

THEOREM 3.2 (NGUYEN'S THEOREM). *If Γ is compact in $\mathbf{C}$, with $m(\Gamma) > 0$, there exists a nonconstant bounded holomorphic Lipschitz function on $\hat{\mathbf{C}} - \Gamma$.*

4. Extreme points of the closed convex hull

The smallest closed convex set that contains U is called the closed convex hull of U and it is denoted by $\overline{co}U$.

THEOREM 4.1. *If $S_f \in U$ and $S_f \in e(\overline{co}U)$, then f cannot omit a set of positive area.*

PROOF. Let $S_f \in U$, and put $\Omega = f(\mathbf{D}), \Gamma = \hat{\mathbf{C}} - \Omega$. There is no loss of generality in assuming that $\infty \in \Omega$. We shall suppose that $m(\Gamma) > 0$, and conclude that $S_f \notin e(\overline{co}U)$.

Since Γ is compact with $m(\Gamma) > 0$, Nguyen's theorem gives a nonconstant bounded holomorphic Lipschitz function F on Ω. There is a point $w_o \in \Omega$ at which $F'''(w_o) \neq 0$, otherwise F would be a quadratic polynomial, which is impossible, because F could not then be bounded. By adding a linear term to F if necessary, we may in addition assume that $F'(w_o) = 0$. The new F will still be a nonconstant holomorphic Lipschitz function. For convenience, we write $G = F'F''' - (3/2)F''^2$.

Let A be the Lipschitz constant of F, and put

$$H_\theta(w) = w + re^{i\theta}F(w), \qquad \text{with } 0 \leq r < 1/A.$$

Then H_θ is conformal on Ω. We have

$$S_{H_\theta} = (re^{i\theta}F''' + r^2e^{2i\theta}G)/(1 + re^{i\theta}F')^2, \tag{5}$$

and so S_{H_θ} depends holomorphically on $re^{i\theta}$. Thus by the mean value theorem

$$\frac{1}{2\pi}\int_0^{2\pi} S_{H_\theta}(w)d\theta = 0.$$

Using (2), we get

$$\frac{1}{2\pi}\int_0^{2\pi} S_{H_\theta \circ f}(z)d\theta = S_f(z)$$

Put

$$\psi_j(z) = \frac{1}{\pi}\int_{\pi j-\pi}^{\pi j} S_{H_\theta \circ f}(z)d\theta \qquad \text{for } j = 1,2.$$

Then clearly $S_f = (1/2)\psi_1 + (1/2)\psi_2$.

Putting $r = 1/(2A)$. By calculation, using $F'(w_o) = 0$, we see that

$$\frac{1}{\pi}\int_{\pi j-\pi}^{\pi j} S_{H_\theta}(w_o)d\theta = (-1)^{j+1}\frac{i}{\pi A}F'''(w_o) \qquad \text{for } j = 1,2.$$

These two integrals are not equal at $w_o \in \Omega$, and so we conclude that $\psi_1 \neq \psi_2$.

To conclude that $S_f \notin e(\overline{co}U)$, it remains to show that $\psi_1, \psi_2 \in \overline{co}U$. This goes in exactly the same way for ψ_1 and ψ_2; we will do it for

$$\psi_1 = \frac{1}{\pi}\int_0^{\pi} S_{H_\theta \circ f}d\theta.$$

Since E is a Banach space and $d\theta/\pi$ is a Borel probability measure, it is enough to show that the mapping $\theta \mapsto S_{H_\theta \circ f}$ is continuous [13, p.74]. For then

$$\psi_1 \in \overline{co}\{S_{H_\theta \circ f} | \theta \in [0,\pi]\} \subseteq \overline{co}U$$

since $H_\theta \circ f$ is conformal.

We have

$$\begin{aligned} \|S_{H_\alpha \circ f} - S_{H_\beta \circ f}\| &= \sup_{z\in\mathbf{D}} |S_{H_\alpha \circ f}(z) - S_{H_\beta \circ f}(z)|(1-|z|^2)^2 \\ &= \sup_{z\in\mathbf{D}} |S_{H_\alpha}(f(z)) - S_{H_\beta}(f(z))|\lambda_\Omega(f(z))^{-2} \\ &= \sup_{w\in\Omega} |S_{H_\alpha}(w) - S_{H_\beta}(w)|\lambda_\Omega(w)^{-2} \\ &= \|S_{H_\alpha} - S_{H_\beta}\|_\Omega \end{aligned}$$

by (1),(2),(3) and (4). Since F satisfies a Lipschitz condition, the inequality $|F'(w)| \leq A$ is valid in Ω. Furthermore, we have $S_f(w) \leq 12\lambda_\Omega(w)^2$ in Ω by the theorem of Gehring. Thus

$$\begin{aligned}|re^{i\theta}F'''(w) + r^2e^{2i\theta}G(w)| &\leq |1 + re^{i\theta}F'(w)|^2 12\lambda_\Omega(w)^2 \\ &\leq 12(1 + Ar)^2\lambda_\Omega(w)^2\end{aligned}$$

in Ω. Putting $r = 1/(2A)$, $\theta = 0, \pi$, and this yields

$$|F'''(w)| \leq 54A\lambda_\Omega(w)^2 \text{ and } G(w) \leq 108A^2\lambda_\Omega(w)^2$$

by the triangle inequality. See [12] for this. Using (5) and these inequalities, straightforward calculations give

$$||S_{H_\alpha} - S_{H_\beta}||_\Omega \leq 24732|e^{i\alpha} - e^{i\beta}|.$$

Thus $\theta \mapsto S_{H_\theta \circ f}$ is continuous, and the proof of theorem 4.1 is complete. □

REMARKS. The method used in proving theorem 4.1 is a combination of the method from [12], using Nguyen's theorem to produce a family of Schwarzians of univalent functions depending on a parameter, and the use of integration to produce a convex decomposition. Since theorem 4.1 deals with the closed convex hull, it does not supersede the result in [12].

Since U is not convex, theorem 4.1 does not necessarily supersede theorem 2.2 even for extreme points. For there might be some $S_f \in e(U)$ with $S_f \notin e(\overline{co}U)$. What the exact relationship between $e(U)$ and $e(\overline{co}U)$ is seems to be unknown.

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