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ON WEAKLY ASSOCIATIVE BCI-ALGEBRAS

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ABSTRACT. In this paper, we introduce the notion of weakly associative BCI-algebras and investigate structure of it. Some of characterizations of elements of the quasi-associative part $Q(X)$ of a BCI-algebra X are shown.

K. Iséki [3] introduced the notion of BCI-algebras as a generalization of one of BCK-algebras. Q. P. Hu and K. Iséki [1], T. D. Lei and C. C. Xi [4], and C. C. Xi [13] introduced respectively the notions of associative, p-semisimple and quasi-associative BCI-algebras. In this note, we introduce the notion of weakly associative BCI-algebras and investigate structure of it. Some of characterizations of elements of the quasi-associative part $Q(X)$ of a BCI-algebra X are shown.

First let us recall some definitions and results.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following axioms: for all $x, y, z \in X$,

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = y * x = 0$ implies $x = y$.

A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X is called a BCK-algebra if it satisfies

(V) $0 * x = 0$ for all $x \in X$.

A subset S of a BCK/BCI-algebra X is called a subalgebra of X if $x * y \in S$ whenever $x, y \in S$.

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In a BCI-algebra X, the following hold:

(1) $x \leq 0$ implies $x = 0$,
(2) $x * 0 = x$,
(3) $(x * y) * z = (x * z) * y$,
(4) $0 * (x * y) = (0 * x) * (0 * y)$,
(5) $x * (x * (x * y)) = x * y$,
(6) $((x * z) * (y * z)) * (x * y) = 0$,
(7) $x * y = 0$ implies $(x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$.

J. Meng and X. L. Xin [9] introduced the notion of atoms. An element a in a BCI-algebra X is called an atom if $x * a = 0$ implies $x = a$ for all x in X. Note that if a is an atom, then $a * x$ is an atom for all $x \in X$. Let $L(X)$ denote the set of all atoms of X. Obviously, $0 \in L(X)$. For all a in $L(X)$, the set $V(a) = \{x \in X : a * x = 0\}$ is called a branch of X.

Proposition 1. (Meng et al. [9]) *Let X be a BCI-algebra. Then the following results are true:*

(8) *For all $x \in X$, $0 * (0 * x) \in L(X)$ and $x \in V(0 * (0 * x))$.*
(9) *If $a, b \in L(X)$, then $a * b \in L(X)$ and for all $x \in V(a)$ and all $y \in V(b)$, $x * y \in V(a * b)$.*
(10) *If $a, b \in L(X)$, then $a * x = a * b$ for all $x \in V(b)$.*
(11) *$L(X)$ is a subalgebra of X.*
(12) *$a \in L(X)$ if and only if $x * (x * a) = a$ for all $x \in X$.*
(13) *If $a, b \in L(X)$, then $0 * (a * b) = b * a$.*

A BCI-algebra X is said to be associative if it satisfies

(14) $x * (y * z) = (x * y) * z$ for all $x, y, z \in X$.

A BCI-algebra X is associative if and only if it satisfies

(14)′ $0 * x = x$ for all $x \in X$.

A BCI-algebra X is said to be quasi-associative if it satisfies

(15) $(x * y) * z \leq x * (y * z)$ for all $x, y, z \in X$.

A BCI-algebra X is quasi-associative if and only if it satisfies

(15)′ $0 * (0 * x) = 0 * x$ for all $x \in X$.

For any BCI-algebra X, the set $B(X) = \{x \in X : 0 * x = 0\}$ is the BCK-part of X, the p-semisimple part of X is the set $P(X) = \{x \in$

$X : 0 * (0 * x) = x\}$, the associative part of X is the set $A(X) = \{x \in X : 0 * x = x\}$, and the quasi-associative part of X is the set $Q(X) = \{x \in X : 0 * (0 * x) = 0 * x\}$. If $B(X) = \{0\}$, we say that X is a p-semisimple BCI-algebra. A BCI-algebra X is p-semisimple if and only if $X = P(X)$.

The following proposition is obvious.

PROPOSITION 2. *Let X be a BCI-algebra. Then the following results are true:*

(16) $L(X) = P(X)$,
(17) $x \in Q(X)$ *implies* $x \in V(0 * (0 * x))$,
(18) $Q(X) = \cup\{V(a) : a \in A(X)\}$.

PROPOSITION 3. (Meng et al. [10] and Wang [12]) *Let X be a BCI-algebra and let $x, y \in X$. Then the following results are true:*

(19) $A(X) = Q(X) \cap P(X)$ *and* $B(X) \subset Q(X)$,
(20) $B(X), Q(X), A(X)$ *and* $P(X)$ *are all subalgebras of* X, *and* $B(X)$ *and* $Q(X)$ *are both ideals of* X.
(21) $x \in Q(X)$ *if and only if* $0 * x \in Q(X)$,
(22) $x * y \in Q(X)$ *if and only if* $y * x \in Q(X)$.

THEOREM 4. *Let X be a BCI-algebra. Then the following conditions are equivalent for all $x, y, z \in X$, $u \in P(X)$ and $a \in A(X)$:*

(23) $y \in Q(X)$,
(24) $a * y \in A(X)$,
(25) $0 * y \in A(X)$,
(26) $u * y = u * (0 * y)$,
(27) $u * (x * y) = (u * x) * y$,
(28) $0 * (x * y) = (0 * x) * y$,
(29) $u * (x * (z * y)) = u * ((x * z) * y)$,
(30) $0 * (x * (z * y)) = 0 * ((x * z) * y)$,
(31) $x * y \leq x * (0 * y)$,
(32) $(x * y) * y \leq x$,
(33) $(x * y) * x \leq y$,
(34) $(x * z) * y \leq x * (z * y)$.

PROOF. (24)⇒(25), (27)⇒(28), (29)⇒(30), (31)⇒(32) and (32)⇒(33) are trivial.

(23)⇒(24). If $y \in Q(X)$, then $0*(0*y) = 0*y$ and hence $0*y \in A(X)$. Note from (8) that $0*(0*y) \in L(X)$ and $y \in V(0*(0*y))$ for all $y \in X$. It follows from (10) that $a*y = a*(0*(0*y)) = a*(0*y) \in A(X)$ for all $a \in A(X) \subseteq L(X)$. Hence (24) holds.

(25)⇒(26). If $0*y \in A(X)$ for $y \in X$, then $0*(0*y) = 0*y$. For all $u \in P(X)$, by (8) and (10) we have $u*y = u*(0*(0*y)) = u*(0*y)$. Hence (26) holds.

(26)⇒(27). If $u*y = u*(0*y)$ for $x, y \in X$ and $u \in P(X)$, then $(u*x)*y = (u*x)*(0*y)$. It follows that

$$\begin{aligned} & u*(x*y) \\ &= u*(0*(0*(x*y))) \qquad \text{[by (8) and (10)]} \\ &= (u*0)*((0*(0*x))*(0*(0*y))) \quad \text{[by (2) and (4)]} \\ &= (u*(0*(0*x)))*(0*(0*(0*y))) \quad \text{[by (8) and (11)]} \\ &= (u*x)*(0*y) \qquad \text{[by (5), (8) and (10)]} \\ &= (u*x)*y. \end{aligned}$$

Hence (27) holds.

(28)⇒(29). For $x, y, z \in X$ and $u \in P(X)$, if $0*(x*y) = (0*x)*y$, then $(0*(x*z))*y = 0*((x*z)*y)$. It follows that

$$\begin{aligned} & u*(x*(z*y)) \\ &= u*(0*(0*(x*(z*y)))) \qquad \text{[by (8) and (10)]} \\ &= u*(0*(((0*x)*0)*((0*z)*(0*y)))) \qquad \text{[by (2) and (4)]} \\ &= u*(0*(((0*x)*(0*z))*(0*(0*y)))) \qquad \text{[by (11)]} \\ &= u*(0*((0*(x*z))*(0*(0*y)))) \qquad \text{[by (4)]} \\ &= u*(0*((0*(x*z))*y)) \qquad \text{[by (8) and (10)]} \\ &= u*(0*(0*((x*z)*y))) \\ &= u*((x*z)*y). \qquad \text{[by (8) and (10)]} \end{aligned}$$

Hence (29) holds.

(30)⇒(31). If $0*(x*(z*y)) = 0*((x*z)*y)$ for $x, y, z \in X$, then

$$\begin{aligned}
&(x*y)*(x*(0*y)) \\
&= (x*(x*(0*y)))*y && \text{[by (3)]} \\
&= (0*y)*y && \text{[by (12)]} \\
&= (0*(0*(0*y)))*y && \text{[by (5)]} \\
&= (0*((0*0)*y))*y && \text{[by (30)]} \\
&= (0*(0*y))*y = 0. && \text{[by (III) and (3)]}
\end{aligned}$$

Hence $x*y \leq x*(0*y)$.

(33)⇒(34). For $x, y, z \in X$, if $(x*y)*x \leq y$, then

$$\begin{aligned}
&((x*z)*y)*(x*(z*y)) \\
&= ((x*(x*(z*y)))*z)*y && \text{[by (3)]} \\
&\leq ((z*y)*z)*y && \text{[by (II) and (7)]} \\
&\leq y*y && \text{[by (33) and (7)]} \\
&= 0.
\end{aligned}$$

It follows from (1) that $((x*z)*y)*(x*(z*y)) = 0$ or $(x*z)*y \leq x*(z*y)$.

(34)⇒(23). For $x, y, z \in X$, if $(x*z)*y \leq x*(z*y)$, then

$$\begin{aligned}
0*y = ((x*z)*y)*(x*z) &\leq (x*(z*y))*(x*z) \\
&= (x*(x*z))*(z*y) \leq z*(z*y) \leq y.
\end{aligned}$$

Hence $y \in Q(X)$. □

THEOREM 5. *Let X be a BCI-algebra. Then for all $x, y \in X$ and $z \in Q(X)$,*

(35) *$x*y \in Q(X)$ if and only if $(0*x)*x = (0*y)*y$.*

(36) *$y \in Q(X)$ if and only if $0*(z*y) = 0*(y*z)$.*

(37) *If $x \in Q(X)$ and $y \notin Q(X)$, then $x*y, y*x \notin Q(X)$.*

PROOF. (35). For all $x, y \in X$, we have

$$
\begin{aligned}
&((0*x)*x)*((0*y)*y) \\
&= ((0*x)*(0*(0*x)))*((0*y)*(0*(0*y))) \qquad \text{[by (8) and (10)]} \\
&= ((0*x)*(0*y))*((0*(0*x))*(0*(0*y))) \qquad \text{[by (11)]} \\
&= (0*(x*y))*(0*(0*(x*y))) \qquad \text{[by (4)]} \\
&= (0*(x*y))*(x*y). \qquad \text{[by (8) and (10)]}
\end{aligned}
$$

If $x*y \in Q(X)$, then $((0*x)*x)*((0*y)*y) = (0*(x*y))*(x*y) = 0$. So $(0*x)*x \leq (0*y)*y$. Similarly, $(0*y)*y \leq (0*x)*x$. Thus $(0*x)*x = (0*y)*y$.

Conversely, if $(0*x)*x = (0*y)*y$, then

$$(0*(x*y))*(x*y) = ((0*x)*x)*((0*y)*y) = 0,$$

hence $0*(x*y) \leq x*y$. Thus $x*y \in Q(X)$

(36). For all $y \in X$ and all $z \in Q(X)$, by (23) and (28) we have $0*(y*z) = (0*y)*z = (0*z)*y$. If $y \in Q(X)$, then from (23) and (28) it follows that $(0*z)*y = 0*(z*y)$. Hence $0*(z*y) = 0*(y*z)$. Conversely, if $0*(z*y) = 0*(y*z)$, then $(0*z)*y = 0*(y*z) = 0*(z*y)$. Hence $y \in Q(X)$.

(37). Suppose $x \in Q(X)$ and $y \notin Q(X)$. Since

$$
\begin{aligned}
&(0*(x*y))*(x*y) \\
&= ((0*x)*(0*y))*((0*(0*x))*(0*(0*y))) \qquad \text{[by (4), (8) and (10)]} \\
&= ((0*x)*(0*(0*x)))*((0*y)*(0*(0*y))) \qquad \text{[by (11)]} \\
&= 0*((0*y)*(0*(0*y))) \\
&= (0*(0*y))*(0*y) \neq 0
\end{aligned}
$$

and

$$
\begin{aligned}
&(0*(y*x))*(y*x) \\
&= ((0*y)*(0*x))*((0*(0*y))*(0*(0*x))) \qquad \text{[by (4), (8) and (10)]} \\
&= ((0*y)*(0*(0*y)))*((0*x)*(0*(0*x))) \qquad \text{[by (11)]} \\
&= (0*y)*(0*(0*y)) \neq 0,
\end{aligned}
$$

we have $x*y, y*x \notin Q(X)$. □

THEOREM 6. *Let X be a BCI-algebra. If $L(X)$ is an ideal of X, then so is $A(X)$.*

PROOF. Straightforward. □

The following example shows that the converse of Theorem 6 does not hold.

EXAMPLE 7. Let $X = \{0,1,2,3\}$. The binary operation $*$ on X is defined by the following table:

$*$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
3	3	3	2	3

Then it is easy to verify that $(X;*,0)$ is a BCI-algebra and $A(X) = \{0\}$ is an ideal of X. But $L(X) = \{0,2,3\}$ is not an ideal of X since $1 * 3 = 2 \in L(X)$, $3 \in L(X)$ and $1 \notin L(X)$.

From [11; Theorems 1 and 3] and [8; Theorems 5 and 6] we have the following theorem.

THEOREM 8. *Let X be a BCI-algebra. Then the following conditions are equivalent: for $x, y \in X$, $a, b \in A(X)$, and $c \in L(X)$*

(38) *$A(X)$ is an ideal of X,*
(39) *$A(X)$ is an ideal of $Q(X)$*
(40) *$Q(X) \cong B(X) \times A(X)$,*
(41) *$x * a = c * a$ implies $x = c$,*
(42) *$x * a = 0 * a$ implies $x = 0$,*
(43) *$x * a = y * a$ implies $x = y$,*
(44) *$x = (x * a) * (0 * a)$,*
(45) *$(x * a) * (y * b) = (x * y) * (a * b)$.*

From [2; Theorem 2] it immediately follows the following theorem.

THEOREM 9. *Let X be a BCI-algebra. If $A(X)$ is an ideal of X, then for all $x \in Q(X)$, there exist unique $u \in B(X)$ and unique $v \in A(X)$ such that $x = u * v$.*

DEFINITION 1. A BCI-algebra X is said to be weakly associative if it satisfies

(46) $0 * (0 * x) = 0 * x$ or $0 * (0 * x) = x$ for all $x \in X$.

THEOREM 10. *A BCI-algebra X is weakly associative if and only if $X = Q(X) \cup L(X)$.*

PROOF. Straightforward. □

LEMMA 11. *Let X be a weakly associative BCI-algebra. Then the following hold: for all $x \in Q(X)$ and $y \in L(X) - A(X)$,*

(47) $x * y = (0 * x) * y$,

(48) $y * x = y * (0 * x)$.

PROOF. Let $x \in Q(X)$ and $y \in L(X) - A(X)$. It follows from Theorem 5 that $x * y, y * x \in L(X) - A(X)$. Using (4) and (12), we have

$$x * y = 0 * (0 * (x * y)) = (0 * (0 * x)) * (0 * (0 * y)) = (0 * x) * y \text{ and}$$

$$y * x = 0 * (0 * (y * x)) = (0 * (0 * y)) * (0 * (0 * x)) = y * (0 * x).$$

Hence $x * y = (0 * x) * y$ and $y * x = y * (0 * x)$. □

LEMMA 12. (Li [5]) *An algebra $(X; *, 0)$ of type (2,0) is a BCI-algebra if and only if it satisfies the conditions (I), (IV) and (1).*

THEOREM 13. *Let Y be a quasi-associative BCI-algebra and Z a p-semisimple BCI-algebra with $A(Y) = A(Z) = Y \cap Z$, and the operations of Y and Z agree on $Y \cap Z$. Define an operation $*$ on $Y \cup Z$ as follows. If $x, y \in Y$(resp. Z), then use the operation on Y(resp. Z) to give $x * y$. If $x \in Y - (Y \cap Z)$ and $y \in Z - (Y \cap Z)$, put $x * y = (0 *_Y x) *_Z y$ and $y * x = y *_Z (0 *_Y x)$, where $*_Y$ and $*_Z$ denote the operations in Y and Z, respectively. Then $Y \cup Z$ is a weakly associative BCI-algebra, and $Q(Y \cup Z) = Y$, $L(Y \cup Z) = Z$ and $A(Y \cup Z) = A(Y) = A(Z) = Y \cap Z$.*

PROOF. To show that $Y \cup Z$ is a BCI-algebra. By Lemma 12, we only need to verify by Lemma 12 that $Y \cup Z$ satisfies (I), (IV) and (1). But by routine calculations we know that $Y \cup Z$ satisfies (I), (IV) and (2). Thus $Y \cup Z$ is a BCI-algebra. Next we show that $Y \cup Z$ is weakly associative. For all $x \in Y \cup Z$, if $x \in Y$, then $0 * x = 0 *_Y x = 0 *_Y (0 *_Y x) = 0 * (0 * x)$;

if $x \in Z$ then $0 * (0 * x) = 0 *_Z (0 *_Z x) = x$. Thus $Y \cup Z$ is a weakly associative BCI-algebra. Obviously $Q(Y \cup Z) = Y$, $L(Y \cup Z) = Z$ and $A(Y \cup Z) = A(Y) = A(Z) = Y \cap Z$. □

EXAMPLE 14. Let $Y = \{0, 1, a, b\}$ and $Z = \{0, a, x, y, u, v\}$ with Cayley tables as follows:

$*_Y$	0	1	a	b
0	0	0	a	a
1	1	0	b	a
a	a	a	0	0
b	b	a	1	0

$*_Z$	0	a	x	y	u	v
0	0	a	y	x	v	u
a	a	0	v	u	y	x
x	x	u	0	y	a	v
y	y	v	x	0	u	a
u	u	x	a	v	0	y
v	v	y	u	a	x	0

By routine calculations we know that Y is a quasi-associative BCI-algebra and Z is a p-semisimple BCI-algebra, and $A(Y) = A(Z) = Y \cap Z = \{0, a\}$. By Theorem 13 we know that the associative union $Y \cup Z$ of Y and Z is a weakly associative BCI-algebra with Cayley table as follows:

$*$	0	1	a	b	x	y	u	v
0	0	0	a	a	y	x	v	u
1	1	0	b	a	y	x	v	u
a	a	a	0	0	v	u	y	x
b	b	a	1	0	v	u	y	x
x	x	x	u	u	0	y	a	v
y	y	y	v	v	x	0	u	a
u	u	u	x	x	a	v	0	y
v	v	v	y	y	u	a	x	0

DEFINITION 2. In Theorem 13, $Y \cup Z$ is called an associative union of a quasi-associative BCI-algebra Y and a p-semisimple BCI-algebra Z, or an associative union of Y and Z (for short).

THEOREM 15. *Let X be a BCI-algebra. Then X is weakly associative if and only if X is an associative union of a quasi-associative BCI-algebra and a p-semisimple BCI-algebra.*

The following example shows that a BCI-algebra may not be weakly associative.

EXAMPLE 16. Let $X = \{0, 1, a, b, c, d\}$. The binary operation $*$ on X is defined as follows:

$*$	0	1	a	b	c	d
0	0	0	c	c	a	a
1	1	0	d	c	b	a
a	a	a	0	0	c	c
b	b	a	1	0	d	c
c	c	c	a	a	0	0
d	d	c	b	a	1	0

By routine calculations we konw that X is a BCI-algebra, and $Q(X) = B(X) = \{0, 1\}$, $P(X) = \{a, c\}$. Hence $X \neq Q(X) \cup P(X)$, and so X is not weakly associative.

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