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COVERING GROUPS IN THE THEORY OF GROUP REPRESENTATION

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ABSTRACT. In this paper, we shall study the gencralized covering group which plays a role for Schur multiplier. We discuss the lifting property over covering group ad product of covering groups.

1. Introduction

The purpose of this paper is to study one of the central topics of the theory of representations; so called covering group.

If F is an algebraically closed field, a device invented by Schur yields a finite group G^* such that all projective representations of G can be lifted to ordinary representations of G^*. Originally Schur referred to G^* as a "representation group (Darstellungsgruppe)" of G, but as a more popular term, it is called "covering group".

A covering group G^* of a group G over a field F satisfies the following:

(1) there is a central extension $1 \to A \to G^* \to G \to 1$ such that any projective representation of G over F lifts projectively to an ordinary representation of G^*. That is, any projective representation of G over F is projectively equivalent to one that can be lifted.

(2) $|A| = |H^2(G, F^*)|$, where $F^* = F - \{0\}$.

Thus, a covering group G^* exists in the case that the second cohomology group $H^2(G, F^*)$ is finite. If F is an algebraically closed field, then a covering group G^* always exists, and this gives a foundation of Schur's theory of projective representation. However, it is also well-known that $H^2(G, F^*)$ need not be finite so there are no covering groups of G over any field.

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In order to generalize Schur's theory over to arbitrary field F, Yamazaki [8] used the torsion subgroup $t(F^*)$ of F^* rather than F^* itself, and used the fact that $t(F^*) \to F^*$ induces an injective homomorphism $H^2(G, t(F^*)) \to H^2(G, F^*)$, and that $H^2(G, t(F^*))$ is always a finite group.

There is another simple device. Let $\alpha \in Z^2(G, F^*)$ be a 2-cocycle and let $A(\alpha)$ be the subgroup of F^* generated by all the values of α. Define a new group $G(\alpha)$ consisting of all pairs $(a, g) \in A(\alpha) \times G$ with multiplicative operation $(a, g)(b, x) = (ab\alpha(g, x), gx)$. Then $G(\alpha) \to G$, $(a, g) \mapsto g$ is a surjective homomorphism whose kernel is a central subgroup isomorphic to $A(\alpha)$. This provides a central group extension

$$1 \to A(\alpha) \to G(\alpha) \to G \to 1$$

and it is proved that any α-representation of G lifts to an ordinary representation of $G(\alpha)$. A consideration of special case where α is of finite order was given by Fong [4] or Reynolds [7]. This additional condition on α says that $G(\alpha)$ is finite. The $G(\alpha)$ is called an α-covering group of G, and plays a role for covering group.

2. Generalized α-covering Groups and Lifting Property

A goal of this section is to generalize an α-covering group and then prove lifting property on that.

Let F be a field of characteristic $p > 0$ and G be a group. Let $\alpha \in Z^2(G, F^*)$ and $l = o(\alpha) < \infty$. Denote a primitive t-th root of unity by ζ_t for integer $t \in Z$.

LEMMA 1. *If p is prime then p does not divide l, and $\zeta_l \in F^*$.*

PROOF. For any $g, x \in G$, $\alpha(g, x)$ can be written as $\alpha(g, x) = \zeta_l^k$ for some $k \in Z$, and $\langle \alpha(g, x) | \ g, x \in G \rangle \subseteq \langle \zeta_l \rangle$. If we suppose that $|\langle \alpha(g, x) | g, x \in G \rangle| = l_1 < l$, then $(\alpha(g, x))^{l_1} = 1$ for all $g, x \in G$, contrary to the minimality of l. Therefore, $l_1 = l$ so that $\zeta_l \in F^*$, this completes the proof.

Let $F^\alpha G$ be a twisted group algebra with an F-basis $\{a_g \mid g \in G\}$ such that $a_g a_x = \alpha(g, x) a_{gx}$ for $g, x \in G$, and $a_1 = 1$. By Lemma 1, $G(\alpha) = \{\zeta_l^i a_g \mid g \in G, \ 0 \leq i \leq l - 1\} \subseteq F^\alpha G$.

Let r be any positive integer divisible by $o(\alpha) = l < \infty$, and assume that F^* contains a primitive r-th root ζ_r of unity. Then p does not divide r, and a multiplicative group $G_r(\alpha) = \{\zeta_r^i a_g \mid 0 \leq i \leq r-1,\ g \in G\}$ in $F^\alpha G$ is called a generalized α-covering group. Of course, $A_r(\alpha)$ is defined as a subgroup consisting of $\zeta_r^i a_1$ for all $0 \leq i \leq r-1$. We may choose $r = o(\alpha)$; thus $G_r(\alpha)$ is a generalization of α-covering group $G(\alpha)$ (we may refer to [3]). The next theorem shows simple relations of (generalized) α-covering groups.

THEOREM 1. *Let r be an integer divisible by $l = o(\alpha)$.*

(1) *A map $\pi : G_r(\alpha) \to G$ by $\zeta_r^i a_g \mapsto g$ is a surjective homomorphism whose kernel is a central subgroup $A_r(\alpha) = \{\zeta_r^i a_1 \mid i \in Z\}$. Hence π gives rise to a central cyclic group extension*

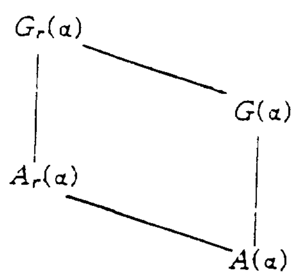
$$1 \to A_r(\alpha) \to G_r(\alpha) \to G \to 1.$$

(2) $|G_r(\alpha)| = r|G| = |A_r(\alpha)||G|$.

(3) *$G(\alpha)$ is a normal subgroup of $G_r(\alpha)$, and $G_r(\alpha)/G(\alpha)$ is cyclic of order r/l. Hence $G(\alpha)$ contains a commutator subgroup of $G_r(\alpha)$, which in fact equals a commutator subgroup of $G(\alpha)$.*

(4) *Furthermore, $G_r(\alpha)$ is a central product of $G(\alpha)$ and $A_r(\alpha)$ with $G(\alpha) \cap A_r(\alpha) = A(\alpha) \subseteq Z(G_r(\alpha))$. Thus $G_r(\alpha)$ is an epimorphic image of $G(\alpha) \times A_r(\alpha)$.*

PROOF. Certainly, $A_r(\alpha)$ and $G(\alpha)$ are normal in $G_r(\alpha)$ and $G \cong G_r(\alpha)/A_r(\alpha)$. And $G_r(\alpha) = G(\alpha)A_r(\alpha)$, $[G(\alpha), A_r(\alpha)] = \langle [u,v] | u \in G(\alpha), v \in A_r(\alpha)\rangle = 1$. Thus $G_r(\alpha)$ is a central product of $G(\alpha)$ and $A_r(\alpha)$. Refer to the diagram.



Denote by $g^* \in G_r(\alpha)$ arbitrary preimage of $g \in G$ by π, i.e., $g^* = \zeta_r^i a_g$ for some $i \in Z$.

In order to show that group extension in (1) of Theorem 1 has a lifting property, recall some definitions and preliminary results.

Let V be a finite dimensional vector space over a field F. A projective representation $P : G \to \mathrm{GL}(V)$ of G over F satisfies

$$P(1_G) = 1_V, \quad P(g)P(x) = \alpha(g,x)P(gx) \text{ for all } g, x \in G,$$

where α is a mapping $G \times G \to F^*$. Then α is in $Z^2(G, F^*)$, and P is called an *α-representation*. Two projective representations $P_i : G \to GL(V_i)$ $(i = 1, 2)$ are said to be projectively equivalent denoted by $P_1 \sim_p P_2$, if there exists a mapping $\mu : G \to F^*$ with $\mu(1) = 1$ and a vector space isomorphism $f : V_1 \to V_2$ such that $P_2(g) = \mu(g) f P_1(g) f^{-1}$ for all $g \in G$.

LEMMA 2. (refer to [5])

(1) *The α-representations P of G correspond bijectively to the representations T of $F^\alpha G$ by $P(g) = T(a_g)$.*

(2) *If P_i is α_i-representations on V_i $(i = 1, 2)$ such that $P_1 \sim_p P_2$ then α_1 is cohomologous to α_2. Conversely, if α_1 is cohomologous to α_2, and P_1 is an α_1-representation on V, then there is an α_2-representation P_2 on V such that $P_1 \sim_p P_2$.*

Let $1 \to A \to B \xrightarrow{f} C \to 1$ be a central group extension of A by C. Given a projective representation $P : C \to GL(V)$, we say P lifts to an ordinary representation $P^* : B \to GL(V)$ if

(1) $P^*(a)$ $(a \in A)$ is a scalar multiple of 1_V

(2) there is a section s of f such that $P(c) = P^*(s(c))$ for $c \in C$.

More generally, we say that P lifts projectively to an ordinary representation $R : B \to GL(V)$ if R satisfies (1) and the next:

(3) there is a section s of f together with a map $t : C \to F^*$ such that $P(c) = t(c)R(s(c))$ for all $c \in C$.

The next theorem shows our goal; the lifting property.

THEOREM 2.

(1) *Every α-representation of G can be lifted to an ordinary representation of $G_r(\alpha)$ using the same section of $\pi : G_r(\alpha) \to G$. So, there is a bijection between the set of all α-representations of G and that of all lifted representations of $G_r(\alpha)$.*

(2) *Let $\beta \in Z^2(G, F^*)$ be cohomologous to α. Then any β-representation of G lifts projectively to an ordinary representation of $G_r(\alpha)$.*

PROOF. Let P be any α-representation of G. Let $s : G \to G_r(\alpha)$ be a map defined by $s(g) = a_g$ for all $g \in G$. Then $s(1) = 1_{G_r(\alpha)}$ and $\pi s = 1$, thus s is a section of π. We now define a map $R : G_r(\alpha) \to GL(V)$ by $R(\zeta_r^i a_g) = \zeta_r^i P(g)$ for all $g \in G, i \in Z$. Consider a map $T : F^\alpha G \to \mathrm{End}_F(V)$ defined by $\sum_{g\in G} k_g a_g \mapsto \sum_{g \in G} k_g P(g)$. Then T is a vector space homomorphism by F-linearity and $T(a_g a_x) = T(\alpha(g,x)a_{gx}) = \alpha(g,x)P(gx) = P(g)P(x) = T(a_g)T(a_x)$ for all $g, x \in G$, by Lemma 2 (1). Hence T is an F-algebra homomorphism. But since R is the restriction of T to $G_r(\alpha)$, T is an ordinary representation on $G_r(\alpha)$. Further $R(s(g)) = R(a_g) = P(g)$ for all $g \in G$, this yields that P lifts to R by the section s of π.

Let $\mathcal{A}$ and $\mathcal{B}$ be the sets of all α-representations of G and of all lifted ordinary representations of $G_r(\alpha)$, respectively. Then using the section s, there is a bijection $\Phi : \mathcal{A} \to \mathcal{B}$ defined by $P \mapsto R$ such that $Rs = P$. This proves (1).

Let U be a β-representation on G. Since β is cohomologous to α, there is a map $\lambda : G \to F^*$ such that $\alpha = \beta(\delta\lambda)$, i.e., $\alpha(g,x) = \lambda(g)\lambda(x)\lambda^{-1}(gx)\beta(g,x)$. Here, δ is of course the boundary map. Define P on G by $P(g) = \lambda(g)U(g)$ for all $g \in G$. Then $P(g)P(x) = \lambda(g)\lambda(x)U(g)U(x) = \lambda(g)\lambda(x)\beta(gx)U(gx) = \alpha(g,x)\lambda(gx)U(gx) = \alpha(g, x)P(gx)$, thus P is an α-representation on G, and P lifts to P^* which is an ordinary representation on $G_r(\alpha)$. Then $P^*(1_{G_r(\alpha)})$ is a scalar multiple of 1_V, and there is a section s of π with $P(g) = P^*(s(g))$ for all $g \in G$. Thus, $T(g) = \lambda^{-1}(g)P(g) = \lambda^{-1}(g)P^*(s(g))$. Taking $t : G \to F^*$ by $t(g) = \lambda^{-1}(g)$, we can prove that T lifts projectively to P^* on $G_r(\alpha)$.

COROLLARY 1. *For any linearly equivalent α-representations of G, their lifted representations of $G_r(\alpha)$ are equivalent.*

PROOF. Suppose that $P_i : G \to GL(V_i)$ (V_i is a vector space $i = 1, 2$) are α-representations of G which are equivalent. There is a vector space isomorphism $\phi : V_1 \to V_2$ such that $P_2(g) = \phi P_1(g)\phi^{-1}$ for all $g \in G$. Let R_i be the lifted representation of P_i ($i = 1, 2$) using the section s in the proof of the Theorem 2. For $g^* \in G_r(\alpha)$, we have

$g^* = \zeta_r^j a_g$ for some $j \in Z$ and $R_2(\zeta_r^j a_g) = \zeta_r^j P_2(g) = \phi \zeta_r^j P_1(g)\phi^{-1} = \phi R_1(\zeta_r^j a_g)\phi^{-1}$, hence R_1 is equivalent to R_2 by taking the same vector space isomorphism ϕ.

Clearly it is not true that any ordinary representation of $G_r(\alpha)$ comes from a projective representation on G.

COROLLARY 2. *Every ordinary representation of $G_r(\alpha)$ whose restriction to $A_r(\alpha)$ is a scalar multiple of the identity mapping in $GL(V)$ comes from some α-representations using the section s.*

PROOF. The proof follows immediately from the fact that $R(\zeta_r^i 1_\Gamma) = \zeta_r^i I_{\deg T} = \zeta_r^i I_{GL(V)}$, where T is a representation of $F^\alpha G$.

Let F be a field of any characteristic p with algebraic closure E. For any group H, denote by $Z_F(H)$ a subgroup of H consisting of all z in the center $Z(H)$ of H satisfying $z^{m(\sigma)} = z$, where $\sigma \in \mathrm{Gal}(E/F)$ and $m(\sigma)$ is an integer such that $\zeta_{n_{p'}}^{\sigma} = \zeta_{n_{p'}}^{m(\sigma)}$ and $m(\sigma) \equiv 1 \pmod{n_p}$, for all positive integer n divisible by $\exp(H)$.

LEMMA 3. (refer to [2]) *Let $1 \to A \to H \to G \to 1$ be a central group extension such that the image of A is a cyclic p'-subgroup of $Z_F(H)$. Then there exists a 2-cocycle $f \in Z^2(G, F^*)$ with finite order dividing $|A| = r$ such that $H \cong G_r(f)$, and the given extension commutes with $1 \to A_r(f) \to G_r(f) \to G \to 1$.*

This lemma provides a reasoning to study the generalized covering group rather than the covering group. Combining with Theorem 2, we have more general situation for lifting.

COROLLARY 3. *Suppose we have a central group extension $1 \to A \to H \to G \to 1$ such that the image of A is a cyclic p'-subgroup of $Z_F(H)$. Then there exists a 2-cocycle $f \in Z^2(G, F^*)$ and every f-representation of G lifts to an ordinary representation on H.*

Certainly the f can be obtained as in Lemma 3.

3. Some Group Algebras

The study of representations of G over F is equivalent to the study of modules over the group algebra FG, and the role played by the group algebra is taken by the twisted group algebra $F^{\alpha}G$ when considering projective representations. The purpose of this section is to provide relations between some group algebras.

LEMMA 4. (refer to [5])

(1) *Let $1 \to A \to H \to G \to 1$ be a finite central group extension. Then*

$$FH \cong \prod_{\chi \in Hom(A,F^*)} F^{\bar{\chi}}G,$$

where $\bar{\chi}$ is the image of $\chi \in Hom(A,F^*)$ by a transgression map $Hom(A,F^*) \to H^2(G,F^*)$.

(2) *For a covering group G^* of G, we have*

$$FG^* \cong \prod_{f \in T} F^f G,$$

where T is a transversal of $Z^2(G,F^*)$ in $B^2(G,F^*)$.

Furthermore, for $\alpha \in Z^2(G,F^*)$ of finite order l, $FG(\alpha)$ is isomorphic to $\prod_{i=0}^{l-1} F^{\alpha^i}G$. We have a similar result with respect to $G_r(\alpha)$.

THEOREM 3. *Let $\alpha \in Z^2(G,F^*)$ be of finite order l. Then $FG_r(\alpha) = FG(\alpha)$, thus $FG_r(\alpha) \cong \prod_{i=0}^{l-1} F^{\alpha^i}G$.*

PROOF. $G(\alpha) \subseteq G_r(\alpha)$ implies that $FG(\alpha) \subseteq FG_r(\alpha)$. Since any element in $FG_r(\alpha)$ is of the form $k\zeta_r^i a_g$ for some $k \in F$ and since $\zeta_r^i \in F^*$, $k\zeta_r^i a_g = ta_g \in FG(\alpha)$ for some $t \in F^*$.

For $\alpha \in Z^2(G,F^*)$, $\beta \in Z^2(H,F^*)$, a map $\alpha \times \beta : (G \times H) \times (G \times H) \to F^*$ defined by $(\alpha \times \beta)((g,h),(g_1,h_1)) = \alpha(g,g_1)\beta(h,h_1)$ with $g, g_1 \in G$; $h, h_1 \in H$ is easily seen to be an element of $Z^2(G \times H, F^*)$. The $\alpha \times \beta$ is called an outer product of α and β.

LEMMA 5. ([5]) *$F^{\alpha}G \otimes F^{\beta}H \cong F^{\alpha \times \beta}(G \times H)$ as F-algebras.*

THEOREM 4. *Let $\alpha \in Z^2(G, F^*)$ and $\beta \in Z^2(H, F^*)$ with finite orders l and m, respectively, which are relatively prime. Then $G(\alpha) \times H(\beta) \cong (G \times H)(\alpha \times \beta)$.*

PROOF. Let $\{a_g |\ g \in G\}$, $\{b_h |\ h \in H\}$ and $\{c_{(g,h)} |\ (g,h) \in G \times H\}$ be bases for $F^\alpha G$, $F^\beta H$ and $F^{\alpha \times \beta}(G \times H)$, respectively. Note that $c_{(g,h)} c_{(g_1,h_1)} = (\alpha \times \beta)((g,h),(g_1,h_1)) c_{(g,h)(g_1,h_1)}$, for $g, g_1 \in G$, $h, h_1 \in H$. Consider a map defined by

$$\psi: \ G(\alpha) \times H(\beta) \to (G \times H)(\alpha \times \beta), \quad \psi(\zeta_l^i a_g, \zeta_m^j b_h) = \zeta_l^i \zeta_m^j c_{(g,h)}.$$

Then

$$\begin{aligned} &\psi((a_g, b_h)(a_{g_1}, b_{h_1})) = \psi(\alpha(g,g_1) a_{gg_1}, \beta(h,h_1) b_{hh_1}) \\ &= \alpha(g,g_1)\beta(h,h_1) c_{(gg_1, hh_1)} = (\alpha \times \beta)((g,h),(g_1,h_1)) c_{(g,h)(g_1,h_1)} \\ &= c_{(g,h)} c_{(g_1,h_1)} = \psi(a_g, b_h)\psi(a_{g_1}, b_{h_1}); \end{aligned}$$

hence ψ is a homomorphism. Since $(\alpha \times \beta)^{lm}((g,h),(g_1,h_1)) = (\alpha(g,g_1)\beta(h,h_1))^{lm} = 1$, $o(\alpha \times \beta)$ divides lm. Let $o(\alpha \times \beta) = k$ for some $k > 0$. Then $1 = \alpha^k(g,g_1)\beta^k(h,h_1)$, so that $\alpha^k(g,g_1) = \beta^{-k}(h,h_1)$. Further since, $o(\alpha^k) = l/\gcd(k,l)$ and $o(\beta^{-k}) = m/\gcd(k,m)$, we have that $l/\gcd(k,l) = m/\gcd(k,m) = 1$ (due to $\gcd(l,m) = 1$). Hence, both l and m divides k, and so does lm. Therefore, $o(\alpha \times \beta) = lm$ and $|(G \times H)(\alpha \times \beta)| = |G \times H| lm = |G||H| lm = |G(\alpha)||H(\beta)|$, this shows that ψ is an isomorphism.

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