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ON THE PARTITION OF UNIPOTENT RADICALS OF PARABOLIC SUBGROUPS IN CHEVALLEY GROUPS

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ABSTRACT. Let P_J be a standard parabolic subgroup of a Chevalley group G and U_J the unipotent radical of P_J. In this paper, we find a certain partition of the set of roots corresponding to root subgroups generating U_J and investigate some properties of the partition.

1. Introduction

Let Φ be a root system in a Euclidean space E equipped with a positive definite scalar product $(\ ,\ )$ and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be a fixed base of the root system Φ [Hu]. We denote Φ^+ the set of positive roots in Φ. Denote W the Weyl group of Φ generated by the simple reflections w_i, $i = 1, \dots, \ell$, corresponding to simple roots α_i. For $x, y \in E$, we denote

$$\langle x, y \rangle = 2\frac{(x, y)}{(y, y)}.$$

Let J be a subset of $I = \{1, \dots, \ell\}$. Then Δ_J is a subset of Δ such as $\{\alpha_j \in \Delta \,|\, j \in J\}$, Φ_J is the subsystem of Φ spanned by $\{\alpha_j \,|\, j \in J\}$ and W_J is the Weyl group of Φ_J generated by $\{w_j \,|\, j \in J\}$. Then W_J acts on $\Phi^+ - \Phi_J$. We assume that Φ is irreducible and $J \neq I$. Throughout this paper, we shall assume that the numberings of $\alpha_i \in \Delta$ are those of [Hu, p. 58].

Consider a Chevalley group G derived from any type representation, over a field F, defined by a complex semisimple Lie algebra with a root system Φ (see [St]).

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Let P_J be the parabolic subgroup of G corresponding to the subset J of I. Then we have a Levi decomposition $P_J = L_J U_J$ with a Levi subgroup L_J and the unipotent radical U_J of P_J. The unipotent radical U_J is generated by $\{x_\alpha(t) \mid \alpha \in \Phi^+ - \Phi_J,\ t \in F\}$, where $x_\alpha(t)$ is an element of G corresponding to a root $\alpha \in \Phi^+ - \Phi_J$ and $t \in F$ (see [Ca]). Gibbs [Gi] characterized the automorphisms of U_J when $J = \emptyset$ and char$(F) \neq 2, 3$, and Khor [Kh] did the same thing when $\Phi = A_\ell$, $J \neq \emptyset$ and char$(F) \neq 2$.

We denote $\mathbb{Z}^+$ the set of all positive integers including 0.

DEFINITION 1.1. [Le] Set $n = \ell - |J|$ and $J' = I - J = \{ i_1, \dots, i_n \}$ with $1 \leq i_1 < i_2 < \cdots < i_n \leq \ell$. For $\alpha \in \Phi^+ - \Phi_J$, we say that α is of type $\mathbf{A}$ if

$$\alpha = a_1\alpha_{i_1} + a_2\alpha_{i_2} + \cdots + a_n\alpha_{i_n} + (\text{ terms in } \alpha_j,\ \ j \in J\ ),$$

where $\mathbf{A} = (a_1, a_2, \dots, a_n) \in (\mathbb{Z}^+)^n$.

Then we know that a W_J-orbit of $\Phi^+ - \Phi_J$ can be characterized by its type (a lattice point in $(\mathbb{Z}^+)^{\ell - |J|}$) if there exist roots only of the same length of the type [Le]. By the way, there may exist different orbits of the same type in some cases, that is, in cases that there exist two kinds of roots of different lengths of the same type.

In this paper, whether the lengths of roots in a type are all equal or not, we show that there exists a unique root with minimal height among roots of the same type and, similarly, there exists a unique root with maximal height even though different orbits of the same type exist.

Also, we introduce J-roots. Let $J = \{j_1, j_2, \dots, j_m\}$ with $1 \leq j_1 < j_2 < \cdots < j_m \leq \ell$, and $J' = I - J = \{i_1, \cdots, i_n\}$ with $1 \leq i_1 < \cdots < i_n \leq \ell$. So $I = J \cup J'$ and $\ell = |I| = |J| + |J'| = m + n$. We define the following:

DEFINITION 1.2. For $\alpha = a_1\alpha_{i_1} + \cdots + a_n\alpha_{i_n} + k_1\alpha_{j_1} + \cdots + k_m\alpha_{j_m} \in \Phi^+ - \Phi_J$ with $k_1, \dots, k_m \in \mathbb{Z}^+$, the set

$$\langle\alpha\rangle = \{a_1\alpha_{i_1} + \cdots + a_n\alpha_{i_n} + t_1\alpha_{j_1} + \cdots + t_m\alpha_{j_m} \in \Phi^+ - \Phi_J \mid t_s \in \mathbb{Z}^+,\quad 1 \leq s \leq m\}$$

is called a J-root.

We then define the addition of J-roots and show that the addition of J-roots can be characterized by the addition of corresponding lattice points. As a result, we obtain a partition of $\Phi^+ - \Phi_J$ by J-roots. Since U_J is generated by $x_\alpha(t)$ for $t \in F$ and $\alpha \in \Phi^+ - \Phi_J$, it seems that the notion of J-roots plays an important role when we study the automorphisms of U_J. (See [Gi], [Kh], and [Je].)

2. A Partition of $\Phi^+ - \Phi_J$

For $\alpha \in \Phi^+ - \Phi_J$, let $[\alpha]$ denote the W_J-orbit containing α. Then,

THEOREM 2.1. *Let α and β be roots in $\Phi^+ - \Phi_J$ with the same length. Assume that α and β are of types* **A** *and* **B**, *respectively. Then $[\alpha] = [\beta]$ if and only if* $\mathbf{A} = \mathbf{B}$.

PROOF. See [Le]. □

COROLLARY 2.2. *Let α and β be roots in $\Phi^+ - \Phi_J$. If $[\alpha] = [\beta]$, then $\langle\alpha\rangle = \langle\beta\rangle$.*

PROOF. Assume α is of type **A** and β is of type **B**. If $[\alpha] = [\beta]$, then $\beta = w \cdot \alpha$ for some $w \in W_J$. Since the action of w_j for $j \in J$ possibly changes the coefficient of α_j, we have $\mathbf{A} = \mathbf{B}$. Since $\mathbf{A} = \mathbf{B}$ if and only if $\langle\alpha\rangle = \langle\beta\rangle$ by definition, even though lengths of roots α and β are different, we show $\langle\alpha\rangle = \langle\beta\rangle$. □

However, the converse of Corollary 2.2 is not true. For example, let Φ be of type B_2 denoting α_2 as the longer root and $J = \{1\}$. Choosing the roots $\alpha = \alpha_1 + \alpha_2$ and $\beta = \alpha_2$, we get a counterexample.

PROPOSITION 2.3. *Assume that α is a root of the maximal height among roots in its orbit $[\alpha]$. Then for any root β in $[\alpha]$, $\alpha - \beta$ is a sum of simple roots α_j for $j \in J$. Therefore, such α is uniquely determined. Also there is a unique root of minimal height in each orbit.*

PROOF. See [Le]. □

In the proof of the following proposition, we exclude the case of the root system $\Phi = G_2$. But the proposition is also true to the case of $\Phi = G_2$, which is easily checked by direct observation of the roots of G_2.

PROPOSITION 2.4. *Let α be a root of $\Phi^+ - \Phi_J$. Then there is a unique root with minimal height in $\langle\alpha\rangle$, and also a unique root with maximal height in $\langle\alpha\rangle$.*

PROOF. Let **A** be the type of the root α. If all roots of type **A** in $\Phi^+ - \Phi_J$ have the same length, we can apply Proposition 2.3 to this case, because $\langle\alpha\rangle = [\alpha]$ by Theorem 2.1.

If roots of type **A** have different lengths, let r_0 be the root with minimal height among short roots of type **A** and r_1 the root with minimal height among long roots of type **A**. This is possible because, in an irreducible root system Φ, at most two root lengths occur [Hu], and by Proposition 2.3, there is a unique root with such property in each orbit. Since r_1 and r_0 are roots with minimal height in their orbits, $\langle r_1, \alpha_j\rangle \leq 0$ and $\langle r_0, \alpha_j\rangle \leq 0$ for any $\alpha_j \in \Delta_J$. Therefore, if $\langle r_0, r_1\rangle \leq 0$, $\{r_0, r_1\} \cup \Delta_J$ is a linearly independent set. We can show this by the argument showing the fact that any set of vectors lying strictly on one side of a hyperplane in E and forming obtuse angles pairwise must be linearly independent. But this is absurd, since $r_1 - r_0 = \sum_{j\in J} k_j\alpha_j$ ($k_j \in \mathbb{Z}$). So $\langle r_0, r_1\rangle > 0$. Therefore $r_0 - r_1$ and $r_1 - r_0$ are roots. Suppose $r_1 - r_0$ is positive. Then

$$\langle r_1, r_1 - r_0\rangle = \frac{2(r_1, r_1 - r_0)}{(r_1 - r_0, r_1 - r_0)} = 2,$$

since r_1 is a long root and $r_1 - r_0$ is a short root. So $r_1 - 2(r_1 - r_0) \in \Phi$. Also, $r_1 - 2(r_1 - r_0) = 2r_0 - r_1$ is of the same type with r_1, that is type **A**, because $r_1 - r_0 \in \sum \mathbb{Z}\Delta_J$.

Now, the length of the root $2r_0 - r_1$ is long and $r_1 = (2r_0 - r_1) +$

$2(r_1 - r_0)$, which is depicted in Figure 1.

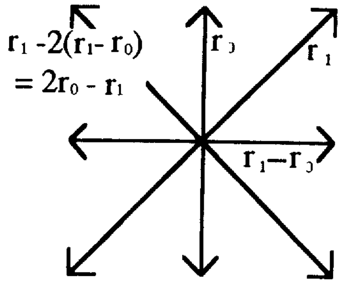

Figure 1

But this makes a contradiction to the minimality of r_1 among the long roots of type **A**.

So $r_0 - r_1$ is positive. And since $r_0 = r_1 + (r_0 - r_1)$, we conclude r_1 is the root with the minimal height among roots of type **A**. Here we note $r_0 - r_1$ is a short root.

Now, consider the maximality. By the method similar to the above, in the case that roots of type **A** have different lengths, we put μ_0 as the root with maximal height among short roots of type **A** and μ_1 as the root with maximal height among long roots of type **A**. Then μ_0 is a member of the orbit $[r_0]$ and μ_1 is in the orbit $[r_1]$. So we can write $\mu_0 = w \cdot r_0$ for some $w \in W_J$. Since $r_0 = r_1 + \eta$ for some short root η in Φ_J^+,

$$\mu_0 = w \cdot r_0 = w \cdot r_1 + w \cdot \eta.$$

Then, since $r_1 \in \Phi^+ - \Phi_J$ and $w \in W_J$, $w \cdot r_1$ is a long root in $\Phi^+ - \Phi_J$ of type **A** and $w \cdot \eta$ is a short root in Φ_J. Now for the notational convenience, put $w \cdot r_1 = \beta$ and $w \cdot \eta = u$, so $\mu_0 = \beta + u$, where β is a long root of type **A** and u is a short root in Φ_J.

If u is negative, $-u$ is in Φ_J^+. Then we can write $\beta = \mu_0 + (-u)$. It means $\beta \succ \mu_0$. (Here the symbol '$\prec$' means that $\beta \prec \alpha$ iff $\alpha - \beta$ is a sum of positive roots or $\beta = \alpha$ for $\alpha, \beta \in \Phi$.) Also $\mu_1 \succ \beta$, because μ_1 is the root with maximal height among long roots of type **A**.

So $\mu_1 \succ \mu_0$ and also we can write $\mu_1 = \mu_0 + \delta$, where δ is a sum of positive roots in Φ_J. Therefore, μ_1 is the unique root with maximal height among roots of type **A**.

If u is positive, consider the value of $\langle \mu_0, u \rangle$. Since μ_0 is a root of short length, $\langle \mu_0, u \rangle \leq 1$. Therefore,

$$\langle \mu_0, u \rangle = \langle \beta + u, u \rangle = \langle \beta, u \rangle + 2 \leq 1.$$

So $\langle \beta, u\rangle \leq -1$. Then $\langle \beta, u\rangle = -2$, because β is a long root and u is a short root.

So $\beta + 2u$ becomes a root and also is a positive root of type $\mathbf{A}$ with long length. Therefore $\mu_1 \succ \beta + 2u$ because μ_1 is a maximal height root among long roots of type $\mathbf{A}$. Also $\beta + 2u = (\beta + u) + u = \mu_0 + u$. So $\beta + 2u \succ \mu_0$. Therefore $\mu_1 \succ \mu_0$ and also we can write $\mu_1 = \mu_0 + \delta$, where δ is a sum of some positive roots in Φ_J. □

From now on, for the convenience of notation, for roots $\alpha \in \Phi^+ - \Phi_J$, let $\overline{\alpha}$ denote the root of maximal height among roots in $\langle\alpha\rangle$ and $\underline{\alpha}$ the root of minimal height among roots in $\langle\alpha\rangle$. By above proposition, those are well-defined. So if the root α is of type $\mathbf{A} = (a_1, \dots, a_n)$, we can write $\overline{\alpha} = a_1\alpha_{i_1} + \cdots + a_n\alpha_{i_n} + b_1\alpha_{j_1} + \cdots + b_m\alpha_{j_m}$ for some $b_1, \dots, b_m \in \mathbb{Z}^+$, and $\underline{\alpha} = a_1\alpha_{i_1} + \cdots + a_n\alpha_{i_n} + c_1\alpha_{j_1} + \cdots + c_m\alpha_{j_m}$ for some $c_1, \dots, c_m \in \mathbb{Z}^+$, where $J = \{j_1, \dots, j_m\}$ and $I - J = \{i_1, \dots, i_n\}$.

Now, we define the addition of the J-roots in $\Phi^+ - \Phi_J$.

DEFINITION 2.5. For α, β, and γ in $\Phi^+ - \Phi_J$, we say $\langle\alpha\rangle + \langle\beta\rangle = \langle\gamma\rangle$ if there exist $\alpha' \in \langle\alpha\rangle$ and $\beta' \in \langle\beta\rangle$ such that $\alpha' + \beta'$ is a root and $\alpha' + \beta' \in \langle\gamma\rangle$.

THEOREM 2.6. *Let Φ be an irreducible root system. And let α and β be roots in $\Phi^+ - \Phi_J$ of types $\mathbf{A}$ and $\mathbf{B}$, respectively. If there exists a root γ of type $\mathbf{A} + \mathbf{B}$ in Φ, then $\langle\alpha\rangle + \langle\beta\rangle$ is the same J-root as $\langle\gamma\rangle$.*

PROOF. Since $\underline{\alpha}$ is a minimal height root of type $\mathbf{A}$, that is, a root of minimal height in the root set $\langle\alpha\rangle$, $\underline{\beta}$ is that of type $\mathbf{B}$ and $\underline{\gamma}$ is that of type $\mathbf{A} + \mathbf{B}$, we can write

$$(1) \qquad \underline{\gamma} = \underline{\alpha} + \underline{\beta} + x_1\alpha_{j_1} + \cdots + x_m\alpha_{j_m} \qquad \text{for some } x_1, \dots, x_m \in \mathbb{Z},$$

where $\alpha_{j_1}, \dots, \alpha_{j_m} \in \Delta_J$ for $J = \{j_1, \dots, j_m\}$.

Moving terms α_{j_k}'s whose coefficients x_k's are negative for some k's belonging to $\{1, \dots, m\}$ in the righthand side of equation (1) to its left-hand side, we can rewrite equation (1) as following:

$$(2) \qquad \underline{\gamma} + \sum_{j_k \in J_1} y_k\alpha_{j_k} = \underline{\alpha} + \underline{\beta} + \sum_{j_k \in J_2} x_k\alpha_{j_k},$$

where $J_1 = \{j_k \in J \,|\, x_k < 0 \text{ in equation (1)}\}$, $J_2 = \{j_k \in J \,|\, x_k > 0 \text{ in equation (1)}\}$ and $y_k = -x_k$ for $j_k \in J_1$.

Therefore, the coefficients $\{y_k \,|\, j_k \in J_1\}$ and $\{x_k \,|\, j_k \in J_2\}$ in equation (2) are all positive integers. And the sets J_1 and J_2 are disjoint.

Let $\epsilon = \underline{\gamma} + \sum_{j_k \in J_1} y_k \alpha_{j_k}$. Then

$$\begin{aligned}(\epsilon, \epsilon) &= (\underline{\gamma} + \sum_{j_k \in J_1} y_k \alpha_{j_k},\ \underline{\alpha} + \underline{\beta} + \sum_{j_k \in J_2} x_k \alpha_{j_k}) \\ &= (\underline{\gamma}, \underline{\alpha}) + (\underline{\gamma}, \underline{\beta}) + \sum_{j_k \in J_2} x_k (\underline{\gamma}, \alpha_{j_k}) + \sum_{j_k \in J_1} y_k (\alpha_{j_k}, \underline{\alpha}) \\ &\quad + \sum_{j_k \in J_1} y_k (\alpha_{j_k}, \underline{\beta}) + \sum_{\substack{j_k \in J_1 \\ j_s \in J_2}} y_k x_s (\alpha_{j_k}, \alpha_{j_s}).\end{aligned}$$

Therefore, if $\langle \underline{\gamma}, \underline{\alpha} \rangle \leq 0$ and $\langle \underline{\gamma}, \underline{\beta} \rangle \leq 0$, we have $(\epsilon, \epsilon) \leq 0$, because $\underline{\gamma}$, $\underline{\alpha}$, and $\underline{\beta}$ are the roots with minimal heights in the roots of their own types, having the property that $(\underline{\gamma}, \alpha_{j_k}) \leq 0$, $(\underline{\alpha}, \alpha_{j_k}) \leq 0$, and $(\underline{\beta}, \alpha_{j_k}) \leq 0$ for all $\alpha_{j_k} \in \Delta_J$, and because $(\alpha_{j_k}, \alpha_{j_s}) \leq 0$ for simple roots $\alpha_{j_k} \neq \alpha_{j_s}$.

It forces that $\epsilon = \underline{\gamma} + \sum_{j_k \in J_1} y_k \alpha_{j_k} = 0$. But, it contradicts the fact that $\underline{\gamma} \in \Phi^+ - \Phi_J$. So it should be that $\langle \underline{\gamma}, \underline{\alpha} \rangle > 0$ or $\langle \underline{\gamma}, \underline{\beta} \rangle > 0$. Therefore, by [Hu, 9.4], $\underline{\gamma} - \underline{\alpha}$ may be a root or $\underline{\gamma} - \underline{\beta}$ may be a root, in that case, they are included in $\Phi^+ - \Phi_J$.

If $\underline{\gamma} - \underline{\alpha}$ is a root, since $\underline{\gamma} - \underline{\alpha}$ has the same type as that of $\underline{\beta}$, the roots $\underline{\alpha}$ in $\langle \alpha \rangle$ and $\underline{\gamma} - \underline{\alpha}$ in $\langle \beta \rangle$ satisfy

$$\underline{\alpha} + (\underline{\gamma} - \underline{\alpha}) = \underline{\gamma} \in \langle \gamma \rangle \quad \text{i.e.} < \alpha > + < \beta > = < \gamma > \text{ by Definition 2.5.}$$

Also, for the case of $\underline{\gamma} - \underline{\beta} \in \Phi^+ - \Phi_J$, we use the same argument. □

COROLLARY 2.7. *Let* α, β, *and* γ *be roots of type* **A**, **B**, *and* **C**, *respectively. Then*

$$\langle \alpha \rangle + \langle \beta \rangle = \langle \gamma \rangle$$

if and only if $\mathbf{A} + \mathbf{B} = \mathbf{C}$.

PROOF. By the above theorem and the definition of the addition, we can easily prove the statement. □

COROLLARY 2.8. *Let α and β be roots in $\Phi^+ - \Phi_J$ of type* **A** *and* **B** *respectively. And γ be the root in $\Phi^+ - \Phi_J$ of type* **A** + **B**. *Then we can write*

$$\underline{\gamma} = \underline{\alpha} + \underline{\beta} + x_1\alpha_{j_1} + \cdots + x_m\alpha_{j_m} \quad \text{for some } x_1, \ldots, x_m \in \mathbb{Z}^+,$$

where $\alpha_{j_1}, \ldots, \alpha_{j_m}$ are roots in Δ_J.

PROOF. First, by assumptions, we can write

$$(1) \qquad \underline{\gamma} = \underline{\alpha} + \underline{\beta} + x_1\alpha_{j_1} + \cdots + x_m\alpha_{j_m} \qquad \text{for some } x_1, \ldots, x_m \in \mathbb{Z}.$$

Then, by above theorem, we know $\langle \underline{\gamma}, \underline{\alpha} \rangle > 0$ or $\langle \underline{\gamma}, \underline{\beta} \rangle > 0$. If $\langle \underline{\gamma}, \underline{\alpha} \rangle > 0$, $\underline{\gamma} - \underline{\alpha}$ is a root of $\Phi^+ - \Phi_J$ with the same type as $\langle \beta \rangle$. Therefore $\underline{\gamma} - \underline{\alpha} = \underline{\beta} + y_1\alpha_{j_1} + \cdots + y_m\alpha_{j_m}$ for some $y_1, \ldots, y_m \in \mathbb{Z}^+$, because $\underline{\beta}$ is the root with minimal height in $\langle \beta \rangle$.

But since $\underline{\gamma} - \underline{\alpha} = \underline{\beta} + x_1\alpha_{j_1} + \cdots + x_m\alpha_{j_m}$ for some $x_1, \ldots, x_m \in \mathbb{Z}$, by the equation (1), $(y_1 - x_1)\alpha_{j_1} + \cdots + (y_m - x_m)\alpha_{j_1} = 0$. Therefore, $y_1 = x_1, \ldots, y_m = x_m$ and $x_j \geq 0$ for all $1 \leq j \leq m$. $\square$

DEFINITION 2.9. [Le] A J-root is called an indecomposable J-root if it cannot be written as a sum of two (not necessarily distinct) J-roots.

DEFINITION 2.10. [Le] Let $\langle \alpha \rangle$ be a J-root. We define the level of $\langle \alpha \rangle$ by the sum of coefficients with respect to the set $\{ \langle \alpha_i \rangle \mid i \in I - J \}$. Here we can easily show that the set $\{ \langle \alpha_i \rangle \mid i \in I - J \}$ is a basis of a real vector space consisting of J-roots. Note that the level of $\langle \alpha \rangle$ is equal to the sum of the coordinates of **A**, where α is of type **A**.

PROPOSITION 2.11. *For any $\alpha \in \Phi^+ - \Phi_J$, $\langle \alpha \rangle$ can be written in the form*

$$\langle \alpha \rangle = \langle \alpha_{k_1} \rangle + \cdots + \langle \alpha_{k_t} \rangle, \qquad k_1, \ldots, k_t \in I - J$$

in such a way that $\langle \alpha_{k_1} \rangle + \cdots + \langle \alpha_{k_s} \rangle$ is a J-root for any $s = 1, \ldots, t$, where $\langle \alpha_{k_i} \rangle$'s for $1 \leq i \leq t$ are not necessarily distinct.

PROOF. Assume that the type of $\langle \alpha \rangle$ is $\mathbf{A} = (a_1, \ldots, a_n)$ and $a_1 + \cdots + a_n \neq 1$.

First, we choose $\underline{\alpha} \in \langle \alpha \rangle$, the root of minimal height in $\langle \alpha \rangle$. Then there exists a root α_i for some $i \in I - J$ such that $\langle \underline{\alpha}, \alpha_i \rangle > 0$, because $\langle \underline{\alpha}, \alpha_j \rangle \leq 0$ for all $j \in J$. Since $\langle \alpha \rangle = \langle \underline{\alpha} \rangle = \langle \underline{\alpha} - \alpha_i \rangle + \langle \alpha_i \rangle$ and $\langle \underline{\alpha} - \alpha_i \rangle$ is a J-root with a level smaller than that of $\langle\alpha\rangle$, we show the proposition, if we use an induction on the level which is the sum of the coordinates of $\mathbf{A}$. $\square$

REMARK 2.12. [Le] If ψ is an automorphism of U_J, then ψ is determined by its action on $\mathfrak{X}_{\langle\alpha\rangle}$ such that the level of $\langle\alpha\rangle = 1$, where $\mathfrak{X}_{\langle\alpha\rangle} = \langle x_\beta(t) \mid \beta \in \langle\alpha\rangle, t \in F \rangle$ and $x_\beta(t)$ is an element of the chevalley group G corresponding to a root $\beta \in \Phi^+ - \Phi_J$ and $t \in F$. It seems that the level function on J-roots plays an important role when we study the automorphisms of U_J.

References

[Ca] R. W. Carter, *Simple Groups of Lie Type*, John Wiley & Sons, London, 1972.

[Gi] J. A. Gibbs, *Automorphisms of certain unipotent groups*, J. Algebra **14** (1970), 203-228.

[Hu] J. E.Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972.

[Je] Y. K. Jeong, *Automorphisms of Unipotent Radicals of Parabolic Subgroups in the Chevalley Groups*, Ph.D. Thesis, Seoul National University, 1995.

[Kh] H. P. Khor, *The automorphisms of the unipotent radical of certain parabolic subgroups of* $\mathrm{GL}(\ell+1,k)$, J. Algebra **96** (1985), 54-77.

[Le] I. S. Lee and B. H. Im, *On the unipotent radicals of parabolic subgroups in Chevalley groups*, Comm. Korean Math. Soc. **8** (1993), 213-223.

[St] R. Sternberg, *Lectures on Chevalley Groups*, Yale University, 1967.

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