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PRIMITIVE ORE EXTENSIONS OVER SPECIAL MATRIX RINGS

JANG-HO CHUN AND JUNE-WON PARK

ABSTRACT. We find an equivalent condition of $M_n(R)[x, \delta]$ to be primitive and characterize a special subring P of $M_n(R)$. Also, we find an equivalent condition of $P[x, \delta]$ to be primitive.

1. Introduction

Jordan[5] introduced two different sufficient conditions for an Ore extensions $R[x, \delta]$ to be primitive, for any right noetherian differential ring R. Goodearl and Warfield[3] showed necessary and sufficient conditions for $R[x, \delta]$, over a commutative noetherian ring R, to be primitive. In this paper, we study the primitivity of $R[x, \delta]$, over a matrix ring and over a special subring of a matrix ring.

Throughout this paper, all rings are associative with identity and δ will be a derivation on R, i.e., δ is an additive map from R to itself which satisfies the product rule, $\delta(ab) = \delta(a)b + a\delta(b)$ for each $a, b \in R$.

The Ore extension $S = R[x, \delta]$ is a ring of polynomials over R in an indeterminate x with multiplication subject to the relation

$$xr = rx + \delta(r)$$

for each $r \in R$.

DEFINITION. (1) An ideal I of R is said to be a δ-ideal if $\delta(I) \subseteq I$.
(2) A δ-ideal I of R is said to be δ-prime if for all δ-ideals A, B of R such that $AB \subseteq I$ either $A \subseteq I$ or $B \subseteq I$.
(3) R is said to be δ-primitive if there exists a maximal right ideal M of R containing no non-zero δ-ideals of R.

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(4) R is said to be δG-ring if it is δ-prime and the intersection of its non zero δ-prime ideals is non zero.

2. Primitive Ore extentions over matrix ring

LEMMA 2.1 [3, THEOREM]. *Let R be a commutative noetherian differential ring with no $\mathbb{Z}$-torsion. Then $R[x,\delta]$ is primitive if and only if $\delta \neq 0$ and R is either δ-primitive or a δG-ring.*

Let $A = M_n(R)$ be the ring of all $n \times n$ matrices over an arbitrary ring R; then any derivation δ_R of R can be extended to A by putting $\delta((a_{ij})) = (\delta_R a_{ij})$. Namely, the derivative of a matrix is the matrix of the derivatives of the entries. We shall denote this derivation also by δ_R.

LEMMA 2.2[1, THEOREM2]. *Let R be a commutative ring. Then the derivations δ of $M_n(R)$ are of the form $\delta = \delta_R + ad_v$, where $v \in M_n(R)$ and $ad_v(r) = rv - vr$ is the inner derivation determined by v.*

For an element v of a ring R if $\delta = ad_v$, then $R[x,\delta]$ is a polynomial ring $R[x+v]$. Thus we have the following proposition.

PROPOSITION 2.3. *Let $A = M_n(R)$ over a commutative ring R and $\delta = \delta_R + ad_v$ be a derivation of A, where $v \in M_n(R)$. Then*

$$A[x,\delta] \cong A[x+v,\delta_R]$$

where δ_R is a trivial extension of R

$$\cong M_n(R[x+v,\delta_R])$$

where δ_R is a restriction of A.

PROPOSITION 2.4. *Let $A = M_n(R)$ over a commutative ring R and $\delta = ad_v + \delta_R$. Then $M_n(I)$ is a δ-ideal of A if and only if I is a δ_R-ideal of R.*

PROOF. Let $M_n(I)$ be a δ -ideal of A and $(a_{ij}) \in M_n(I)$ such that $a_{11}(\neq 0) \in I$ and $a_{ij} = 0$ for i or $j \neq 1$. Then $\delta((a_{ij})) \in M_n(I)$. But $\delta_R(a_{11}) = (1,1)$ entry of $\delta_R((a_{ij})) = (1,1)$ entry of $(ad_v + \delta_R((a_{ij}))) = \delta((a_{ij}))_{11} \in I$. So I is a δ_R-ideal of R. The converse is trivial.

COROLLARY 2.5. *Let $A = M_n(R)$ and $\delta = ad_v + \delta_R$. Then*
(1) *A is δ-primitive if and only if R is δ_R-primitive.*
(2) *A is a δG-ring if and only if R is a $\delta_R G$-ring.*

THEOREM 2.6. *Let R be a commutative noetherian differential ring with no $\mathbb{Z}$-torsion and $A = M_n(R)$. Then $A[x,\delta]$ is primitive if and only if $\delta \neq$ inner and A is either δ-primitive or a δG-ring.*

PROOF. Since $A[x,\delta] \cong A[x+v,\delta_R] \cong M_n(R[x+v,\delta_R])$ where $\delta = ad_v + \delta_R$,
$A[x,\delta]$ is primitive
$\Leftrightarrow R[x+v,\delta_R]$ is primitive because primitivity is Morita-invariant
$\Leftrightarrow \delta_R \neq 0$ and R is either δ_R-primitive or a $\delta_R G$-ring by Lemma 2.1
$\Leftrightarrow \delta \neq$ inner and A is either δ-primitive or a δG-ring by Corollary 2.5.

3. Ore extensions over special subrings of matrix rings

For a reflexive and transitive relation ρ on the set $I_n = \{1,2,\ldots,n\}$ and $M_n(R)$, we use the following conventions:
E^{ij} = the element of the standard basis of $M_n(R)$,
$M_n(R)_\rho = \{(a_{ij}) \in M_n(R) | a_{ij} = 0 \text{ for } (i,j) \notin \rho\}$.
It is clear that $M_n(R)_\rho$ is a subring of $M_n(R)$. Conversely, if σ is a reflexive relation on I_n and $M_n(R)_\sigma$ is a subring of $M_n(R)$ then σ is transitive. We say that a subring P of $M_n(R)$ is special with the relation ρ iff $P \cong M_n(R)_\rho$.
If $\{U_{ij} | i,j \in I_n\}$ is a family of subsets of R, then we denote by $[U_{ij}]$ the set $\{(a_{ij}) \in M_n(R) | a_{ij} \in U_{ij} \text{ for all } i,j \in I_n\}$.
The following lemma describes all (two-sided) ideals of special subrings of $M_n(R)$.

PROPOSITION 3.1 [6, LEMMA 2.1]. *Let P be a special subring of $M_n(R)$ with the relation ρ and let U be a subset of P. The following conditions are equivalent:*

(1) *U is an ideal of P,*
(2) *$U = [U_{ij}]$, where U_{ij} are ideals of R such that*
  (a) *$U_{ij} = 0$ if $(i,j) \notin \rho$,*
  (b) *$U_{ij} + U_{jk} \subseteq U_{ik}$, if $(i,j),(j,k) \in \rho$.*

LEMMA 3.2. *Let P be a special subring of $M_n(R)$ with the relation ρ. If there exist $\rho_k \equiv I_k \times I_k \subset \rho$ where $I_k = \{i_1, i_2, \ldots, i_k\} \subset I_n$ such that $(i,j) \in \rho$ and i or $j \in I_k$ implies $(i,j) \in \rho_k$. Let $U = \{(a_{ij}) \in P \mid a_{ij} = 0$ if $(i,j) \notin \rho_k\}$. Then U is an ideal of P.*

Moreover, U is a direct summand of P.

PROOF. By Proposition 3.1, U is an ideal of P.

Moreover, let

$$e = (e_{ij}), \text{where } e_{ij} = \begin{cases} 1 & \text{if } i = j \in I_k \\ 0 & \text{otherwise.} \end{cases}$$

Then e is a central idempotent of P and $eP = U$. Thus U is a direct summand of P.

For the relation ρ, let

$$C_s = \{(i,s) \in \rho \mid i \in I_n\}$$

and

$$D_t = \{(t,j) \in \rho \mid j \in I_n\}.$$

Then there exist maximal elements $C_{s'}$ and $D_{t'}$ for cardinal numbers in $\{C_s | s \in I_n\}$ and $\{D_t | t \in I_n\}$, respectively. Let

$$C = \bigcup \{C_s | (i,s) \in \rho \Leftrightarrow (i,s') \in \rho \text{ for all } i \in I_n\}$$

and

$$D = \bigcup \{D_t | (t,j) \in \rho \Leftrightarrow (t',j) \in \rho \text{ for all } j \in I_n\}.$$

Then we have the following lemma.

LEMMA 3.3. *Let P be a special subring of $M_n(R)$ with the relation ρ and let U_C(or U_D) $= [U_{ij}]$ where*

$$U_{ij} = \begin{cases} R & \text{if } (i,j) \in C (\text{or } D) \\ 0 & \text{if } (i,j) \notin C (\text{or } D) \end{cases}$$

Then U_C(or U_D) is an ideal of P.

PROOF. Since $P \cdot U_C \subseteq U_C$ obviously, U_C is a left ideal.

Now, assume $C = \{i_1, i_2, \ldots, i_i\} \times \{j_1, j_2, \ldots, j_j\}$ and suppose $U_C \cdot P \nsubseteq U_C$. Then $U_C \cdot P$ has an element whose (i,j) entry is non zero for some $(i,j) \in \rho - C$. So there exists $k \in I_n$ such that $E^{ik} \cdot E^{kj} = E^{ij}$ where $(i,k) \in C$, $(k,j) \in \rho$ and $(i,j) \in \rho - C$. This means that $E^{i_1 j}, E^{i_2 j}, \ldots, E^{i_i j}$ are elements of P and $\{(i_1,j),(i_2,j),\ldots,(i_i,j)\} \subseteq \rho - C$. This is a contradiction to the definition of C. Thus U_C is a right ideal of P. Therefore U_C is an ideal of P.

Similarly, U_D is an ideal of P.

THEOREM 3.4. *Let P be a special subring of $M_n(R)$ with relation ρ. Then P is prime (simple) if and only if $P = M_n(R)$ and R is prime (simple).*

PROOF. ($\Rightarrow$) Suppose $P \neq M_n(R)$. By Lemma 3.3, we can take U_C and U_D ideals of P.

If $U_C \cdot U_D = 0$, then P is not prime.

Suppose $U_C \cdot U_D \neq 0$ and assume $a \in U_C \cdot U_D$, where (α,β) entry of $a \neq 0$. Then for some $k, E^{\alpha k} \cdot E^{k\beta} = E^{\alpha\beta} \in U_C \cdot U_D$ where $(\alpha,k) \in C$ and $(k,\beta) \in D$. Since $E^{\alpha\beta}$ is contained in U_C and U_D, $E^{ij} \in U_C \cdot U_D$ if $(i,j) \in C_\beta$ or D_α. Let

$$\rho' = \{(i,j) | (i,\beta) \in C_\beta \text{ and } (\alpha,j) \in D_\alpha\}.$$

Then $E^{ij} \in U_C \cdot U_D$ if $(i,j) \in \rho'$.

Now, since ρ is reflexive,

$$\rho' = I_t \times I_t \text{ for some } I_t = \{i_1, i_2, \ldots, i_t\} \subset I_n.$$

By Lemma 3.2, $U_C \cdot U_D$ is a direct summand of P. Therefore P is not prime.

($\Leftarrow$) It is obvious.

COROLLARY 3.5. *Let P be a special subring of $M_n(R)$ with relation ρ. Then $P[x,\delta]$ is prime if and only if $P = M_n(R)$ and R is δ-prime.*

PROOF. c.f [4,Theorem 2.2].

THEOREM 3.6. *Let R be a commutative noetherian Ore extension with no $\mathbb{Z}$-torsion and let P is a special subring of $M_n(R)$ with relation ρ . Then $P[x,\delta]$ is primitive if and only if $P = M_n(R), \delta \neq$ inner and P is either δ-primitive or a δG-ring.*

PROOF. It is immediate from Theorem 2.6 and Theorem 3.4.

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Jang-Ho Chun
Department of mathematics
Yeungnam University
Kyongsan 712-749, Korea

June-Won Park
Department of mathematics
Kyungpook Sanup University
Kyongsan 712-701, Korea