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CHOW GROUPS ON COMPLETE REGULAR LOCAL RINGS II

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ABSTRACT. We study some special cases of Chow groups of a ramified complete regular local ring R of dimension n. We prove that (a) for codimension 3 Gorenstein ideal I, $[I] = 0$ in $A_{n-3}(R)$ and (b) for a particular class of almost complete intersection prime ideals P of height i, $[P] = 0$ in $A_{n-i}(R)$.

1. Introduction

We define the i-th Chow group $A_i(R)$ for a Noetherian Cohen-Macaulay ring R of dimension(dim) n by $Z_i(R)/Rat_i(R)$, where $Z_i(R)$ is the free abelian group generated by prime ideals in R of height(ht) $n-i$ and $Rat_i(R)$ is the subgroup of $Z_i(R)$ generated by the cycles of the form $\sum l(R_{P_i}/(q+(x))R_{P_i})[P_i]$, where q is a prime ideal of height $n-i-1, x \notin q$ and P_i ranges over the minimal associated prime ideals of $R/(q+(x))$ satisfying dim $R/P =$ dim $R/(q+(x))$. When M is a finitely generated R-module, then $[M] = \sum l(M_{P_i})[P_i]$, where P_i ranges over the minimal associated prime ideals of M satisfying dim $R/P_i =$ dim M. From Claborn and Fossum [4], if R is a regular local ring, then the above definition is equivalent to the group $Z_i(R)/ < R/(x_1, \cdots, x_{n-i}) >$, where $< R/(x_1, \cdots, x_{n-i}) >$ is the subgroup of $Z_i(R)$ generated by $\sum l(R/(x_1, \cdots, x_{n-i}))_P[P]$, P ranges over the associated prime ideals of $R/(x_1, \cdots, x_{n-i})$, for all R-sequence $x_1, \cdots, x_{n-i}$. From this definition, when R is a regular local ring of dimension n, we have $A_0(R) = 0$, $A_{n-1}(R) = 0$, $A_n(R) = Z$ and from [7] and [6], we get $A_{n-2}(R) = 0, A_1(R) = 0$. Moreover, if P is a complete intersection ideal of height $n-i$ in a regular local ring R

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of dimension n, then $[P]=0$ in $A_i(R)$. In the dimension 5 case, to calculate $A_2(R)$ still remains open.

In this paper, we shall discuss $A_{n-3}(R)$ for Gorenstein ideals I of R of codimension 3 and vanishing part of some almost complete intersection prime ideals.

2. Lifting of Gorenstein Rings

Our main purpose of this section is to introduce and prove Theorem 2.1. We use the lifting property of a Gorenstein ideal of codimension 3 and a lifting Theorem.

THEOREM 2.1. *Let R be a regular local ring of dimension n, and let I be an ideal in R of height 3 such that R/I is a Gorenstein ring. Then $[I]=0$ in $A_{n-3}(R)$.*

Before proving Theorem 2.1, we describe some terminology: Let R be a commutative ring with 1 and let F be a finitely generated free R-module. Then a map $g : F^* \to F$ is said to be alternating with respect to some basis of F and F^*, if the matrix representation of g is skew symmetric and all its diagonal entries are 0. When the rank of F is even and $g : F^* \to F$ is alternating, we define Pfaffian of g which is denoted by $Pf(g)$ as $\det(g)=(Pf(g))^2$. More generally, when F has any rank, and G is a free summand of F of even rank, with projection $\pi : F \to G$, then the composite

$$\phi : G^* \xrightarrow{\pi^*} F^* \xrightarrow{g} F \xrightarrow{\pi} G$$

is alternating and we say that $Pf(\phi)$ is a Pfaffian of order n of g and $Pf_n(g)$ is the ideal generated by all the n-th order Pfaffians of g.

In the proof of Theorem 2.1, we need two results due to Buchsbaum and Eisenbud, and Dutta.

THEOREM 2.2 (BUCHSBAUM AND EISENBUD[3], THEOREM 2.1). *Let (R, m, k) be a Noetherian local ring.*
(1) Let $n \geq 3$ be an odd interger, and let F be a free R-module of rank n. Let $g : F^* \to F$ be an alternating map whose image is contained in

mF. Suppose $Pf_{n-1}(g)$ has grade 3. Then $Pf_{n-1}(g)$ is a Gorenstein ideal, minimally generated by n elements.
(2) Every Gorenstein ideal of grade 3 arises as in (1).

Moreover, they showed that when g is alternating, then grade $(Pf_{n-1}\ (g)) \leq 3$ and grade$(Pf_{n-1}(g)) = 3$ if and only if $Pf_{n-1}(g)$ is a Gorenstein ideal. If $Pf_{n-1}(g)$ is a Gorenstein ideal, then the following sequence is exact.

$$0 \to R \xrightarrow{l^*} F^* \xrightarrow{g} F \xrightarrow{l} R \to R/Pf_{n-1}(g) \to 0$$

In the course of our proof of Theorem 2.1 we will also need the following result due to Dutta:

PROPOSITION 2.3. (S. P. Dutta[6], Proposition 2.2) *Let (S, m, k) be a Cohen-Macaulay local ring of dimension n and let f be a non-zero-divisor in m. Let I be an ideal in S of height i such that $[I] = 0$ in $A_{n-i}(S)$ and f be not a zero-divisor on S/I. Then $[I + fS/fS] = 0$ in $A_{n-1-i}(S/fS)$.*

PROOF OF THEOREM 2.1. Let $R = V[[X_1, \cdots, X_n]]/(f) = S/(f)$, where $S = V[[X_1, \cdots, X_n]]$ and f is an Eisenstein polynomial in X_n over $V[[X_1, \cdots, X_{n-1}]]$. By Theorem 2.2, since I is a Gorenstein ideal of height 3, we have a minimal resolution of R/I as follows:

$$C_1 : 0 \to R \xrightarrow{l^*} F^* \xrightarrow{g} F \xrightarrow{l} R \to R/I \to 0.$$

Let n = rank of F = minimal number of generators of I. In C_1, g is an alternating map and $I = Pf_{n-1}(g)$. Write $G = S^n$.
Let $\widetilde{g}$ be a lift of g such that $\widetilde{g}$ is also alternating(i.e., $\widetilde{g} \otimes id_R = g$).
Now consider the following sequence

$$C_2 : 0 \to S \xrightarrow{\widetilde{l}^*} G^* \xrightarrow{\widetilde{g}} G \xrightarrow{\widetilde{l}} S \to S/J \to 0$$

where $J = Pf_{n-1}(\widetilde{g})$.
Note that $C_2 \otimes id_R = C_1$.
From the proof of Theorem 2.2, we know that (a) C_2 is a complex and (b) C_2 is exact if $htJ = 3$.

First, we want to show that $htJ = 3$.

Note that $J + (f)/(f) \cong I$. We also know that $htJ \leq 3$. If $htJ \leq 2$ and Q is a minimal prime ideal of J such that $htQ = htJ$, then $ht(Q+(f)/(f)) \leq 2$. But this would imply that $htI = ht(J+(f)/(f)) \leq ht(Q+(f)/(f)) \leq 2$ - which is a contradiction. Hence C_2 is exact and since $C_2 \otimes id_R = C_1$ we drive that $J \cap (f) = fJ$ i.e., f is a non-zero-divisor on S/J. This shows that J is a lifting of I to S. Since $[J] = 0$ in $A_{n-2}(S)$. Hence by Proposition 2.3, $[I] = 0$ in $A_{n-3}(R)$.

Now, we are going to study the case of almost complete intersection prime ideals.

3. The case when P is an almost complete intersection prime ideal

As we discussed at the introduction, complete intersection ideals of height $n - i$ in a regular local ring R of dimension n vanish in $A_i(R)$. In this section, we will discuss the vanishing part of the Chow group of almost complete intersection prime ideals in a regular local ring.

Let $P = (x_1, x_2, \cdots, x_{i+1})$ be an almost complete intersection prime ideal of height i in a regular local ring R of dimension n. Then we have to consider following two cases:

Case 1. There are i elements in a minimal generating set of P which do not form an R-sequence,

Case 2. Any set of i elements in a minimal generating set of P form an R-sequence.

PROPOSITION 3.1. *Let R, P be as above. Then $[P] = 0$ in $A_{n-i}(R)$ in Case 1.*

PROOF. Suppose that $x_1, x_2, \cdots, x_i$ is not an R-sequence. Let $I = (x_1, x_2, \cdots, x_i)$ and let Q be a minimal prime ideal containing I such that $htI = htQ = i - 1$. Then $x_{i+1} \notin Q$. By the definition of the $(n-i)$-th Chow group, we have $[Q + (x_{i+1})] = 0$ in $A_{n-i}(R)$. Since P is contained in $Q + (x_{i+1})$ and $ht(Q + (x_{i+1})) = i$, we have $Q + (x_{i+1}) = P$ and therefore $[P] = 0$ in $A_{n-i}(R)$.

The following is an immediate Corollary of Proposition 3.1.

COROLLARY 3.2. *Let R, P be as above. If there exists a subset of i elements, say, $x_1, x_2, \cdots, \check{x}_j, \cdots x_{i+1}$ which forms an R-sequence (i.e., in Case 2), but $x_1, x_2, \cdots, \check{x}_j, \cdots, x_i, ax_{i+1}+x_j$ do not form an R-sequence for some $a \in R$ (where $\check{x}_j$ means deleting x_j), then $[P] = 0$ in $A_{n-i}(R)$.*

PROOF. Let $I = (x_1, x_2, \cdots, \check{x}_j, \cdots, x_i, ax_{i+1} + x_j)$ and let Q be a minimal prime ideal containing I such that $ht\ I = ht\ Q = i - 1$. Then, by assumption on the R-sequence, $x_{i+1} \notin Q$. Thus $Q + (x_{i+1}) = P$ and $[P] = [Q + (x_{i+1})] = 0$ in $A_{n-i}(R)$.

References

1. Bourbaki, *Algebra, Chap.III*, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1989.
2. D. A. Buchsbaum and D. Eisenbud, *Lifting modules and a theorem on finite free resolutions*, in "Ring theory"(Robert Gorden ed.), Academic press, New York, 1972.
3. ———, *Algebra Structures for finite free resolutions and some structure theorems for ideals of codimension* 3, Amer. J. Math **99** (1977), 447-485.
4. L. Claborn and R. Fossum, *Generalization of the notion of class group Ill.*, J. Math. **12** (1968), 228-253.
5. S. P. Dutta, *A note on Chow groups and intersection multiplicity of modules*, J. Algebra **161** (1993), 186-198.
6. ———, *On Chow group and intersection multiplicity of modules II*, J. Algebra **171** (1995), 370-382.
7. S. Lee, *Chow groups on complete regular local rings*, to appear.
8. H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, Cambridge, 1986.

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