Comm. Korean Math. Soc. **11** (1996), No. 3, pp. 595–600

ON DEDUCTIVE SYSTEMS OF HILBERT ALGEBRAS

SUNG MIN HONG AND YOUNG BAE JUN

ABSTRACT. We give a characterization of a deductive system. We introduce the concept of maximal deductive systems and show that every bounded Hilbert algebra with at least two elements contains at least one maximal deductive system. Moreover, we introduce the notion of radical and semisimple in a Hilbert algebra and prove that if H is a bounded Hilbert algebra in which every element is an involution, then H is semisimple.

1. Introduction

In 1966, Diego [6] introduced the notions of Hilbert algebra and deductive system and proved various properties. The theory of Hilbert algebras and deductive systems was further developed by Busneag in [2 - 5]. The second author [9] gave some characterizations of deductive systems. The aim of this paper are

1. To give a characterization of a deductive system,
2. To introduce the concept of maximal deductive systems, involutions, radicals and semisimple Hilbert algebras,
3. To show that every bounded Hilbert algebra with at least two elements contains at least one maximal deductive system, and
4. To verify that if H is a bounded Hilbert algebra in which every element is an involution then H is semisimple.

Received February 16, 1996. Revised June 21, 1996.

1991 AMS Subject Classification: 03G25, 06F99.

Key words and phrases: (Bounded, semisimple) Hilbert algebra, (maximal) deductive system, involution, radical.

The second author was supported by the LG Yonam Foundation (1995), and the present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1995, Project No. BSRI-95-1406.

2. Preliminaries

DEFINITION 2.1. (Busneag [4] and Diego [6]) A *Hilbert algebra* is a triple $(H, \rightarrow, 1)$, where H is a nonempty set, $\rightarrow$ is a binary operation on H, $1 \in H$ is an element such that the following three axioms are satisfied for every $x, y, z \in H$:

(i) $x \rightarrow (y \rightarrow x) = 1$,

(ii) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,

(iii) If $x \rightarrow y = y \rightarrow x = 1$ then $x = y$.

If H is a Hilbert algebra, then the relation $x \leq y$ iff $x \rightarrow y = 1$ is a partial order on H, which will be called the *natural ordering* on H, with respect to this ordering 1 is the largest element of H.

A *bounded* Hilbert algebra is a Hilbert algebra with a smallest element 0 relative to the natural ordering. If H is a bounded Hilbert algebra and $x \in H$, we denote by $x^* = x \rightarrow 0$. In a bounded Hilbert algebra, the following holds:

(1) $0^* = 1$ and $1^* = 0$.

EXAMPLE 2.2. (Busneag [4]) If $(H, \leq)$ is a poset, then $(H, \rightarrow, 1)$ is a Hilbert algebra,where 1 is the largest element of H and

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases}$$

for $x, y \in H$.

EXAMPLE 2.3. (Busneag [4]). If $(H, \vee, \wedge, \neg, 0, 1)$ is a Boolean lattice, then $(H, \rightarrow, 1)$ is a bounded Hilbert algebra, where $\rightarrow$ is defined by $x \rightarrow y = (\neg x) \vee y$ for $x, y \in H$.

PROPOSITION 2.4. (Busneag [4] and Diego [6]) *If H is a Hilbert algebra and $x, y, z \in H$, then the following hold:*

(i) $x \leq y \rightarrow x$.

(ii) $x \rightarrow 1 = 1$.

(iii) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

(iv) $(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y)$.

(v) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(vi) $x \leq (x \rightarrow y) \rightarrow y$.

(vii) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.

(viii) $1 \to x = x$.

3. Deductive systems

DEFINITION 3.1. (Diego [6]) If H is a Hilbert algebra, a subset D of H is called a *deductive system* of H if it satisfies:

(i) $1 \in D$,

(ii) $x \in D$ and $x \to y \in D$ imply $y \in D$.

We denote by $\mathcal{D}(H)$ the set of all deductive systems of H. If $X \subseteq H$, we denote by

$$(X) = \cap\{D \in \mathcal{D}(H) : X \subseteq D\},$$

and we call (X) the *deductive system generated* by X.

If $X = \{x_1, x_2, ..., x_n\}$, we denote by $(x_1, x_2, ..., x_n) = (\{x_1, x_2, ..., x_n\}$); the deductive system generated by one element $a \in H$, will be denoted by $[a)$ and it is easy to verify that $[a) = \{x \in H : a \leq x\}$, which is called a *principal deductive system* (see [2]).

PROPOSITION 3.2.. *Let H be a Hilbert algebra and $x, y, z \in H$. If $z \leq x \to y$ and $z \leq x$, then $z \leq y$.*

PROOF. Assume that $z \leq x \to y$ and $z \leq x$ for any $x, y, z \in H$. Then $x \to y \in [z)$ and $x \in [z)$. Since $[z)$ is a deductive system, it follows from Definition 3.1(ii) that $y \in [z)$ or $z \leq y$. □

PROPOSITION 3.3. (Diego [6]) *Any deductive system D of a Hilbert algebra H has the property: $x \leq y$ and $x \in D$ imply $y \in D$.*

We now give an equivalent condition of a deductive system.

THEOREM 3.4. *Let D be a nonempty subset of a Hilbert algebra H. Then D is a deductive system of H if and only if for x and y in D, $x \leq y \to z$ implies $z \in D$.*

PROOF. Let D be a deductive system of H and let $x, y \in D$. If $x \leq y \to z$, then $x \to (y \to z) = 1$. Using Definition 3.1(i)-(ii), we have $y \to z \in D$. Using Definition 3.1(ii) again, then $z \in D$.

Conversely assume that $x \leq y \to z$ implies $z \in D$ for all $x, y \in D$ and $z \in H$. Since D is nonempty, we may assume $x \in D$. We note from Proposition 2.4(ii) that $x \le x \to 1$ so that $1 \in D$ by assumption. Let $x \in D$ and $x \to y \in D$. Combining Proposition 2.4(vi) and the assumption, we get $y \in D$. Hence D is a deductive system of H. This completes the proof. □

THEOREM 3.5. (Busneag [2] and Diego [6]) *For any deductive system D of a Hilbert algebra H and any $a \in H$, the set $D_a = \{x \in H | a \to x \in D\}$ is the least deductive system of H containing D and a.*

DEFINITION 3.6. A proper deductive system of a Hilbert algebra H is said to be *maximal* if it is not contained in any other proper deductive system of H.

THEOREM 3.7. *Let D be a maximal deductive system of a Hilbert algebra H. Then for any $x, y \in H$, we have $x \to y \in D$ or $y \to x \in D$.*

PROOF. Let $x, y \in H$. If $x \in D$, then $x \le y \to x$ implies $y \to x \in D$. Similarly, if $y \in D$, then $x \to y \in D$. Finally, assume that $x \notin D, y \notin D$ and $x \to y \notin D$. Then

$$D_{x \to y} = \{z \in H | (x \to y) \to z \in D\}$$

is a deductive system containing D and $x \to y$. Since D is maximal, $D_{x \to y} = H$. Hence $(x \to y) \to (y \to x) \in D$, which implies from Proposition 2.4(v) that $y \to ((x \to y) \to x) \in D$. Using Definition 2.1(i) and Propositions 2.4(iii) and 2.4(viii), we have

$$\begin{aligned} y \to x &= 1 \to (y \to x) \\ &= (y \to (x \to y)) \to (y \to x) \\ &= y \to ((x \to y) \to x) \in D, \end{aligned}$$

which completes the proof. □

LEMMA 3.8. *Let H be a bounded Hilbert algebra with at least two elements. Then a deductive system D of H is proper if and only if $0 \notin D$.*

PROOF. Assume D is a proper deductive system of H and $0 \in D$. Let $x \in H$. Since $0 \le x$, we have $0 \to x = 1 \in D$, which implies $x \in D$. This means that $H \subset D$ or $H = D$ which is absurd. The converse is clear. □

LEMMA 3.9. *Let H be a bounded Hilbert algebra with at least two elements. Each proper deductive system D of H is contained in a maximal deductive system.*

PROOF. Straightforward. □

We note that $\{1\}$ is always a proper deductive system of any Hilbert algebra. Combining Lemmas 3.8 and 3.9, we have the following theorem.

THEOREM 3.10. *Every bounded Hilbert algebra with at least two elements contains at least one maximal deductive system.*

Let H be a bounded Hilbert algebra. By the *radical* of H, written $\mathrm{Rad}(H)$, we shall mean the set

$$\cap\{D|D \text{ is a maximal deductive system of } H\}.$$

In view of Theorem 3.10, $\mathrm{Rad}(H)$ always exists for a bounded Hilbert algebra H. Following a standard terminology in the contemporary algebra, we shall call a Hilbert algebra H *semisimple* if $\mathrm{Rad}(H) = \{1\}$.

DEFINITION 3.11. Let H be a bounded Hilbert algebra. If an element x of H satisfies $x^{**} = x$, then x is called an *involution*.

LEMMA 3.12. *Let H be a bounded Hilbert algebra in which every element is an involution. Then for each $x \in H$ with $x \neq 1$, there exists a maximal deductive system D of H such that $x \notin D$.*

PROOF. Let $x \in H$ be such that $x \neq 1$. Consider the principal deductive system generated by x^*, i.e., $[x^*) = \{y \in H|x^* \leq y\}$. We claim that $[x^*)$ is a proper deductive system of H. For if not, then $[x^*) = H$. Hence $0 \in [x^*)$, i.e., $x^* \leq 0$, and so $x^* = 0$. It follows that $x = x^{**} = 0^* = 1$ because x is an involution. This is a contradiction. Therefore $[x^*)$ is a proper deductive system of H. By Lemma 3.9 there is a maximal deductive system D of H such that $[x^*) \subset D$. It is clear that $x^* \in D$. Now we show that $x \notin D$. Suppose that $x \in D$. Since then D is a deductive system and $x^* \in D$, it follows that $0 \in D$, contrary to the Lemma 3.8. This completes the proof. □

THEOREM 3.13. *Let H be a bounded Hilbert algebra. If every element of H is an involution, then H is semisimple.*

PROOF. By Lemma 3.12 and the definition of semisimplicity. □

References

1. R. Balbes and P. Dwinger, *Distributive lattices*, University of Missouri Press. (1974).
2. D. Busneag, *A note on deductive systems of a Hilbert algebra*, Kobe J. Math. **2** (1985), 29-35.
3. ———, *On the maximal deductive systems of a bounded Hilbert algebra*, Bull. Math. de la Soc. des Sciences Mathematiques de la Roumanie, Tome. **31** (1987), 9-21.
4. ———, *Hilbert algebras of fractions and maximal Hilbert algebras of quotients*, Kobe J. Math. **5** (1988), 161-172.
5. ———, *Hertz algebras of fractions and maximal Hertz algebras of quotients*, Math. Japon. **39** (1993), 461-469.
6. A. Diego, *Sur les algèbras de Hilbert*, Gauthier-Villars, Colléction de Logique Math. Serie A. **21** (1966).
7. G. Grätzer, *Lattice theory*, W. H. Freeman and Company, San Francisco. (1979).
8. J. Hashimoto, *Ideal theory for lattices*, Math. Japon. **2** (1952), 149-186.
9. Y. B. Jun, *Deductive systems of Hilbert algebras*, Math. Japon. (to appear).

Department of Mathematics and Research Institute of Natural Science
Gyeongsang National University
Chinju 660-701, Korea
E-mail address: : ybjun@nongae.gsnu.ac.kr