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LIE–ADMISSIBLE ALGEBRAS AND THE VIRASORO ALGEBRA

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1. Introduction

Let A be an (nonassociative) algebra with multiplication xy over a field F, and denote by A^- the algebra with multiplication $[x,y] = xy - yx$ defined on the vector space A. If A^- is a Lie algebra, then A is called Lie-admissible. Lie-admissible algebras arise in various topics, including geometry of invariant affine connections on Lie groups and classical and quantum mechanics(see [2,5,6,7] and references therein). The main approach to the structure theory has been to determine all Lie-admissible products on a prescribed Lie algebra under certain conditions. Flexibility or, more generally, third power-associativity is commonly used in the study of Lie-admissible algebras [1,2,4,5,7].

An algebra A is termed *flexible* if A satisfies the flexible law

$$(xy)x = x(yx) \tag{1}$$

for all $x, y \in A$. More generally, A is called *third power-associative* if

$$x^2x = xx^2 \tag{2}$$

for all $x \in A$. Much of the structure theory for Lie-admissible algebras has focused on finite-dimensional algebras [1,5]. The primary concern of this note is to determine all third power-associative, Lie-admissible products on the Virasoro algebra.

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2. Main section

Let L be a Lie algebra with multiplication $[x,y]$ over F of characteristic $\neq 2$. A multiplication xy defined on the vector space L is said to be *compatible* with L if $xy - yx = [x,y]$ for all $x, y \in L$; i.e., $L^- = L$. If xy is compatible with L, then

$$xy = \frac{1}{2}[x,y] + x \circ y \tag{3}$$

for $x, y \in L$, where $x \circ y = \frac{1}{2}(xy + yx)$. In particular, L with product xy is Lie-admissible. A central problem in the study of Lie-admissible algebras is to determine all compatible multiplications defined on Lie algebras. This problem has been resolved for finite-dimensional third power-associative Lie-admissible algebras A with A^- semisimple over an algebraically closed field of characteristic 0 [1,5], whereas this is open for the infinite-dimensional case. In this note, we consider this problem for the Virasoro algebra, which is important in theoretical physics and provides a basic example of graded Lie algebras [3].

Let $\mathfrak{W}$ be the (full) Witt algebra over F with multiplication

$$[e_i, e_j] = (j - i)e_{i+j}, \quad i, j \in \mathbb{Z}, \tag{4}$$

where $\{e_i : i \in \mathbb{Z}\}$ is a basis of $\mathfrak{W}$. The Virasoro algebra $\mathfrak{V}$ over F(of characteristic 0) is a one-dimensional extension $\mathfrak{V} = \mathfrak{W} \oplus Fc$ of $\mathfrak{W}$ with multiplication

$$\begin{gathered}[e_i, e_j] = (j - i)e_{i+j} + \frac{1}{12}(i^3 - i)\delta_{i+j,0}\, c, \\ [c, \mathfrak{V}] = [\mathfrak{V}, c] = 0, \quad i, j \in \mathbb{Z}.\end{gathered} \tag{5}$$

The following is our principle result.

THEOREM. *A multiplication xy defined on $\mathfrak{V}$ is third power-associative and compatible with $\mathfrak{V}$ if and only if it is given by*

$$\begin{gathered}xy = \frac{1}{2}[x,y] + \tau(x)y + \tau(y)x + \sigma(x,y)c, \\ cx = xc = \alpha x + \lambda(x)c, \quad c^2 = \beta c, \quad x, y \in \mathfrak{W},\end{gathered} \tag{6}$$

where $\lambda, \tau : \mathfrak{W} \to F$ are linear functionals, $\sigma : \mathfrak{W} \times \mathfrak{W} \to F$ is a symmetric bilinear form and $\alpha, \beta \in F$ are fixed scalars. Moreover, the multiplication (6) is flexible if and only if $\lambda = \tau = 0$, $\alpha = 0$ and σ is identically zero on $\mathfrak{W} \times \mathfrak{W}$.

Proof. Assume that xy is third power-associative and compatible with $\mathfrak{V}$. In view of (3), it suffices to determine the commutative product $x \circ y$ on $\mathfrak{V}$. Since $xy - yx = [x, y]$, (2) expresses as $[x, x \circ x] = 0$ for $x \in \mathfrak{V}$ which is linearized to the identity

$$2[x, x \circ y] + [y, x \circ x] = 0, \quad x, y \in \mathfrak{V}. \tag{7}$$

If $\phi(e_i, e_j) = \frac{1}{12}(i^3 - i)\delta_{i+j,0}$ for $i, j \in \mathbb{Z}$, then ϕ is an F-valued 2-cocycle of $\mathfrak{W}$. Since the product $x \circ y$ is commutative, we can let

$$e_i \circ e_j = \sum_{k \in \mathbb{Z}} \gamma_{ij}^k e_k + \sigma(e_i, e_j)c, \quad i, j \in \mathbb{Z} \tag{8}$$

with $\gamma_{ij}^k = \gamma_{ji}^k \in F$ for a symmetric bilinear form $\sigma : \mathfrak{W} \times \mathfrak{W} \to F$.

Letting $x = y = e_i$ in (7), one has from (5) and (8)

$$0 = [e_i, e_i \circ e_i] = \sum_k \gamma_{ii}^k (k - i) e_{i+k} + \sum_k \gamma_{ii}^k \phi(e_i, e_k) c,$$

and hence

$$\gamma_{ii}^k = 0, \quad i \neq k, \quad i, k \in \mathbb{Z}.$$

If $x = e_i$ and $y = e_j$ in (7), then, as above,

$$2 \sum_k \gamma_{ij}^k [e_i, e_k] = \gamma_{ii}^i [e_i, e_j],$$

$$2 \sum_k \gamma_{ij}^k (k - i) e_{i+k} + 2 \sum_k \gamma_{ij}^k \phi(e_i, e_k) c = \gamma_{ii}^i (j - i) e_{i+j} + \gamma_{ii}^i \phi(e_i, e_j) c,$$

which imply

$$\gamma_{ij}^k = 0, \quad i \neq k \neq j,$$
$$2\gamma_{ij}^j = \gamma_{ii}^i = 2\gamma_{ji}^j, \quad i \neq j$$

for $i, j, k \in \mathbb{Z}$. Therefore, (8) becomes

$$e_i \circ e_j = \frac{1}{2}\gamma_{ii}^i e_j + \frac{1}{2}\gamma_{jj}^j e_i + \sigma(e_i, e_j)c,$$

and the first relation of (6) follows from this with the linear functional $\tau : \mathfrak{W} \to F$ defined by $\tau(e_i) = \frac{1}{2}\gamma_{ii}^i$ $(i \in \mathbb{Z})$.

Next, we linearize (7) to the identity

$$[u, v \circ w] + [w, u \circ v] + [v, w \circ u] = 0 \tag{9}$$

for $u, v, w \in \mathfrak{V}$. Since $2[e_i, c \circ e_i] = [e_i \circ e_i, c] = 0$, there exist linear functionals $\mu, \lambda : \mathfrak{W} \to F$ such that

$$ce_i = e_i c = c \circ e_i = e_i \circ c = \mu(e_i)e_i + \lambda(e_i)c.$$

If $u = e_j$, $v = c$ and $w = e_i$ in (9), then it follows that

$$\mu(e_i - e_j)[(i-j)e_{i+j} + \phi(e_i, e_j)c] = 0$$

and hence $\mu(e_i) = \mu(e_j)$ for $i \neq j$. Since $[e_i, c \circ c] = 0$ by (7) for all $i \in \mathbb{Z}$, $cc = c \circ c = \beta c$ for some β. Letting $\mu(e_i) = \alpha \in F$ for all $i \in \mathbb{Z}$, we have the last relations of (6). Since third power-associativity is equivalent to (9), it is easy to see that any multiplication on $\mathfrak{V}$ given by (6) is third power-associative.

Notice that flexibility (1) is equivalent to the identity

$$u \circ [v, u] = [u \circ v, u], \quad u, v \in \mathfrak{V}. \tag{10}$$

Assume that $\mathfrak{V}$ is flexible under the product in (6). If $u = x$ and $v = y$ in (10), then

$$\tau(x)[y, x] + \tau([y, x])x + \sigma(x, [y, x])c = \tau(x)[y, x]$$

and hence

$$\tau([y, x]) = \sigma(x, [y, x]) = 0 \tag{11}$$

for all $x, y \in \mathfrak{W}$. Since $[\mathfrak{W}, \mathfrak{W}] = \mathfrak{W}$, $\tau = 0$ and the linearization of $\sigma(x, [y, x]) = 0$ gives the invariance of σ:

$$\sigma([x, y], z) = \sigma(x, [y, z]), \quad x, y, z \in \mathfrak{W}. \tag{12}$$

We now show that σ is identically zero on $\mathfrak{W} \times \mathfrak{W}$. If $i \neq 0$, then by (4) and (12)

$$\begin{aligned}\sigma([e_i, e_{-i}], e_0) &= -2i\sigma(e_0, e_0)\\ &= \sigma(e_i, [e_{-i}, e_0]) = i\sigma(e_i, e_{-i})\end{aligned}$$

and hence $\sigma(e_i, e_{-i}) = -2\sigma(e_0, e_0)$. For a $j \neq 0$ with $i + j \neq 0$, one has by (4) and (12)

$$\begin{aligned}\sigma([e_i, e_j], e_{-(i+j)}) &= -2(j - i)\sigma(e_0, e_0)\\ &= \sigma(e_i, [e_j, e_{-(i+j)}]) = -(i + 2j)\sigma(e_i, e_{-i})\\ &= 2(i + 2j)\sigma(e_0, e_0),\end{aligned}$$

which implies $\sigma(e_0, e_0) = 0$ and so, by the above $\sigma(e_i, e_{-i}) = -2\sigma(e_0, e_0) = 0$ for all $i \in \mathbb{Z}$. Assume $i + j \neq 0$. By (4) and (12), $\sigma([e_0, e_i], e_j) = i\sigma(e_i, e_j) = -\sigma([e_i, e_0], e_j) = -\sigma(e_i, [e_0, e_j]) = -j\sigma(e_i, e_j)$, and thus $\sigma(e_i, e_j) = 0$. This shows that $\sigma(e_i, e_j) = 0$ for all $i, j \in \mathbb{Z}$.

The linearization of (10) implies the identity

$$[u, v \circ w] = v \circ [u, w] + [u, v] \circ w \tag{13}$$

for all $u, v, w \in \mathfrak{V}$. To verify $\lambda = 0$ and $\alpha = 0$, for any $i \in \mathbb{Z}$, choose $j \in \mathbb{Z}$ such that $i + j \neq 0$ and $j \neq 0, \pm 1$, so $\frac{1}{12}(j^3 - j) \neq 0$. If $u = e_j$, $v = e_i$ and $w = e_{-j}$ in (13), then

$$\begin{aligned}0 &= [e_j, e_i \circ e_{-j}] = e_i \circ [e_j, e_{-j}] + [e_j, e_i] \circ e_{-j}\\ &= \frac{1}{12}(j^3 - j)e_i \circ c = \frac{1}{12}(j^3 - j)(\alpha e_i + \lambda(e_i)c),\end{aligned}$$

and hence $\lambda(e_i) = \alpha = 0$ for all $i \in \mathbb{Z}$.

Conversely, it is easy to see that the multiplication (6) with $\sigma = \lambda = \tau = 0$ and $\alpha = 0$ satisfies (10) and thus is flexible. □

As a corollary to the proof of the Theorem, we have :

COROLLARY. *Let $\mathfrak{W}$ be the Witt algebra given by (4) over a field F of characteristic 0. A multiplication xy defined on $\mathfrak{W}$ is third power-associative and compatible with $\mathfrak{W}$ if and only if it is given by*

$$xy = \frac{1}{2}[x,y] + \tau(x)y + \tau(y)x, \quad x, y \in \mathfrak{W} \tag{14}$$

for a linear functional $\tau : \mathfrak{W} \to F$. The multiplication (14) is flexible if and only if $\tau = 0$, i.e., $xy = \frac{1}{2}[x,y]$ for all $x, y \in \mathfrak{W}$.

Proof. This follows from the same calculation as in the proof of the Theorem. □

Multiplications similar to (6) and (14) have been obtained for matrix algebras, octonion algebras and finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0 [1,4,5]. It is reasonable to pose the conjecture that the Theorem holds for any affine Kac-Moody algebra.

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References

1. G. M. Benkart, *Power-associative Lie-admissible algebras*, J. Algebra **90** (1984), 37-58.
2. A. Elduque and H. C. Myung, *Mutations of Alternative Algebras*, Math. Appl. Vol. 278, Kluwer Acad. Publ., Boston, 1994.
3. V. Kac, *Infinite Dimensional Lie Algebras*, 3rd ed., Cambridge Univ. Press, New York, 1990.
4. H. C. Myung, *Power-associative products on octonions*, Comm. Algebra **13** (1985), 1681-1697.
5. H. C. Myung, *Malcev-Admissible Algebras*, Prog. Math. Vol. 64, Birkhäuser, Boston, 1986.
6. H. C. Myung, *Non-Unital Composition Algebras*, Lecture Notes Ser. no. 22, Global Analysis Research Center, Seoul, 1994.
7. S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics*, Cambridge Univ. Press, New York, 1995.

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