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PROPERTY OF LATTICE ON LATTICE VARIETIES I

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ABSTRACT. This paper is a contribution to the study of the properties of the lattice of all lattice varieties. Among the properties, that of finite base is investigated here. The question whether the join of two finitely based modular lattice varieties is finitely based is investigated under certain conditions.

1. Introduction

It was first shown by K. Baker that the join of two finitely based lattice varieties need not be finitely based. The first example can be found in [11], but Baker's example, which was published in [4], is more relevant here, for in it the two varieties are modular. Here the question whether the join of two finitely based modular lattice varieties is finitely based is investigated under certain condition.

The rest of this paper is divided into two sections. In section 1 we will give some preliminary definitions and facts. And finally in the last section we shall prove the main theorem. For standard concepts and facts from lattice theory we refer the reader to Grätzer[5]. However we use $+$ and $\times$ instead of $\vee$ and $\wedge$ respectively for the lattice operations.

2. Preliminaries

A class C of algebras is *finitely based* if it is the class of all models of some finite set Σ of identities.

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DEFINITION 2.1. A class $\mathcal{C}$ of first order $\mathcal{L}$-structure is *an elementary class* if there is a set Σ of first order formulas such that

$$A \in \mathcal{C} \text{ if and only if } \Sigma \text{ holds in } A.$$

And an elementary class $\mathcal{C}$ is *a strictly elementary class* if Σ can be taken to be finite.

Now we will review a well known theorem in Universal Algebra, which is very useful here.

THEOREM 2.2. *Let $\mathcal{V}$ and $\mathcal{V}'$ be varieties. Then the following are equivalent.*

(1) *$\mathcal{V} + \mathcal{V}'$ is finitely based.*
(2) *$\mathcal{V} + \mathcal{V}'$ is strictly elementary.*
(3) *The complement of $\mathcal{V} + \mathcal{V}'$ is elementary.*
(4) *The complement of $\mathcal{V} + \mathcal{V}'$ is closed under ultraproduct.*

Here the complements can be taken relative to any finitely based supervariety $\mathcal{U}$.

In Jónsson [9], the following criterion for membership in the join of two congruence distributive varieties is obtained.

THEOREM 2.3. *Suppose $\mathcal{U}$ is a congruence distributive variety, and let $\mathcal{V}$ and $\mathcal{V}'$ be subvarieties of $\mathcal{U}$ defined, relative to $\mathcal{U}$, by the identities $\alpha = \beta$ and $\gamma = \delta$, respectively. Then for an algebra A the following are equivalent.*

(1) $A \in \mathcal{V} + \mathcal{V}'$.
(2) $A \leq B \times B'$ *with $B \in \mathcal{V}$ and $B' \in \mathcal{V}'$.*
(3) $\theta \cap \theta' = 0_A$, *null congruence relation in A, where $\theta, \theta' \in Con(A)$, the lattice of all congruences over A, are the smallest congruence relations with $A/\theta \in \mathcal{V}$ and $A/\theta' \in \mathcal{V}'$, respectively.*
(4) $con(\alpha(\mu), \beta(\mu)) \cap con(\gamma(\nu), \delta(\nu)) = 0_A$ *for all $\mu, \nu \in {}^{w}A$.*

This theorem applies in particular to modular lattice varieties. The closer study of congruence relations on lattices is based on Dilworth's concept of projectivity. Consider two quotients a/b and c/d in a lattice L. If $a + d = c$ and $ad = b$, then we say that a/b *transposes up onto* c/d and that c/d *transposes down onto* a/b(in symbols, $a/b \nearrow$

c/d and $c/d \searrow a/b$). If there exists a sequence of quotients $a/b = a_0/b_0, a_1/b_1, \cdots, a_n/b_n = c/d$ such that for $i = 0, 1, 2, \cdots, n-1, a_i/b_i \nearrow a_{i+1}/b_{i+1}$ or $a_i/b_i \searrow c_{i+1}/d_{i+1}$, then we say that *$a/b$ projects onto c/d* in n steps.

For modular lattices, principal congruences can be described in terms of projectivities. Therefore the criterion for membership in $\mathcal{V} + \mathcal{V}'$ in Theorem 2.3 can be expressed in terms of this notion for modular lattice varieties.

COROLLARY 2.4[9]. *Suppose $\mathcal{U}$ is a modular lattice variety. Let $\mathcal{V}$ and $\mathcal{V}'$ be subvarieties of $\mathcal{U}$ defined, relative to $\mathcal{U}$, by the identities $\alpha = \beta$ and $\gamma = \delta$ respectively, where the inclusions $\beta \leq \alpha$ and $\delta \leq \gamma$ hold in $\mathcal{U}$. Then a lattice $L \in \mathcal{U}$ belongs to $\mathcal{V} + \mathcal{V}'$ if and only if two nontrivial subquotients of $\alpha(\mu)/\beta(\mu)$ and $\gamma(\nu)/\delta(\nu)$ with $\mu, \nu \in {}^{w}L$ never project onto each other.*

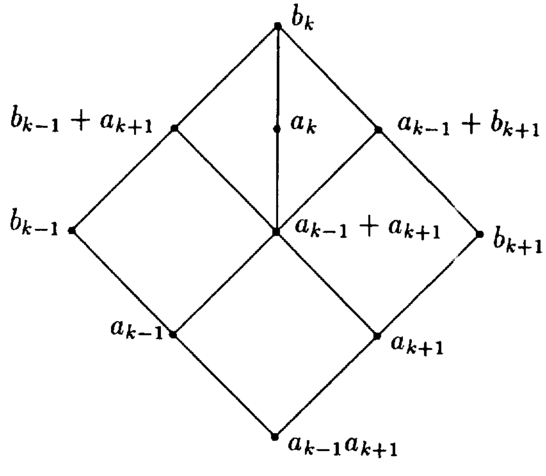


FIGURE 1

By a *strongly normal* sequence of transposes in L we mean a (finite) sequence of quotients

$$(1) \qquad a_0/b_0,\ a_1/b_1,\ \cdots, a_n/b_n$$

such that if for $0 \leq k \leq n$, $a_{k-1}/b_{k-1} \nearrow a_k/b_k \searrow a_{k+1}/b_{k+1}$, $a_k = a_{k-1} + a_{k+1}$ and $a_{k-1}a_{k+1} < b_k$ or if $a_{k-1}/b_{k-1} \searrow a_k/b_k \nearrow a_{k+1}/b_{k+1}$,

$b_k = b_{k-1}b_{k+1}$ and $b_{k-1} + b_{k+1} > a_k$. Suppose (1) is a strongly normal sequence of transposes in L. We see from Figure 1 that if a quotient a_{k-1}/b_{k-1} is nontrivial, the figure contains a nontrivial diamond

$$D_k = [v_k < x_k,\ y_k,\ z_k\ < u_k]$$

where

$$D_k = [b_{k-1} + b_{k+1} < a_{k-1} + b_{k+1}, b_k, b_{k-1} + a_{k+1} < a_k]$$

if

$$a_{k-1}/b_{k-1} \nearrow a_k/b_k \searrow a_{k+1}/b_{k+1}$$

and

$$D_k = [b_k < b_{k-1}a_{k+1}, a_k, a_{k-1}b_{k+1} < a_{k-1}a_{k+1}]$$

if

$$a_{k-1}/b_{k-1} \searrow a_k/b_k \nearrow a_{k+1}/b_{k+1}.$$

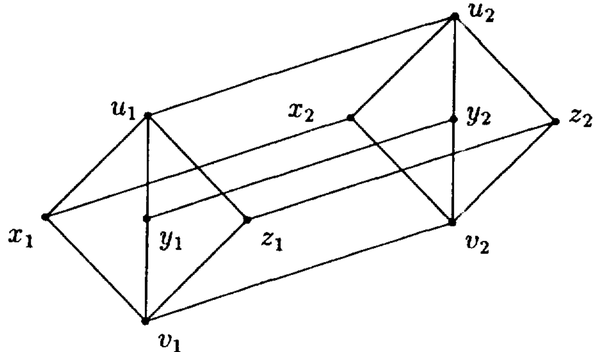
By this argument, a strongly normal sequence of transposes (1) in L generates a sequence of $(n-1)$ diamonds $D_1, D_2, \cdots, D_{n-1}$. This said to be the associated sequence of diamonds.

In Jónsson [10], it was proved that if two quotients in a modular lattice L project onto each other in n steps, then there exist nontrivial subquotients of them which project strongly normally onto each other in $\leq n$ steps. Therefore, with each sequence of projectivities there is always an associated sequence of diamonds. We must now investigate how these diamonds fit together. First, we define some notations. Given two diamonds

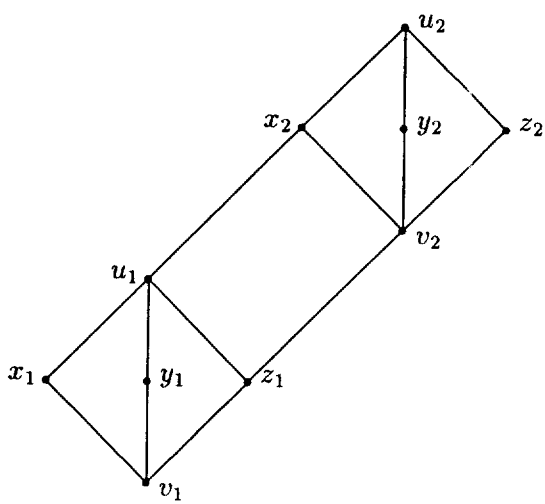
$$D_i = [v_i < x_i, y_i, z_i < u_i], \quad i = 1, 2,$$

we say that D_1 *transposes down onto* D_2 (in symbols $D_1 \searrow_{(1)} D_2$) or that D_2 *transposes up onto* D_1 (in symbols $D_2 \nearrow_{(1)} D_1$) if $u_1/v_1 \searrow u_2/v_2$, and under this transposition the vertices x_1, y_1, z_1 are mapped onto the corresponding vertices x_2, y_2, z_2 (see Figure 2). Also, we say that D_1 *translates up onto* D_2 (in symbols $D_1 \nearrow_{(2)} D_2$) and that D_1 *translates down onto* D_2 (in symbols $D_1 \searrow_{(2)} D_2$) if $u_1/z_1 \nearrow x_2/v_2$ and if $z_1/v_1 \searrow u_2/x_2$, respectively (see Figure 2). Note that $D_1 \nearrow_{(2)} D_2$

does not imply $D_2 \searrow_{(2)} D_1$. If $D = [v < x, y, z < u]$ is a diamond, then D^* is defined to be the diamond $[v < z, x, y < u]$. So $D_1 \searrow_{(1)} D_2^*$ means that $u_1/v_1 \searrow u_2/v_2$, $x_1 u_2 = z_2$, $y_1 u_2 = x_2$ and $z_1 u_2 = y_2$. The investigation of how these associated diamonds fit together was done by D. X. Hong [7]. That contains the following useful theorem. We call it Hong's Theorem in this paper.



$D_1 \nearrow_{(1)} D_2$



$D_1 \nearrow_{(2)} D_2$

FIGURE 2

THEOREM 2.5 (HONG'S THEOREM). *Let a/b and c/d be nontrivial quotients in a modular lattice such that $P(a/b, c/d) = n$, $2 \leq n \leq \infty$. Then some nontrivial subquotients a'/b' and c'/d' of a/b and c/d, respectively, can be connected by a strongly normal sequence of transposes*

$$a'/b' = a_0/b_0,\ a_1/b_1,\ \cdots,a_n/b_n = c'/d'$$

such that the associated diamonds $D_1, D_2, \cdots, D_{n-1}$ satisfy

(i) $D_k \nearrow_{(1)} D^*_{k+1}$ *or* $D_k \nearrow_{(2)} D_{k+1}$ *if* $a_k/b_k \nearrow a_{k+1}/b_{k+1}$ *and* $D_k \searrow_{(1)} D^*_{k+1}$ *or* $D_k \searrow_{(2)} D_{k+1}$ *if* $a_k/b_k \searrow a_{k+1}/b_{k+1}$, $k = 1, 2, \cdots, n-2$.

(ii) *If* $D_k \nearrow_{(1)} D^*_{k+1}$ *or* $D_k \searrow_{(1)} D^*_{k+1}$, *then* $D_k = D^*_{k+1}$, $k = 2, \cdots, n-2$.

(iii) *If* $D_k \nearrow_{(1)} D^*_{k+1}$ *or* $D_k \searrow_{(1)} D^*_{k+1}$, *then it can not happen that* $D_{k+1} \searrow_{(1)} D^*_{k+2}$ *or* $D_{k+1} \nearrow_{(1)} D^*_{k+2}$, *respectively.*

If the conditions (i), (ii) and (iii) are satisfied, then we refer to the strongly normal sequence of transposes in Hong's Theorem as a Hong sequence.

THEOREM 2.6 (JÓNSSON[11]). *Suppose $\mathcal{U}$ is a lattice variety, and let $\mathcal{V}$ and $\mathcal{V}'$ be subvarieties of $\mathcal{U}$ defined, relative to $\mathcal{U}$, by the identities $\alpha = \beta$ and $\gamma = \delta$, respectively, where the inclusions $\beta \leq \alpha$ and $\delta \leq \gamma$ hold in $\mathcal{U}$. In order for $\mathcal{V} + \mathcal{V}'$ to be finitely based relative to $\mathcal{U}$, it is necessary and sufficient that there exists a positive integer n with the following property :*

$P(n)$: For any lattice $L \in \mathcal{U}$, if there exist $\mu, \nu \in {}^{w}L$ and $c, d \in L$ with $c < d$ such that both $\alpha(\mu)/\beta(\mu)$ and $\gamma(\nu)/\delta(\nu)$ project weakly onto d/c, then there exist $\mu', \nu' \in {}^{w}L$ and $c', d' \in L$ with $c' < d'$ such that both $\alpha(\mu')/\beta(\mu')$ and $\gamma(\nu')/\delta(\nu')$ project weakly onto d'/c' in n steps.

COROLLARY 2.7. *Suppose $\mathcal{M}$ is a moduar lattice variety, and let $\mathcal{V}$ and $\mathcal{V}'$ be subvarieties of $\mathcal{M}$ defined, relative to $\mathcal{M}$, by the identitiesα $\alpha =$ β and $\gamma = \delta$, respectively, where the inclusions $\beta \leq \alpha$ and $\delta \leq \gamma$ hold in $\mathcal{M}$. In order for $\mathcal{V} + \mathcal{V}'$ to be finitely based relative to $\mathcal{M}$, it is necessary and sufficient that there exists a positive integer n with the following property :*

$P(n)$: For any lattice $L \in \mathcal{M}$, if there exist $\mu, \nu \in {}^{w}L$ such that a non-trivial subquotient of $\alpha(\mu)/\beta(\mu)$ projects onto a non-trivial subquotient of $\gamma(\nu)/\delta(\nu)$, then there exist μ', $\nu' \in {}^{w}L$ such that a non-trivial subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto $\gamma(\nu')/\delta(\nu')$ in n steps.

NOTATION. For any lattice L, if there exists a non-negative integer n such that, for all $a, b, c, d \in L$ with $a > b$ and $c > d$, where a/b projects weakly onto c/d, then a/b projects weakly onto a non-trivial subquotient of c/d in n steps, then the smallest such n is denoted by $R(L)$. If no such n exists, then we write $R(L) = \infty$. For a class $\mathcal{K}$ of lattices, $R(\mathcal{K})$ denotes the supremum of $R(L)$ for $L \in \mathcal{K}$. Also, let $\bar{x}$ denotes the image of each $x \in L$ in the homomorphic image $\bar{L}$ of L, and we shall use this notion for any homomorphic images of a given lattice.

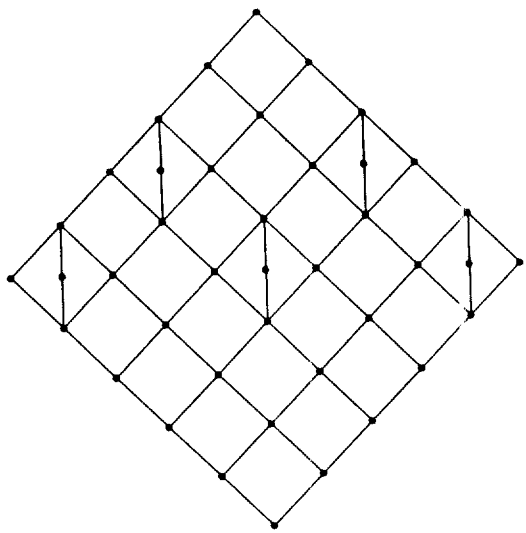
LEMMA 2.8 (BAKER [4]). *For two arbitrary lattices L and M, let $f : L \to M$ be a surjective lattice homomorphism, and let a/b and c/d be non-trivial quotients in L. Suppose $f(a)/f(b)$ projects weakly onto $f(c)/f(d)$ with $f(d) < f(c)$ in k steps in M for some $k \geq 0$. The there exist $c', d' \in L$, with $d \leq d' < c' \leq c$, such that $f(c) = f(c')$ and $f(d) = f(d')$, and such that a/b projects weakly onto c'/d' in $(k+1)$ steps if $k > 0$ and in 2 steps when $k = 0$.*

From above lemma 2.8, we get easily the following as its corollary.

COLLOARY 2.9. *For two arbitrary modular lattices M_1 and M_2, let $f : M_1 \to M_2$ be a surjective lattice homomorphism, and let a/b and c/d be non-trivial quotients in M_1. Suppose $f(a)/f(b)$ projects onto $f(c)/f(d)$ with $f(d) < f(c)$ in k steps in M_2 for some $k \geq 0$. Then there exist $c', d' \in M_1$ with $d \leq d' < c' \leq c$, such that $f(c) = f(c')$ and $f(d) = f(d')$, and such that a subquotient of a/b projects onto c'/d' in $(k+1)$ steps if $k > 0$ and in 2 steps when $k = 0$.*

3. Main Result

By a *critical* quotient of a lattice L we mean a quotient that is collapsed by every nontrivial congruence relation on L.



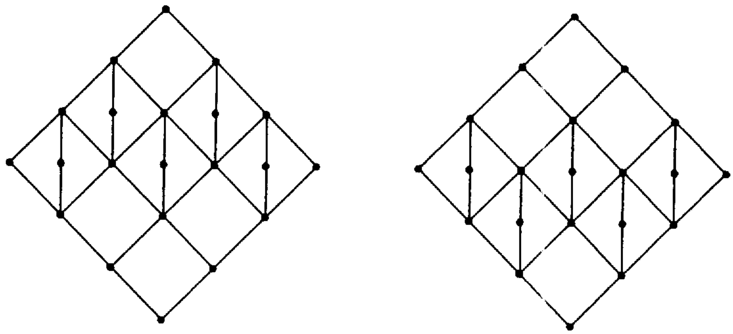
G_5

FIGURE 3

LEMMA 3.1 [8]. *The modular lattice L generated by five diamonds D_1, D_2, D_3, D_4 and D_5 with the property that $D_1 \nearrow_{(2)} D_2 \searrow_{(2)} D_3 \nearrow_{(2)} D_4 \searrow_{(2)} D_5$ has a homomorphic image of G_5 as a homomorphic image. Furthermore, L has the finite simple lattice A_5 as a homomorphic image (See Figure 3 and Figure 4)*

OBSERVATION. If A and B are sublattices of a lattice L, and if a filter F of A projects up onto an ideal I of B , then $A \cup B$ is a sublattice of L, containing A as an ideal and B as a filter.

LEMMA 3.2. *Given a sequence of Hong's associated diamonds D_1, D_2, D_3, D_4, D_5, and D_6 in a modular lattice L. If $D_1 \nearrow_{(2)} D_2$, and if the numbers below the arrows alternate, then the sublattice L_0 of L generated by $D_1, D_2, \cdots, D_6$ has the finite simple lattice B_5 pictured in Figure 5 as a homomorphic image.*



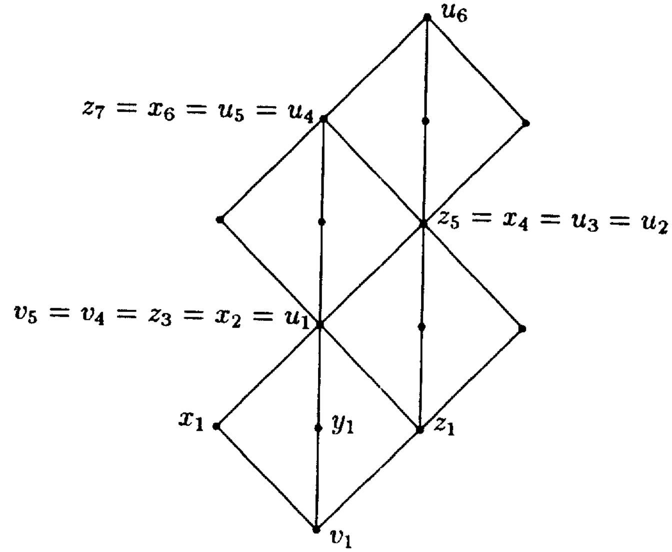
A_5 A_5^d

FIGURE 4

PROOF. By Hong's theorem, $D_2 = D_3^*$ and $D_4 = D_5^*$. Thus D_1, D_2, D_4, D_6 generates L_0. Trivially, u_1/z_1 is a filter of the sublattice D_1 of L and x_2/v_2 is an ideal of the sublattice D_2 of L. Also, by assumption, $u_1/z_1 \nearrow x_2/v_2$. Hence by the Observation, $D_1 \cup D_2$ is a sublattice of L. Also u_2/x_2 is a filter of $D_1 \cup D_2$ and x_4/v_4 is an ideal of the sublattice D_4 of L. By hypothesis, $u_3/z_3 \nearrow x_4/v_4$. Since $D_2 = D_3^*$, we have $u_2/x_2 = u_3/z_3$. Thus we have $u_2/x_2 \nearrow x_4/v_4$. Again, by the Observation, $D_1 \cup D_2 \cup D_4$ is a sublattice of L. Also u_4/x_4 is a filter of $D_1 \cup D_2 \cup D_4$ and x_6/v_6 is an ideal of the sublattice D_6 of L. By hypothesis, $u_5/z_5 \nearrow x_6/v_6$. Since $D_4 = D_5^*$, we have $u_4/x_4 = u_5/z_5$. Thus we have $u_4/x_4 \nearrow x_6/v_6$. By the Observation, $D_1 \cup D_2 \cup D_4 \cup D_6$ is a sublattice of L which is a homomorphic image of the lattice B_5' pictured in Figure 6. Therefore, L_0 has the finite lattice B_5 as a homomorphic image. The proof is complete.

LEMMA 3.3. *Let B_5 be the simple lattice of length 5 pictured in Figure 5. Then $R(B_5) = 9$.*

PROOF. By Lemma 3.5 [14], $k = 5$.

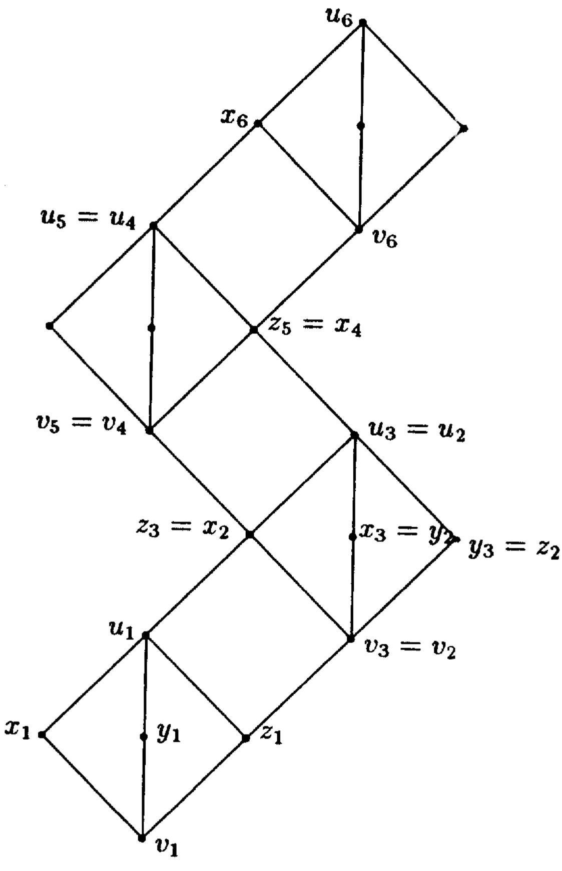


B_5

FIGURE 5

THEOREM 3.4. *Let $\mathcal{V}$ and $\mathcal{V}'$ be finitely based modular lattice varieties. If $A_2, B_5 \notin \mathcal{V}$ and $A_5 \notin \mathcal{V}'$, then $\mathcal{V} + \mathcal{V}'$ is finitely based.*

PROOF. Let $\mathcal{V}$ and $\mathcal{V}'$ be defined by the identities $\alpha = \beta$ and $\gamma = \delta$, respectively, relative to $\mathcal{M}$. We may assume that the inclusions $\beta \leq \alpha$ and $\delta \leq \gamma$ hold in every modular lattice. Letting $\mathcal{U} = \mathcal{M}$, we are going to show that the condition $P(n)$ in corollary 2.7 holds for $n = 25$. Since $\mathcal{M}$ is finitely based, it follows that $\mathcal{V} + \mathcal{V}'$ is finitely based. Consider a lattice $L \in \mathcal{M}$, and suppose there exist $\mu, \nu \in {}^{w}L$ such that a nontrivial subquotient of $\alpha(\mu)/\beta(\mu)$ projects onto a nontrivial subquotient of $\gamma(\nu)/\delta(\nu)$ in m steps.



B_5'

FIGURE 6

Assuming that the two quotients have been so chosen that m is as small as possible, we shall show that the assumption $m > n$ leads to a contradiction. There exists, by Hong's Theorem, a Hong sequence

$$b/a = b_0/a_0, b_1/a_1, \ldots, b_m/a_m = d/c$$

for some nontrivial subquotients b/a and d/c of $\alpha(\mu)/\beta(\mu)$ and

$\gamma(\nu)/\delta(\nu)$, respectively. Let $D_1, D_2, \ldots, D_{m-1}$ be the associated sequence of diamonds. Then we have the following two cases:

(1) There exists a subsequences D_k, D_{k+1}, D_{k+2}, D_{k+3} and D_{k+4} with $0 < k < m-10$ such that $D_k \nearrow_{(2)} D_{k+1}$ and the sequence of numbers below the arrows is $(2, 2, 2, 2)$.

(2) There does not exist any such subsequences with $0 < k < m-10$.

Case (1): By lemma 3.1, the lattice A_5 is a homomorphic image of the sublattice L_0 of L generated by D_k, D_{k+1}, D_{k+2}, D_{k+3} and D_{k+4}. Furthermore b_k/a_k is a nontrivial quotient in L_0 and $\overline{b}_k/\overline{a}_k$ is a critical quotient A_5. Since $A_5 \notin V'$, $\delta(\overline{\nu'}) < \gamma(\overline{\nu'})$ for some $\nu' \in {}^wL_0$. Observe that $\gamma(\overline{\nu'}) = \overline{\gamma(\nu')}$ and $\delta(\overline{\nu'}) = \overline{\delta(\nu')}$. Also $R(A_5) = 8$. Hence $\overline{\gamma(\nu')}/\overline{\delta(\nu')}$ projects weakly onto $\overline{b}_k/\overline{a}_k$ in 8 steps. Since b_k/a_k and $\gamma(\nu')/\delta(\nu')$ are nontrivial quotients in L_0, by corollary 2.9, there exists a nontrivial subquotient y/x, with $a_k \leq x < y \leq b_k$, such that $\overline{y} = \overline{b}_k$ and $\overline{x} = \overline{a}_k$, and such that a subquotient of $\gamma(\nu')/\delta(\nu')$ projects onto y/x in 9 steps. Since L_0 is a modular lattice, therefore a nontrivial subquotient of b/a projects onto a nontrivial subquotient of $\gamma(\nu')/\delta(\nu')$ in $(k+9)$ steps. Since $0 < k < (m-10)$, it leads to a contradiction

Case (2): By Hong's Theorem, we have two subcases.

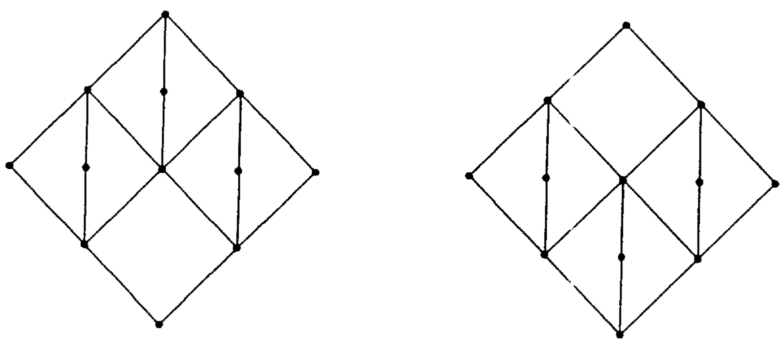
(2.1) there exists a subsequence $D_k, D_{k+1}, D_{k+2}, D_{k+3}, D_{k+4}, D_{k+5}$ with $5 < k < m-10$ such that $D_k \nearrow D_{k+1}$ and the sequence of numbers below the arrows is $(2, 1, 2, 1, 2)$.

(2.2) there exists a subsequence D_k, D_{k+1}, D_{k+2} with $5 < k < m-10$ such that $D_k \nearrow D_{k+1}$ and the sequence of numbers below the arrows is $(2, 2)$.

Case (2.1) : By lemma 3.2, the lattice B_5 is a homomorphic image of the sublattice L_1 of L generated by D_k, D_{k+1}, D_{k+2}, D_{k+3}, D_{k+4} and D_{k+5}. Furthermore $k+6 \to k+5$ is a nontrivial quotient in L_1 and $k+6 \to k+5$ is a critical quotient B_5. Since $B_5 \notin \mathcal{V}$, $\alpha(\overline{\mu'}) < \beta(\overline{\mu'})$ for some $\mu' \in {}^wL_1$. Since, by lemma 3.3, $R(B_5) = 9$, $\overline{\alpha(\mu')/\beta(\mu')}$ projects weakly onto $k+6 \to k+5$ in 9 steps. Therefore, by corollary 2.9, a nontrivial subquotients of $\alpha(\mu')/\beta(\mu')$ projects onto $k+6 \to k+5$ in 10 steps. Since L_1 is a modular lattice, a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto a nontrivial subquotient of $k+6 \to k+5$ in 10 steps. So a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto a nontrivial subquotient of d/c in $(m-k+4)$ steps. This too leads to a contradiction.

Case (2.2) : By the construction of the diamonds, we have $v_{k+1} = v_{k+2} + v_k$ and $z_k x_{k+2} = (u_k z_{k+1})(x_{k+1} u_{k+2}) = u_k u_{k+2}$. Hence $D_k \cup D_{k+1} \cup D_{k+2} \cup \{u_k u_{k+2}, u_k v_{k+2}, v_k u_{k+2}, v_k v_{k+2}\}$ forms a sublattice L_1 of L containing the lattice A_2 pictured in Figure 7 as a homomorphic image. Since A_2 is a simple lattice,

$\overline{b_{k+2}}/\overline{a_{k+2}}$ is a critical quotient in A_2. Since $A_2 \notin \mathcal{V}$, $\alpha(\overline{\mu'}) < \beta(\overline{\mu'})$ for some $\mu' \in {}^w L_1$. Since $R(A_2) < 7$, $\overline{\alpha(\mu')}/\overline{\beta(\mu')}$ projects weakly onto $\overline{b}_{k+2}/\overline{a}_{k+2}$ in 6 steps. Therefore, by corollary 2.9, a nontrivial subquotients of $\alpha(\mu')/\beta(\mu')$ projects onto b_{k+2}/a_{k+2} in 7 steps. Since L_1 is a modular lattice, a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto a nontrivial subquotient of b_{k+2}/a_{k+2} in 7 steps. So a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto a nontrivial subquotient of d/c in $(m-k+5)$ steps. This too leads to a contradiction. Thus the proof is completed.



A_2 A_3

FIGURE 7

COROLLARY 3.5. *Let $\mathcal{V}$ and $\mathcal{V}'$ be finitely based modular lattice varieties. If $A_2, B_5 \notin \mathcal{V}$ and A_5^d, the dual of A_5, is not contained in $\mathcal{V}'$, then $\mathcal{V} + \mathcal{V}'$ is finitely based.*

COROLLARY 3.6. *Let $\mathcal{V}$ and $\mathcal{V}'$ be finitely based modular lattice varities. If $A_3, B_5 \notin \mathcal{V}$ and A_5 is not contained in $\mathcal{V}'$, then $\mathcal{V} + \mathcal{V}'$ is finitely based.*

COROLLARY 3.7. *Let $\mathcal{V}$ and $\mathcal{V}'$ be finitely based modular lattice varities. If $A_3, B_5 \notin \mathcal{V}$ and A_5^d, the dual of A_5, is not contained in $\mathcal{V}'$, then $\mathcal{V} + \mathcal{V}'$ is finitely based.*

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