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MONOMIAL CURVES WHICH ARE SET-THEORETIC COMPLETE INTERSECTIONS

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ABSTRACT. We prove that some monomial curves are set-theoretic complete intersection of two surfaces. We also give explicitly the equations of corresponding surfaces

Throughout the paper we assume that the ground field k is of arbitrary characteristic.

For an ordered triple $p \leq q \leq r$ of nonnegative integers, let $C_{p,q,r}$ denote the curve in $\mathbf{P}^3$ given parametrically by

$$(*) \qquad \begin{cases} w = s^r \\ x = s^{r-p}t^p \\ y = s^{r-q}t^q \\ z = t^r, \end{cases}$$

where $(s:t) \in \mathbf{P}^1$. We will call such a curve monomial.

Note first that if $d = gcd(p,q,r)$, then $C_{p,q,r} = C_{p',q',r'}$, where $p' = \frac{p}{d}$, $q' = \frac{q}{d}$, $r' = \frac{r}{d}$. So from now on we assume $gcd(p,q,r) = 1$. This assumption implies that the parametrization $(*)$ is injective and hence can be viewed as a resolution of singularities of the curve $C_{p,q,r}$.

It can be shown that $C_{p,q,r}$ is smooth if and only if $(p,q,r) = (1,1,1)$, $(1, 2, 2)$, or $(1, r-1, r)$.

In characteristic zero it is not known whether the smooth curves $C_{1,r-1,r}$ $(r \geq 4)$ are set-theoretic complete intersections. (In fact, no smooth curve C with degree $(C) >$ genus $(C) + 3$ has ever been described as the set-theoretic complete intersection of two surfaces.)

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In positive characteristic it was Hartshorne who gave mysterious calculations showing that $C_{1,r-1,r}$ are set-theoretic complete intersections [H2]. In this short note we will deal with singular monomial curves over arbitrary charactericstic. Our main result is as follows.

THEOREM. (i) *If $r = aq - b$ for some integers a, b such that $0 \leq b \leq q$, $1 + b \leq a$, then the monomial curve $C_{1,q,r}$ is a set-theoretic complete intersection.*

(ii) *If $r = a(r-p) - b$ for some integers a, b such that $0 \leq b \leq r-p$, $1 + b \leq a$, then the monomial curve $C_{p,r-1,r}$ is a set-theoretic complete intersection.*

The second result is contained in [RV], but we give here explicitly the equations of corresponding surfaces (See Theorem 1 and 2).

1. THEOREM. *If $r = aq - b$, for some integers a, b such that $0 \leq b \leq q$, $1 + b \leq a$, then the monomial curve $C_{1,q,r}$ is the set-theoretic complete intersection of the two surfaces with equations*

$$yw^{q-1} = x^q$$

and

$$z^q w^{r-q} + \sum_{k=0}^{q-1} (-1)^{q-k} \binom{q}{k} z^k y^{r-ka} x^{kb} w^{k(a-b-1)} = 0.$$

PROOF. Note first that the second equation makes sense, because all the exponents are nonnegative by the condition on q and r. We will show that the intersection of these two surfaces is $C_{1,q,r}$.

If $w = 0$, then from the first equation $x = 0$ and from the second equation $y = 0$, so there is only one point with $w = 0$, and that is on the curve $C_{1,q,r}$.

If $w \neq 0$, we can set $w = 1$, $x = t$, and it is sufficient to show that the only common solution of those two equations is $y = t^q$ and $z = t^r$. Substituting $w = 1$ and $x = t$, the first equation becomes $y = t^q$. Substituting $w = 1$, $x = t$, and $y = t^q$, the second equation becomes

$$z^q + \sum_{k=0}^{g-1} (-1)^{g-k} \binom{q}{k} z^k (t^r)^{q-k} = 0.$$

This is simply

$$(z - t^r)^q = 0.$$

So we have $z = t^r$. □

REMARK. (i) The curve $C_{1,q,r}$ in the above theorem is obtained as an intersection of multiplicity q.

(ii) The following is the table of the triple $(1, q, r)$ covered by the theorem.

$(1,2,3)$	$(1,3,5)$	$(1,4,7)$	$\dots$	$(1,q,2q-1)$
$(1,2,4)$	$(1,3,6)$	$(1,4,8)$		$(1,q,2q)$
$(1,2,5)$	$(1,3,7)$	$(1,4,10)$		$(1,q,3q-2)$
$\vdots$	$\vdots$	$(1,4,11)$		$(1,q,3q-1)$
		$\vdots$		$(1,q,3q)$
				$(1,q,4q-3)$
				$(1,q,4q-2)$
				$(1,q,4q-1)$
				$(1,q,4q)$
				$(1,q,5q-4)$
				$\vdots$

(iii) The missing triples in the above table are

$$(1,3,4), (1,4,5), (1,4,6), \cdots etc.$$

2. THEOREM. *If $r = a(r-p) - b$ for some integers a, b such that $0 \le b \le r-p$, $1 + b \le a$, then the monomial curve $C_{p,r-1,r}$ is the set-theoretic complete intersection of the two surfaces*

$$xz^{r-p-1} = y^{r-p}$$

and

$$z^p w^{r-p} + \sum_{k=0}^{r-p-1} (-1)^{r-p-k} \binom{r-p}{k} w^k x^{r-ka} y^{kb} z^{k(a-b-1)} = 0.$$

PROOF. Via the inversions $(s : t) \to (t : s)$ and $(w : x : y : z) \to (z : y : x : w)$, one can identify $C_{p,r-1,r}$ with $C_{1,r-p,r}$. Now the result follows from Theorem 1. The equations for $C_{p,r-1,r}$ can be obtained from the equations for $C_{1,r-p,r}$ by taking the inversion $(w : x : y : z) \to (z : y : x : w)$ □

3. COROLLARY. *$C_{r-2,r-1,r} (r \geq 3)$ is a set-theoretic complete intersection. The equations are*

$$xz = y^2 \quad \text{and} \quad z^{r-2}w^2 - 2wx^{r-a}y^bz^{a-b-1} + x^r = 0$$

where $r = 2a - b$, $0 \leq b \leq 2$, $1 + b \leq a$.

In particular, the affine curve in $\mathbb{A}^3$

$$C_{r-2,r-1,r} \cap (w \neq 0) = \{(t^{r-2}, t^{r-1}, t^r) | t \in k\}$$

is the set-theoretic complete intersection of

$$xz = y^2 \quad \text{and} \quad z^{r-2} - 2x^{r-a}y^bz^{a-b-1} + x^r = 0$$

where $r = 2a - b$, $0 \leq b \leq 2$, $1 + b \leq a$.

4. EXAMPLE. The equations for $C_{3,4,5}$ are

$$xz = y^2 \quad \text{and} \quad z^3w^2 - 2wx^2yz + x^5 = 0.$$

So the affine curve

$$C_{3,4,5} \cap (w \neq 0) = \{(t^3, t^4, t^5) | t \in k\}$$

is the intersection of the two surfaces in $\mathbb{A}^3$

$$xz = y^2 \quad \text{and} \quad z^3 - 2x^2yz + x^5 = 0.$$

Compare this pair with Hartshorne's pair ([H1] 3.4.5)

$$z^2 = x^2y \quad \text{and} \quad x^4 + y^3 - 2xyz = 0$$

whose homogenized equations define a reducible curve in $\mathbb{P}^3$ having two components $C_{3,4,5}$ and the line $(w = x = 0)$.

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