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A NOTE ON COMPATIBLE VALUATIONS WITH HIGHER LEVEL COMPLETE PREORDERINGS AND HIGHER LEVEL ORDERINGS

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ABSTRACT. In this paper we give some results on higher level complete preorderings and higher level orderings in a field. Further we find some properties which hold between compatible valuations and above preoderings and orderings.

1. Introduction

The notion of orderings of a field was systematically studied by E.Artin and O.Schreier in 1920s. Especially the notion of preorderings is a generalization of that of orderings. Concepts of orderings and preorderings were developed successfully to level 2^n [2,3] ,and partially to level $2n$ [3,4] . In this paper ,comparing these two types, we proceed further to find and supplement some properties that are related to level $2n$ which were already shown in the case of level 2^n [2] miscellaneously. A great part of this paper was written basically on [2].

2. Preliminaries

Let R be a ring with unity. A subset $P \subset R$ is called an *preprime*[1,4] if it satisfies the following conditions : (1) $P + P \subset P$ (2)$PP \subset P$ (3)$0, 1 \in P$ (4)$-1 \notin P$. So char $(R) = 0$. Let K be a field and a subset $T \subset K$ is called a *preordering* if it satisfies (1),(2),(3) in K and $T^\times = T - \{0\}$ is a subgroup of $K^\times = K - \{0\}$. This T is called *proper*

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if $-1 \notin T$ [2], in this case $T \cap -T = \{0\}$. If $a^2 \in T$ always implies $a \in T \cup -T$, T is said to be *complete* [1,4]. A complete and proper preordering T is said to be an *ordering* if $K^\times/T^\times$ is a cyclic group [4]. Especially a preordering T is said to be a *preordering of level* $2n$ if $K^{2n} \subset T$[3]. A complete and proper preordering T of level $2n$ is called an *ordering of level* $2n$ [4] if $K^\times/T^\times$ is a cyclic group. A preprime T in a field K is called a *torsion preprime* [1] if for each $a \in K$, there exists a natural number m such that $a^m \in T$. Every torsion preprime $T \subset K$ is clearly a proper preordering and $K^\times/T^\times$ is a torsion group. We always denote K, T and P be a field, a preordering of level $2n$, and an ordering of level $2n$ respectively unless otherwise stated. Then T is a torsion preordering and by [1,(3.3)Propositon] $K = T - T$ if T is proper.

THEOREM 2. 1. *The followings are equivalent: (1) T is proper. (2) $T \cap -T = \{0\}$ (3) char$(K) = 0, T \neq K$.*

Let P_{2n} be the set of all finite sums of $2n$-th powers in K. If $-1 \notin P_{2n}$, then P_{2n} is a proper preordering of level $2n$. By an application of Zorn's Lemma, we have a maximal proper one. Clearly $P_{2n} \subset T$ for any T in K.

COROLLARY 2.2. *For any field K, the following statements are equivalent. (1) $-1 \notin P_{2n}$. (2) char$(K) = 0, P_{2n} \neq K$.*

THEOREM 2.3. *[4,Satz1.4,Kor2.3,Satz2.17] Let K be char$(K)= 0$. Then the followings hold. (1) $-1 \notin P_{2n}$ if and only if there exists a complete and proper T. In this case $P_{2n} = \cap T$, where T runs over all which are complete and proper. (2) $-1 \notin P_{2n}$ if and only if K is real. (3) Every T which is complete and proper is the intersection of all orderings of level $2n$ containing T.*

For T and $a_1, a_2, \ldots, a_k$ in a field K, we define $T[a_1, a_2, \ldots, a_k]$ to be the set of all polynomial expressions in $a_1, a_2, \ldots, a_k$ with coefficients from T: $\Sigma t_{i1\ldots ik} a_1^{i1} \ldots a_k^{ik}$. Then $T[a_1, a_2, \ldots, a_k]$ is the smallest preordering containing T and $a_1, a_2, \ldots, a_k$. Especially if there exists an element $x \in K$ satisfying $x \notin T \cup -T$ and $x^2 \in T$ for some proper T, then by [4,Lemma 1.3] we have $T = \cap_{a \notin T \cup -T, a^2 \in T} T[a]$. If T is complete and proper, we can get a maximal one which is complete and proper containing T by an application of Zorn's Lemma. By Theorem 2.3, any $T[a_1, a_2, ..., a_k]$ which is complete and proper is the intersection of all P satisfying $T \subset P, a_1, a_2, ..., a_k \in P$. Restricting to $T = P_{2n}$, any $P_{2n}[a_1, a_2, .., a_i, .., a_k]$ that is complete and proper is the set of all elements which lie precisely in those orderings of level $2n$ containing $a_1, a_2, ..., a_k$.

3. Main Results

Let K be as above with a Krull valuation v [5] . We denote by A, I, U, k, Γ its valuation ring,maximal ideal, group of units, the residue field A/I, value group respectively. Let $\psi : A \to k$ be the canonical ephimorphism. Then T induces the preordering $\bar{T} := \overline{A \cap T} = \psi(A \cap T) = \{a + I : a \in T \cap A\} \subset k$. One easily verifies $k^{2n} \subset \bar{T}, \bar{T} + \bar{T} \subset \bar{T}, \bar{T}\bar{T} \subset \bar{T}$. Since every valuation ring is integrally closed [5,(10.6)Theorem], $\bar{T}$ is complete when T is complete. A is said to be *compatible* with a complete and proper T, written $A \sim T$, if $\bar{T}$ is a proper preordering. In that case k is necessarily a real field by Theorem 2.3 and by definition A is a real valuation ring(i.e. $k = A/I$ is formally real [cf.9].). Put $\bar{P} := \{a + I : a \in P \cap A\} \subset k$. This $\bar{P}$ is a complete preordering. P is called *compatible* with v if $1 + I \subset P$.

LEMMA 3.1. *If a valuation ring A is compatible with P in K, then it must be real.*

PROOF. $\bar{P}$ is an ordering of level $2n$ [4, Satz 2.1,8]. Then by Theorem 2.3, $k = A/I$ is real.

A field K with a valuation v is said to be *2-Henselian* if Hensel's Lemma holds for quadratic monic polinomials over the valuation ring of v. Some properties related to this notion are explained in [5,7].

THEOREM 3.2. *Let K be a field with a 2- Henselian valuation v and P be in K. Then P is compatible with v.*

PROOF. By Theorem 2.3, K is a real field and by [7,Theorem 3.16] k is also real. Since v is non-dyadic [7,Lemma3.15], $1 + I = (1 + I)^{2^n}$ for all $n \in N$ [2]. But every $2n = 2^l + \cdots + 2^m$ for some natural number $l, \cdots, m$. So $1 + I \in P$.

Denote Q be the set of rational numbers and Q^+ the positive rational numbers. Set $A(T) = \{a \in K : r \pm a \in T$ for some $r \in Q^+\}$ and $I(T) = \{a \in K : r \pm a \in T$ for any $r \in Q^+\}$ where T is complete and proper. Becker showed in [4] that $A(T)$ is a real valuation ring with the maximal ideal $I(T)$. Especially a valuation ring A is compatible with P if $A(P) \subset A$ [5,(6.6)Theorem]. Let F be a subfield of K. We set $A(P,F) := \{a \in K : r \pm a \in P$ for some $r \in F \cap P^\times\}$ and $I(P,F) := \{a \in K : r \pm a \in P$ for any $r \in F \cap P^\times\}$. Then $A(P,F)$ is a valuation ring compatible with P and its maximal ideal is $I(P,F)$ [9].

THEOREM 3.3. *$A(P,F)$ is the valuation ring containing F .*

PROOF. Since $P \cap F \subset A(P,F)$ and $P \cap F$ is an ordering of F , then $F = P \cap F - P \cap F \subset A(P,F)$ [1].

Clearly $A(P,Q) = A(P)$ and $I(P,Q) = I(P)$. Let $k(P,F)$ be the residue field of $A(P,F)$. Then $k(P,F)$ is an extention of $\bar{F} = \{a + I(P,F) : a \in F\}$ and contains the ordering $\bar{P} = \{a + I(P,F) : a \in P \cap A(P,F)\}$ because $A(P,F)$ is compatible with P.

Let E/K be a field extension, P an ordering of level $2n$ in E. E/K is called *archimedean relative to* P if for any $a \in E$, there is $r \in K \cap P^\times$ such that $r \pm a \in P$, or equivalently ,if $A(P,K) = E$ [2] .

THEOREM 3.4. *Any extention E of K satisfying $E \neq A(P,K)$ and $P \subset E$ is transcendental.*

PROOF. Assume E/K is an algebraic extension and $P \subset E$. Since $K \subset A(P,K)$, we have $A(P,K) = E$ by [5,(9.8)Corollary]. So $A(P,K)$ is archimedian. Hence any extention E of K satisfying $E \neq A(P,K)$ is transcendental.

THEOREM 3.5. *Let $F_1 \subset F_2$ be two subfields of K, v the valuation associated with $A(P,F_1)$. Then (1) $A(P,F_1) \subset A(P,F_2)$ (2)$A(P,F_2) = A(P,F_1)_\Sigma := \{a \in K : a = 0$ or $v(a) \geq r$ for some $r \in \Sigma\}$ where $\Sigma = v({F_2}^\times)$.*

PROOF. (1) Since $F_1 \subset F_2$,we have $P^\times \cap F_1 \subset P^\times \cap F_2$, so $A(P,F_1) \subset A(P,F_2)$. (2) Take $\Sigma := v(F_2^\times)$ a subgroup of Γ. Let $A_1 := A(P,F_1)_\Sigma = \{a \in K : a = 0$ or $v(a) \geq r$ for some $r \in \Sigma\}$. Clearly this A_1 is a valuation ring containing $A(P, F_1)$. If $a \in A_1 (a \neq 0)$, there exists $r \in \Sigma$ with $v(a) \geq r = v(b)$ for some $b \in F_2^\times$. Then $v(a) - v(b) = v(ab^{-1}) \geq 0$,and $ab^{-1} \in A(P, F_1)$, so there exists $s \in F_1 \cap P^\times \subset F_2 \cap P^\times$ with $s \pm (ab^{-1})^{2n} = s \pm a^{2n}b^{-2n} = s \pm a^{2n}t^{-1} \in P$, taking $t := b^{2n} \in F_2 \cap P$. Since $t \in P^\times \cap F_2$,we get $ts \pm a^{2n} \in P$, so $a^{2n} \in A(P, F_2)$. But every valuation ring is integrally closed [5], we get $a \in A(P, F_2)$. This implies $A_1 \subset A(P, F_2)$. Conversely let $a \in A(P, F_2)$. Then $a^{2n} \in A(P, F_2)$, so there exists $r \in P^\times \cap F_2$ such that $r - a^{2n} \in P$.Therefore $v(a^{2n}) \geq v(r) \in \Gamma$ [9,Proposition 2.4] and we have $a^{2n} \in A_1$. Since every valuation ring is integrally closed,we have $a \in A_1$.

THEOREM 3.6. *Let $T, T[1+I]$ be complete and proper . Then the following statements are equivalent. (1) $A \sim T$ (2) $T[1+I] \neq K$ (3) $A \sim P$ for some $P \supset T$.*

PROOF. (1) $\Rightarrow$ (2).$-\bar{1} \notin \bar{T}$ implies $T \cap -(1+I) = \phi$. We shall prove $T[1+I] = T \cdot (1+I)$, which obviously implies $T[1+I] \neq K$. To this end we show that $T \cdot (1+I)$ is a proper preordering. Since other conditions clearly hold, we shall only prove that $T \cdot (1+I)$ is additively closed . Let v be the valuation associated to A , let $t, t' \in T, \epsilon, \eta \in 1+I$; we have to show $x := t\epsilon + t'\eta \in T \cdot (1+T)$. If $v(t\epsilon) \neq v(t'\eta)$, say $v(t\epsilon) > v(t'\eta)$, then one gets $x := t\epsilon\omega$, where $\omega \in 1+I$, hence $x \in T \cdot (1+I)$. If $v(t\epsilon) = v(t'\eta)$, then $t' = t\omega, \omega \in U \cap T$, and $x = t(\epsilon + \eta\omega)$. We see $\epsilon + \eta\omega = 1 + \omega + i, i \in I$. Assume $1 + \omega \in I$, then $\omega = -[1 - (1+\omega)] \in T \cap -(1+I)$ induces a contradiction. Therefore $\epsilon + \eta\omega = (1+\omega)[1 + (1+w)^{-1}i] \in T(1+I)$ and $x \in T \cdot (1+I)$ holds. (2)$\Rightarrow$ (3). Since $T[1+I]$ is complete and proper , there exists an P with $T \subset T[1+I] \subset P$, in particular $A \sim P$. (3) $\Rightarrow$ (1). From $A \sim P$ it follows that $\bar{P} \neq k$. But we have $\bar{T} \subset \bar{P}$, so this implies $\bar{T} \neq k$. This implies that $\bar{T}$ is proper.

A valuation ring A is defined *fully compatible* [cf 2,7] with T if $1 + I \subset T$. Clearly fully compatiblity implies compatibility . Let B be a preordering of level $2n$ containing $\bar{T}$ in k . Then $T \cdot \phi^{-1}(B^\times)$ is a preordering of level $2n$ on K with $\phi(T \cdot \phi^{-1}(B^\times)) = B$[7,8]. We can generalize this notion. We call a subgroup V of $K^\times$ *a subgroup of level 2n* if $-1 \notin V$ and ${K^\times}^{2n} \subset V$ holds.

PROPOSITION 3.7. *Let $\hat{T}$ be a proper preordering of level $2n$ of $k = A/I$, V be as above satisfying $\psi(V \cap U) \subset \hat{T}$. Then $T := V \cdot \psi^{-1}(\hat{T}^\times) \cup \{0\}$ is a proper preordering of level $2n$ in K with $\bar{T} = \hat{T}$ which is fully compatible with A.*

PROOF. $\psi(V \cap U) \subset \hat{T}$ reduces $\bar{T} = \hat{T}$. Clearly $K^{2n} \subset T, TT \subset T$ hold. Take $a, b \in V, \epsilon, \eta \in \psi^{-1}(\hat{T}^\times)$; let v be the valuation associated to A . If $v(a) \neq v(b)$, then we get $a\epsilon + b\eta \in T$ as in the proof of Theorem3.6. But if $v(a) = v(b)$, then $a = b\omega, \omega \in V \cap U$ holds,so the result $a\epsilon + b\eta = b(\omega\epsilon + \eta) \in T$ is followed by $\overline{\omega\epsilon + \eta} = \bar{\omega}\bar{\epsilon} + \bar{\eta} \in \hat{T}^\times$. Therefore T is a proper preordering of level $2n$. Since $\psi^{-1}(\bar{1}) = 1 + I \subset T, T$ is fully compatible with A by definition .

REMARK. If a complete and proper T is fully compatible with A, then for every ordering $P \supset T, \bar{P}$ is an ordering over $\bar{T}$. Furthermore for every ordering $\hat{P} \supset T$ in $k, T_1 := T \cdot \psi^{-1}(\hat{P}^\times)$ is a proper preordering with $\bar{T_1} = \hat{P}$ by Lemma 3.7. If S be an ordering with $S \supset T_1$, then clearly $\hat{P} \subset \bar{S}$ holds.

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ON GARDNER'S PROBLEM

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ABSTRACT. A positive, disjoint linear map $\phi : \mathfrak{A} \to \mathfrak{B}$ of C^*-algebras preserves absolute values if any $*$-anti-homomorphism $\psi : \mathfrak{A} \to \mathfrak{B}$ is skew-hermitian with respect to every commutators of unitary elements.

Throughout this note suppose $\mathfrak{A}$ and $\mathfrak{B}$ are unital C^*-algebras and suppose the set $\mathfrak{A}^+ = \{a^*a : a \in \mathfrak{A}\}$ is a closed convex cone of all positive elements of $\mathfrak{A}$. Every positive element a has a unique square root $a^{\frac{1}{2}}$ in $\mathfrak{A}^+$. If $a \in \mathfrak{A}$, $|a| = (a^*a)^{\frac{1}{2}}$ is called the *absolute value* of a. A linear map $\phi : \mathfrak{A} \to \mathfrak{B}$ is called *positive* if $\phi(\mathfrak{A}^+) \subset \mathfrak{B}^+$, and is called *2-positive* if the map $\phi \otimes \mathrm{id}_2$ is positive on the C^*-algebra $\mathfrak{A} \otimes M_2(\mathbb{C})$ to $\mathfrak{B} \otimes M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ is the C^*-algebra of 2×2 complex matrices. A linear map $\phi : \mathfrak{A} \to \mathfrak{B}$ is called *a Jordan homomorphism* if $\phi(a^2) = \phi(a)^2$ for all $a \in \mathfrak{A}$ and is called *a $*$-homomorphism* if $\phi(a^*) = \phi(a)^*$ and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathfrak{A}$. A linear map $\phi : \mathfrak{A} \to \mathfrak{B}$ is called *unital* if $\phi(I_{\mathfrak{A}}) = I_{\mathfrak{B}}$ and is called *disjoint* if $xy = 0$ in $\mathfrak{A}$ implies $\phi(x)\phi(y) = 0$ in $\mathfrak{B}$.

In 1979, L.T. Gardner [2, Theorem 1] has shown that a 2-positive, disjoint linear map of C^*-algebras preserves absolute values. Also, in [2], he gave the following problem:

GARDNER'S PROBLEM. : Can "2-positive" be replaced by "positive" in the Gardner's theorem?

In this note we give some partial solutions to Gardner's problem.

We begin with:

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LEMMA 1. *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is linear then the followings are equivalent:*
(i) $\phi(I)\phi(a^2) = \phi(a)^2$ *for all $a \in \mathfrak{A}$.*
(ii) $\phi(I)\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ *for all $a, b \in \mathfrak{A}$.*

PROOF. (i) $\Rightarrow$ (ii): Take $a + b$ in place of a.
(ii) $\Rightarrow$ (i): Take $b = a$. □

LEMMA 2. *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is a positive, disjoint linear map then we have*
(i) $\phi(I)\phi(a^2) = \phi(a)^2$
(ii) $\phi(I)$ *centralizes $\phi(\mathfrak{A})$ and $\phi(I)^{-1}$ exists.*
In particular, if ϕ is unital then ϕ is a Jordan homomorphism.

PROOF. If ϕ is a positive, disjoint linear map then an argument of Gardner [2, Lemma 2] gives that ϕ preserves absolute values on self-adjoint elements. Thus by another argument of Gardner [2, Corollary 7] (by way of imbedding the codomain space into its bi-dual space), there exists a Jordan homomorphism $\psi : \mathfrak{A} \to \mathfrak{B}$ such that

$$\phi(a) = \phi(I)\psi(a) \qquad \text{for all } a \in \mathfrak{A}, \tag{1}$$

$\phi(I)$ commutes with $\phi(a)$ for all $a \in \mathfrak{A}$, and $\phi(I)^{-1}$ exists. Thus we have that

$$\begin{aligned} \phi(I)\phi(a^2) = \phi(I)^2\psi(a^2) &= \phi(I)^2\psi(a)^2 \\ &= \phi(I)\phi(a)\psi(a) = \phi(a)\phi(I)\psi(a) = \phi(a)^2. \end{aligned}$$ □

REMARK 1. It was known [2, Theorem 2] that if (2) $\phi : \mathfrak{A} \to \mathfrak{B}$ is a positive linear map then

ϕ preserves absolute values if and only if $\phi(I)\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathfrak{A}$.

Thus if ϕ is unital then ϕ preserves absolute values if and only if ϕ is a $*$-homomorphism. Observe that if the equality in (2) holds for all self-adjoint elements then it also holds for all elements in $\mathfrak{A}$.

We now have

COROLLARY 1. *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is a positive, disjoint linear map and if either $\mathfrak{A}$ or $\phi(\mathfrak{A})$ is commutative then ϕ preserves absolute values.*

PROOF. From the argument for Lemma 2, we can see that $\phi(a) = \phi(I)\psi(a)$ for all $a \in \mathfrak{A}$, where ψ is a Jordan homomorphism. Since Jordan homomorphisms of C^*-algebras are $*$-homomorphisms if either the domain or range is commutative, we have that ψ is a $*$- homomorphism (Note that if $\phi(\mathfrak{A})$ is commutative then $\psi(\mathfrak{A})$ is also commutative). Further since $\phi(I)$ centralizes $\phi(\mathfrak{A})$, it follows that $\phi(I)\phi(ab) = \phi(I)^2\psi(ab) = \phi(I)^2\psi(a)\psi(b) = \phi(I)\psi(a)\phi(I)\psi(b) = \phi(a)\phi(b)$, which, by (2), gives the results. □

As usual $[a,b]$ denotes the commutator $ab - ba$ and $\big[[a,b],c\big]$ is called the *Lie triple product.* It was well known ([3]) that any Jordan homomorphism preserves arbitrary powers, squares of commutators, and Lie triple products. We have an extended version to the nonunital case.

LEMMA 3. *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is a positive, disjoint linear map then we have:*

(i) $\phi(I)^{n-1}\phi(a^n) = \phi(a)^n$ *for all $n \in \mathbb{N}$ and $a \in \mathfrak{A}$.*
(ii) $\phi(I)^2\phi(aba) = \phi(a)\phi(b)\phi(a)$ *for all $a, b \in \mathfrak{A}$.*
(iii) $\phi(I)^2\phi([a,b]^2) = [\phi(a),\phi(b)]^2$ *for all $a, b \in \mathfrak{A}$.*
(iv) $\phi(I)^2\phi\big([[a,b],c]\big) = \big[[\phi(a),\phi(b)],\phi(c)\big]$ *for all $a, b, c \in \mathfrak{A}$.*

PROOF. By Lemma 2, $\phi(I)\phi(a^2) = \phi(a)^2$ and $\phi(I)$ commutes with $\phi(a)$ for all $a \in \mathfrak{A}$.
(i) Apply Lemma 1 with $b = a^2$ and then use an inductive step.
(ii) Use the identity $2aba = 4(a+b)^3 - (a+2b)^3 - 3a^3 + 4b^3 - 2(a^2b + ba^2)$.
(iii) Use (i) and (ii).
(iv) Use the identity $abc + cba = (a+c)b(a+c) - aba - cbc$. □

COROLLARY 2. *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is a positive, disjoint, injective linear map and if either $\mathfrak{A}$ or $\phi(\mathfrak{A})$ is commutative then the other is also commutative.*

Corollary 2 is a corollary of the well known result in the unital case. But for completeness we give a proof: By Lemma 2, $\phi(I)\phi(a^2) = \phi(a)^2$ for all $a \in \mathfrak{A}$. If $\phi(\mathfrak{A})$ is commutative then $\big[[\phi(a),\phi(b)],\phi(c)\big] = 0$.

But since $\phi(I)$ is invertible, it follows from Lemma 3(iii) and 3(iv) that $\phi([a,b]^2) = 0$ and $\phi([[a,b],c]) = 0$. Since ϕ is injective we have that $[a,b]^2 = 0$ and $[[a,b],c] = 0$, which implies that $[a,b]$ is a nilpotent contained in the center of $\mathfrak{A}$. Since an abelian C^*-algebra has no non-zero nilpotents, it follows that $[a,b] = 0$, and hence $\mathfrak{A}$ is commutative. Conversely, if $\mathfrak{A}$ is commutative then a similar argument gives that $\phi(\mathfrak{A})$ is commutative, in which case the condition of injection of ϕ is not needed.

THEOREM 1. *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is a positive, disjoint linear map satisfying (3) $\phi(I)\phi(ab+b^*a^*) = \phi(a)\phi(b)+\phi(b^*)\phi(a^*)$ for all unitary elements $a, b \in \mathfrak{A}$ then ϕ preserves absolute values.*

PROOF. The restriction of ϕ to an abelian C^*- subalgebra of $\mathfrak{A}$ is completely positive (cf. [4, Theorem 3.10]). Since every normal element $a \in \mathfrak{A}$ is contained in an abelian C^*-subalgebra, it follows from Gardner's theorem that ϕ preserves absolute values on normal elements, and hence on unitary elements. Thus if u is unitary then $|\phi(u)| = \phi(|u|) = \phi(I)$. Recall that the unitary group of $\mathfrak{A}$ spans $\mathfrak{A}$. More specifically, if $h \in \mathfrak{A}$ is any self-adjoint element then the spectral radius of $\frac{h}{2||h||}$ is less than 1, so that we may write ([1, Proposition I.14]) that $h = ||h||(u+u^*)$ for some unitary element $u \in \mathfrak{A}$. Thus if $a \in \mathfrak{A}$ is arbitrary then it can be written by a constant multiple of a sum of four unitaries: $a = \sum_{i-1}^{4} c_i u_i$, where $c_1, c_2, c_3, c_4 \in \mathbb{R}$ and $u_i's$ are unitary. We now have that

$$
\begin{aligned}
|\phi(a)|^2 &= \sum_{i=1}^{4} c_i \phi(u_i)^* \sum_{i=1}^{4} c_i \phi(u_i) \\
&= \sum_{i=1}^{4} {c_i}^2 |\phi(u_i)|^2 + \sum_{\substack{i,j \in \{1,\cdots,4\} \\ i<j}} c_i c_j (\phi(u_i^*)\phi(u_j) + \phi(u_j^*)\phi(u_i)) \\
&= \sum_{i=1}^{4} {c_i}^2 \phi(I)^2 + \phi(I) \sum_{\substack{i,j \in \{1,\cdots,4\} \\ i<j}} c_i c_j \phi(u_i^* u_j + u_j^* u_i) \quad ((\text{by } (3))
\end{aligned}
$$

$$= \phi(I)\phi\left(\sum_{i=1}^{4} c_i{}^2 I + \sum_{\substack{i,j\in\{1,\cdots,4\}\\ i<j}} c_i c_j (u_i^* u_j + u_j^* u_i)\right)$$
$$= \phi(I)\phi\left(\sum_{i=1}^{4} c_i u_i^* \sum_{i=1}^{4} c_i u_i\right)$$
$$= \phi(I)\phi(|a|^2)$$
$$= \phi(|a|)^2,$$

which implies that ϕ preserves absolute values because ϕ is positive. □

THEOREM 2. *Suppose $\phi : \mathfrak{A} \to \mathfrak{B}$ is a positive, disjoint linear map. If any $*$-anti-homomorphism $\psi : \mathfrak{A} \to \mathfrak{B}$ is skew-hermitian with respect to every commutators of unitary elements, in the sense that*

(4) $\qquad \psi\big([a,b]\big)^* = -\psi\big([a,b]\big)$ *for all unitary elements* $a, b \in \mathfrak{A}$

then ϕ preserves absolute values.

PROOF. We first assume that ϕ is unital and hence, by Lemma 2, it is a Jordan homomorphism. Recall ([5, Theorem 3.3]) that every Jordan homomorphism of C^*-algebras is a direct sum of a $*$-homomorphism and a $*$- anti-homomorphism: say, $\phi = \phi_1 + \phi_2$ where ϕ_1(resp. ϕ_2) is a $*$ $-$homomorphism (resp. $*$-anti-homomorphism). Then our condition (4) gives that for any unitary elements $a, b \in \mathfrak{A}$,

$$\begin{aligned}\phi(ab + b^*a^*) &= \phi_1(a)\phi_1(b) + \phi_2(a)\phi_2(b) + \phi_1(b^*)\phi_1(a^*) + \phi_2(b^*)\phi_2(a^*)\\ &\quad + \big(\phi_2(ab - ba) + (\phi_2(ab - ba))^*\big)\\ &= \phi(a)\phi(b) + \phi(b^*)\phi(a^*).\end{aligned}$$

Thus by Theorem 1, ϕ preserves absolute values. If ϕ is not unital then in view of (1), ϕ can be written as: $\phi = \phi(I)\psi$, where ψ is a Jordan homomorphism (Note that ψ is a unital, positive, disjoint map). then by what we have just proved, ψ is a $*$-homomorphism. Therefore we have that $\phi(I)\phi(ab) = \phi(I)^2\psi(ab) = \phi(I)\psi(a)\phi(I)\psi(b) = \phi(a)\phi(b)$, which, by (2), gives the result. □

REMARK 2. If either $\mathfrak{A}$ or $\phi(\mathfrak{A})$ is commutative then evidently (4) holds: thus we recapture Corollary 1.

We recall that a linear map $\phi : \mathfrak{A} \to \mathfrak{B}$ is called a *derivation* if $\phi(ab) = a\phi(b) + \phi(a)b$ for all $a, b \in \mathfrak{A}$ and is called a *Jordan derivation* if $\phi(a^2) = a\phi(a) + \phi(a)a$ for all $a \in \mathfrak{A}$.

EXAMPLE. Let ψ_i $(i = 1, 2)$ be linear maps of $\mathfrak{A}$ into itself such that $\psi_1\psi_2 = \psi_2\psi_1 = 0$. Define $\phi : \mathfrak{A} \to M_2(\mathfrak{A})$ by

$$\phi(a) = \begin{pmatrix} a & \psi_1(a) \\ \psi_2(a) & a \end{pmatrix} \quad \text{for each } a \in \mathfrak{A}.$$

If ϕ is a unital, positive, disjoint linear map then ϕ preserves absolute values.

PROOF. By lemma 2, ϕ is a Jordan homomorphism. Thus since $\psi_1\psi_2 = \psi_2\psi_1 = 0$,

$$\begin{pmatrix} a^2 & \psi_1(a^2) \\ \psi_2(a^2) & a^2 \end{pmatrix} = \phi(a^2) = \phi(a)^2$$
$$= \begin{pmatrix} a^2 & a\psi_1(a) + \psi_1(a)a \\ \psi_2(a)a + a\psi_2(a) & a^2 \end{pmatrix},$$

which implies that each ψ_i is a Jordan derivation. But since every Jordan derivation of a C^*-algebra into itself is a derivation ([3]), it follows that each ψ_i is a derivation. Therefore

$$\phi(ab) = \begin{pmatrix} ab & \psi_1(ab) \\ \psi_2(ab) & ab \end{pmatrix}$$
$$= \begin{pmatrix} ab & a\psi_1(b) + \psi_1(a)b \\ \psi_2(a)b + a\psi_2(b) & ab \end{pmatrix} = \phi(a)\phi(b)$$

which, by (2), implies that ϕ preserves absolute values. □

REMARK 3. The transposition map on $M_2(\mathbb{C})$ is a unital, positive Jordan injective map. But this map is not a $*$-homomorphism in general (in fact, it is a $*$-anti-homomorphism). This example shows that if the answer to Gardner's problem is affirmative then the passage from Jordan-ness to $*$-ness use disjoint-ness necessarily although it was already used in the passage from positive-ness to Jordan-ness.

REMARK 4. Suppose $\phi : \mathfrak{A} \to \mathfrak{B}$ is a unital, positive, disjoint linear map and write

$$\phi = \phi_1 + \phi_2, \text{ where } \phi_1 \text{ (resp. } \phi_2) \text{ is a } *\text{-homomorphism (resp. } *\text{-anti-homomorphism).}$$

Then we can see that $\phi(ab) - \phi(a)\phi(b) = \phi_2([a,b])$. Thus if $ab = 0$ then $\phi_2(a)\phi_2(b) = \phi_2(ba) = 0$, which says that ϕ_2 is also a positive, disjoint linear map. Thus Gardner's problem reduces to the followings:

Does there exist a $*$-anti-homomorphism which is a positive, disjoint linear map?

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