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EXCISIONS IN HERMITIAN K-THEORY

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ABSTRACT. We make the definition of hermitian K-theory for nonunital rings which makes as many senses as possible. We next show that the excision property in rational hermitian K-theory implies the nullity of rational HB^{-}-homology which is the antisymmetric part of Bar homology.

1. Introduction

Karoubi's (Quillen-like higher) hermitian K-theory gives rise to many implications. One of those is the fact that the antisymmetric part of the hermitian K-theory is isomorphic to the obstruction group of Wall-Witt in surgery theory. For an involutive unital ring R, The i-th hermitian K-theory of R is defined by ${}_{\varepsilon}L_i(R) := \pi_i B_{\varepsilon}O(R)^+$ for $i \geq 1$. Here $\varepsilon = 1$ or -1. In this paper, we create the defnintion of hermitian K-theory for nonunital rings and show that the excision property in rational hermitian K-theory implies the nullity of rational HB^{-}-homology which is the antisymmetric part of Bar homology. A nonunital ring I is said to have the excision property if for every unital ring R that contains I as an ideal, there exists a natural long exact sequence

$$\cdots \to {}_{\varepsilon}L_n(I) \to {}_{\varepsilon}L_n(R) \to {}_{\varepsilon}L_n(R/I) \to {}_{\varepsilon}L_{n-1}(I) \to \cdots$$

One of the main reasons why we are going to deal with hermitian K-theory of nonunital rings (ideals) is that we wish to get some computational informations from this long exact sequence.

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On the other hand, for a nonunital ring, Bar homology, Hochschild homology, cyclic homology can be defined exactly as in the unital case. Again in the homolgy case we need to consider the excision property. It is known that for a unital ring R, $HB_*(R) = 0$(cf.[4]). For a nonunital ring I, however, this is not always true. A nonunital ring I satisfying $HB_*(I;V) = 0$ for every I-module V is called a H-unital ring by Wodzicki. He in [6] explored the excision properties of K-theory and various homologies. It turns out that H-unitality plays a key role in the excision. Comparing a (hermitian) K-theory with some homology theory is often very significant, since it is, in general, very difficult to compute the algebraic K-theories. The general line of the proof of Theorem 3.5 is analogous to [6], but in this we should be very careful in dealing with involutions and $\mathbb{Z}/2$-equivariant maps.

2. Preliminaries

A nonunital k-algebra means a k-algebra that does not contain unit 1. Let A be a k-algebra (nonunital or unital). The cyclic double complex $\mathcal{C}(A)$ is as follows:

$$
\begin{array}{ccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A^{\otimes n+1} & \xleftarrow{1-x} & A^{\otimes n+1} & \xleftarrow{L} & A^{\otimes n+1} & \xleftarrow{1-x} & A^{\otimes n+1} & \xleftarrow{L} & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & & \\
A^{\otimes n} & \xleftarrow{1-x} & A^{\otimes n} & \xleftarrow{L} & A^{\otimes n} & \xleftarrow{1-x} & A^{\otimes n} & \xleftarrow{L} & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & & \\
A & \xleftarrow{1-x} & A & \xleftarrow{L} & A & \xleftarrow{1-x} & A & \xleftarrow{L} &
\end{array}
$$

The map $x : A^{\otimes n+1} \to A^{\otimes n+1}$ is defined by

$$x(a_0, a_1, \cdots, a_n) = (-1)^n (a_n, a_0, a_1, \cdots, a_n).$$

L is defined by $L = 1 + x + \cdots + x^n$. The homology of the total complex of $\mathcal{C}(A)$ is called *cyclic homology* and denoted by $HC_*(A)$.

The first column makes a chain complex $(C'_*(A, A), b)$, where $C'_n(A, A) = A^{\otimes n+1}$ and $b = \sum_{i=0}^{n}(-1)^i d_i$. Its homology is denoted by $HH'_*(A)$. The first two columns make a double complex, whose homology is denoted by $HH''_*(A)$. These two homologies $HH'_*(A)$ and $HH''_*(A)$ are skew-Hochschild homologies.

From the second column we get

$$B_*(A) : \cdots \xrightarrow{b'} A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \xrightarrow{b'} \cdots \to A^{\otimes 2}$$

whose homology $HB_*(A)$ is called Bar homology. Here $b' : A^{\otimes n+1} \to A^{\otimes n+1}$ is defined by $b' = \sum_{i=0}^{n-1}(-1)^i d_i$. We can easily see that there is a long exact sequence

$$\cdots \to HH'_n(A) \to HH''_n(A) \to HB_n(A) \to HH'_{n-1}(A) \to \cdots$$

In most existing literatures, the cyclic homology and the Hochschild homology are defined for *unital* k-algebras. If A is unital, $HH'_n(A)$ is simply denoted by $HH_n(A)$ and called Hochschild homology. In fact, if A is unital, then the Bar homology $HB_*(A)$ vanishes, because the map $s : A^{\otimes n+1} \to A^{\otimes n+2}$, $(a_0, \cdots, a_n) \mapsto (1, a_0, \cdots, a_n)$ is the contracting homotopy i.e., $sb' + b's = 1$. Hence from the above exact sequence $HH'_n(A)$ is isomorphic to $HH''_n(A)$ are isomorphic, so they can be identified and simply called Hochschild homology. On the other hand if A is nonunital these two homologies do not necessarily coincide.

Suppose A is an involutive k-module, where k is a commutative unital ring. Then we can define involutions on the chain complexes $C'_*(A, A)$ and $B_*(A)$ as follows: Define $y : A^{\otimes n+1} \longrightarrow A^{\otimes n+1}$ by

$$y(a_0, a_1, \cdots, a_n) = (-1)^{\frac{n(n+1)}{2}}(\bar{a}_n, \bar{a}_0, \bar{a}_1, \cdots, \bar{a}_{n-1}).$$

Note that this map y satisfies the dihedral relation $x^{n+1} = y^2 = 1$ and $yxy^{-1} = x^{-1}$, and it gives rise to an involution on $C'_*(A, A)$. We can endow $B_*(A)$ with an involution $z = -yx : A^{\otimes n+1} \longrightarrow A^{\otimes n+1}$, $(a_0, a_1, \cdots, a_n) \longmapsto (-1)^{\frac{(n+1)(n+2)}{2}}(\bar{a}_n, \bar{a}_{n-1}, \cdots, \bar{a}_0)$. For more details of definitions, see [2], [3], [4]. We can easily show the following equalilties hold.

LEMMA 2.1.

(a) $yb = by$ (b) $zb' = b'z$ (c) $zL = -Ly$

(d) $y(1-x) = (1-x)z$

DEFINITION 2.2. Let A be an involutive k-module, where k is a commutative unital ring with $\frac{1}{2} \in k$. The double complex $\mathcal{C}(A)$ has an involution that is induced by y (resp. $-z, -y, z$) on the 4m-th (resp. (4m+1)-st, (4m+2)-nd, (4m+3)-rd) column. Then $\mathcal{C}(A)$ decomposes into $\mathcal{C}(A) = \mathcal{C}(A)^+ \oplus \mathcal{C}(A)^-$, where $\mathcal{C}(A)^+$ is the following:

$$\begin{array}{ccccccccc}
b\downarrow & & -b'\downarrow & & \downarrow & & \downarrow & & \\
C_2(A)^- & \xleftarrow{1-x} & C_2(A)^+ & \xleftarrow{L} & C_2(A)^- & \xleftarrow{1-x} & C_2(A)^+ & \longleftarrow & \\
b\downarrow & & -b'\downarrow & & \downarrow & & \downarrow & & \\
C_1(A)^- & \xleftarrow{1-x} & C_1(A)^+ & \xleftarrow{L} & C_1(A)^- & \xleftarrow{1-x} & C_1(A)^+ & \longleftarrow & \\
b\downarrow & & -b'\downarrow & & \downarrow & & \downarrow & & \\
C_0(A)^- & \xleftarrow{1-x} & C_0(A)^+ & \xleftarrow{L} & C_0(A)^- & \xleftarrow{1-x} & C_0(A)^+ & \longleftarrow &
\end{array}$$

$C_n(A)$ denotes $A^{\otimes n+1}$ for $n \geq 0$. The homology of the total complex of $\mathcal{C}(A)^+$ (resp. $\mathcal{C}(A)^-$) is denoted by $HC_*^+(A)$ (resp. $HC_*^-(A)$) and called the dihedral homology $HD_*(A)$ (resp. the skew-dihedral homology $HD'_*(A)$). Note that the first column of $\mathcal{C}(A)^+$ is antisymmetric part of Hochschild complex. This definition was recently suggested by Loday. It is reverse to that in [3]. His new definition seems to make more sense. For example, the product $HC_p(A) \times HC_q(A) \to HC_{p+q+1}(A)$ sends $HD_p(A) \times HD_q(A)$ to $HD_{p+q+1}(A)$, $HD_p(A) \times HD'_q(A)$ to $HD'_{p+q+1}(A)$ and $HD'_p(A) \times HD'_q(A)$ to $HD_{p+q+1}(A)$. Another incentive for this new choice of notation is that hermitian K-theory is closely related to $HD_*(\ )$, rather than $HD'_*(\ )$.

We have the following Connes exact sequence:

$$\cdots \to HH_n^-(A) \to HD_n^+(A) \to HD_{n-2}^-(A) \to HH_{n-1}^-(A) \to \cdots$$

DEFINITION 2.3. Let A be a nonunital involutive k-algebra. Let $\tilde{A} = k \ltimes A$ be a unital k-algebra obtained by attaching unit to A. Its multiplication and involution are : $(m,a)(n,b) = (mn, mb + na + ab)$, $\overline{(m,a)} = (m, \overline{a})$. The functor $HH_n(\ )$ is defined to be from *unital* k-algebra to k-modules. We can extend this definition for *nonunital* k-algebras as follows:

$$HH_n(A) := ker(HH_n(\tilde{A}) \to HH_n(k))$$

When A is unital, this definition agrees with the above one, since Hochschild homology preserves product. The cyclic homology can be similarly defined. At the begining of this section we defined $HC_*(A)$ for a nonunital ring A. It is easy to see that these two definitions agree.

3. Excisions

In this section we define hermitian K-theory of nonunital rings and prove the main theorem that the excision property in rational hermitian K-theory implies the nullity of HB^--homology. For the definitions of unital case, the reader refers of [1], [5].

DEFINITION 3.1. Let I be a 2-sided ideal of an involutive unital ring R. Let $F(R,I)$ be the homotopy fiber of

$$B_\varepsilon O(R)^+ \longrightarrow B_\varepsilon \overline{O}(R/I)^+$$

where ${}_\varepsilon\overline{O}(R/I) = Im({}_\varepsilon O(R) \to {}_\varepsilon O(R/I)))$. Then we define the relative hermitian K-theory by

$${}_\varepsilon L_i(R,I) := \pi_i F(R,I) \qquad \text{for } i \geq 1$$

We have a long exact sequence:

$$\begin{aligned} &\to {}_\varepsilon L_n(R,I) \to {}_\varepsilon L_n(R) \to {}_\varepsilon L_n(R/I) \to {}_\varepsilon L_{n-1}(R,I) \to \cdots \\ &\to {}_\varepsilon L_1(R) \to {}_\varepsilon L_1(R/I) \end{aligned}$$

The reason why we do not take $F(R,I)$ as a homotopy fiber of $B_\varepsilon O(R)^+ \to B_\varepsilon O(R/I)^+$ is that this fiber is, in general, not connected. In fact the connected component of this fiber is $F(R,I)$ above. As a matter of higher homotopy groups, $B_\varepsilon O(R/I)^+$ and $B_\varepsilon O(R/I)$ are not so much distinct. They just have different fundamental groups. In fact, $B_\varepsilon \overline{O}(R/I)^+ \to B_\varepsilon O(R/I)^+ \to B({}_\varepsilon O(R/I)/{}_\varepsilon\overline{O}(R,I))$ is a homotopy fibration, since ${}_\varepsilon\overline{O}(R/I)$ is a normal subgroup of ${}_\varepsilon O(R/I)$ and ${}_\varepsilon E(R/I) \subset {}_\varepsilon\overline{O}(R/I)$. Thus we have

$$\pi_i B_\varepsilon \overline{O}(R/I)^+ = {}_\varepsilon L_i(R/I), \quad i \geq 2$$
$$0 \to \pi_1 B_\varepsilon \overline{O}(R/I)^+ \to {}_\varepsilon L_1(R/I) \to {}_\varepsilon O(R/I)/{}_\varepsilon\overline{O}(R/I) \to 0.$$

DEFINITION 3.2. For an involutive nonunital ring I, the i-th hermitian K-theory is defined by

$${}_\varepsilon L_i(I) := {}_\varepsilon L_i(\tilde{I}, I) \quad \text{for } i \geq 1$$

DEFINITION 3.3. We say I satisfies *excision* in hermitian K-theory if for every unital ring R which contains I as a two-sided ideal, the canonical map ${}_\varepsilon L_*(I) \to {}_\varepsilon L_*(R,I)$ is a natural isomorphism.

If I satisfies excision then we have a long exact sequence

$$\cdots \to {}_\varepsilon L_n(I) \to {}_\varepsilon L_n(R) \to {}_\varepsilon L_n(R/I) \to {}_\varepsilon L_{n-1}(I) \to \cdots$$

Let $F_*(\ )$ represent the homologies $HH_*(\ ), HC_*(\ )$ and also their symmetric and anti-symmetric parts. $\mathcal{C}_*(\ )$ denotes the corresponding chain complex. For any split k-extension $A \to R \to S$, define $\mathcal{C}_*(R,A) := Ker(\mathcal{C}_*(R) \to \mathcal{C}_*(S))$. We say A satisfies *excision* in F_*-homology if the inclusion $\mathcal{C}_*(A) \hookrightarrow \mathcal{C}_*(R,A)$ is a quasi-isomorphism, that is, if there exists a long exact sequence

$$\cdots \to F_n(A) \to F_n(R) \to F_n(S) \to F_{n-1}(A) \to \cdots$$

It is known that hermitian K-theory is closely related with dihedral homology (cf [5]). For our main theorem (Theorem 3.5) we first recall the theorem which seems to be the most general result for the relation between hermitian K-theory and dihedral homology.

THEOREM 3.4 ([5]). *Let $f : R \to S$ be a surjective map of involutive unital rings such that $Ker f$ is nilpotent. Then relative rational hermitian K-theory is isomorphic to relative rational dihedral homology. Precisely speaking,*

$$ {}_{\varepsilon}L_n(f) \otimes \mathbb{Q} \cong HD_{n-1}(f) \otimes \mathbb{Q} $$

THEOREM 3.5. *Let I be an involutive nonunital ring. Suppose I satisfies excision in rational hermitian K-theory. Then $HB_i^-(\tilde{I} \otimes \mathbb{Q}) = 0$ for all $i \geq 1$.*

PROOF. Let $R = \widetilde{I \oplus J}$, where J is a nilpotent nonunital ring that acts trivially on I. Then there is a short exact sequence $0 \to I \to R \to \tilde{J} \to 0$. We have the following commutative diagram of short exact sequences.

$$\begin{array}{ccccc} & & J & = & J \\ & & \downarrow & & \downarrow \\ I & \longrightarrow & R & \longrightarrow & \tilde{J} \\ \| & & \downarrow & & \downarrow \\ I & \longrightarrow & \tilde{I} & \longrightarrow & \mathbb{Z} \end{array}$$

From this diagram we get the following diagram of hermitian K-theory and dihedral homology (Here ${}_{\varepsilon}L_*(R)$ means ${}_{\varepsilon}L_*(R) \otimes \mathbb{Q}$ and $HD_*(R)$ means $HD_*(R \otimes \mathbb{Q})$).

$$\begin{array}{ccccc} & & {}_{\varepsilon}L_*(R,J) & \xrightarrow{\alpha_*} & {}_{\varepsilon}L_*(J) \\ & & \downarrow & & \downarrow \\ {}_{\varepsilon}L_*(R,I) & \longrightarrow & {}_{\varepsilon}L_*(R) & \longrightarrow & {}_{\varepsilon}L_*(\tilde{J}) \\ {\scriptstyle \beta_*}\downarrow & & \downarrow & & \downarrow \\ {}_{\varepsilon}L_*(I) & \longrightarrow & {}_{\varepsilon}L_*(\tilde{I}) & \longrightarrow & {}_{\varepsilon}L_*(\mathbb{Z}). \end{array}$$

$$\begin{array}{ccccc}
 & & HD_*(R,J) & \xrightarrow{\gamma_*} & HD_*(J) \\
 & & \downarrow & & \downarrow \\
HD_*(R,I) & \longrightarrow & HD_*(R) & \longrightarrow & HD_*(\tilde{J}) \\
{\scriptstyle \delta_*}\downarrow & & \downarrow & & \downarrow \\
HD_*(I) & \longrightarrow & HD_*(\tilde{I}) & \longrightarrow & HD_*(\mathbb{Z}).
\end{array}$$

The assumption that I satisfies excision in rational hermitian K-theory implies β_* is an isomorphism. From the above diagram we have $Ker\ \alpha_* = Ker\ \beta_*$. On the other hand, $Ker\ \alpha_* \cong Ker\ \gamma_{*-1}$ by Theorem 3.4. From the lower diagram again we have $Ker\ \gamma_* = Ker\ \delta_*$. Thus we have $Ker\ \delta_n = Ker\ \beta_{n+1} = 0$ for all n. Note that $Ker\delta_n$ equals the symmetric part of $Ker\ (HC_n(R,I) \to HC_n(I))$. In the case when $J^2 = 0$ (note that this assumption does not harm the generality), it is shown by Wodzicki([6]) that $\bigoplus_{i=1}^{n} HB_i(\tilde{I} \otimes \mathbb{Q}) \otimes (J \otimes \mathbb{Q})^{n-i}$ is isomorphic to a direct summand of $Ker\ (HC_n(R,I) \to HC_n(I))$. A careful computation shows that this isomophism is antisymmetric, that is, by taking their antisymmetric parts we have

$$HB_i^-(\tilde{I} \otimes \mathbb{Q}) = 0 \qquad \text{for all} \quad i \qquad \square$$

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