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WICK DERIVATIONS ON WHITE NOISE FUNCTIONALS

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1. Introduction

The white noise analysis, initiated by Hida [3] in 1975, has been developed to an infinite dimensional distribution theory on Gaussian space (E^*, μ) as an infinite dimensional analogue of Schwartz distribution theory on Euclidean space with Lebesgue measure. The mathematical framework of white noise analysis is the Gel'fand triple $(E) \subset (L^2) \subset (E)^*$ over (E^*, μ) where μ is the standard Gaussian measure associated with a Gel'fand triple $E \subset H \subset E^*$.

The concept of Wick product was first introduced by Hida and Ikeda [4]. Based on white noise analysis, the Wick product of white noise functionals was defined in terms of S-transform by Meyer and Yan [10]. In recent years, the Wick product has been used extensively in the study of stochastic integrals, stochastic differential equations and white noise integral equations, see e.g., [7], [9] and the references cited therein. It is proved in [10] that (E) and $(E)^*$ are topological algebras under Wick product.

In white noise analysis, the set $\{x(t)\ ;\ t \in T\}$ is taken as a coordinate system of (E^*, μ) and Hida differential operator ∂_t is the coordinate differential operator. For $y \in E^*$ the differential operator D_y on (E):

$$D_y\phi = \int_T y(t)\partial_t\phi\, dt, \qquad \phi \in (E) \tag{1.1}$$

is an infinite dimensional analogue of the constant coefficient first order differential operator $\sum_{j=1}^n c_j \frac{\partial}{\partial x_j}$. The expression $y(t)\partial_t\phi$ in (1.1)

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can be interpreted as (1) the pointwise multiplication of $y(t)$ and $\partial_t\phi$, and (2) the Wick product of $y(t)$ and $\partial_t\phi$. Based on viewpoint (1), Obata [13] extended D_y to a first order differential operator with variable coefficients and showed that the operator is indeed a continuous derivation on (E).

In this paper, based on viewpoint (2), we first introduce the concept of a first order Wick differential operator with variable coefficient (Theorems 3.3 and 3.5) and that of Wick derivation on the topological algebras (E) and $(E)^*$ under the multiplication of Wick product and then show that a Wick derivation is nothing more than a first order Wick differential operator (Theorem 4.5). We next characterize all the Wick derivations in terms of their Fock expansions (Theorem 4.6). Finally we discuss Lie algebras of Wick derivations acting on $(E)^*$ (Theorems 5.2 and 5.3).

2. Preliminaries

Let T be a topological space with a Borel measure $d\nu(t) = dt$. We assume that $H = L^2(T, dt; \mathbb{R})$ is a real separable Hilbert space with norm $|\cdot|_0$. Let A be a positive self-adjoint operator on H such that $\rho = \|A^{-1}\|_{\mathrm{OP}} < 1$ and $\|A^{-1}\|_{\mathrm{HS}} < \infty$. Let $E = \bigcap_{p\geq 0} \mathrm{Dom}(A^p)$. For $p \geq 0$, let $E_p = \mathrm{Dom}(A^p)$ and E_{-p} the completion of E with respect to the norm $|\xi|_{-p} = |A^{-p}\xi|_0$. Then the dual of E_p is E_{-p}. Let $E^* = \bigcup_{p\geq 0} E_{-p}$. Then we have a Gel'fand triple $E \subset H \subset E^*$.

Let μ be the standard Gaussian measure on E^*, i.e., its characteristic function is given by

$$\int_{E^*} e^{i\langle x,\xi\rangle}\mu(dx) = e^{-\frac{1}{2}|\xi|_0^2},$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E^* \times E$. Then (E^*, μ) is called a *Gaussian space*. We denote by (L^2) the complex Hilbert space of μ-square integrable functions on E^*. By the Wiener-Ito decomposition theorem, each $\phi \in (L^2)$ admits an expansion:

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle, \tag{2.1}$$

where $f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$, the n-fold symmetric tensor product of the complexification of H. Moreover, the (L^2)-norm $\|\phi\|_0$ of ϕ is given by $\|\phi\|_0^2 = \sum_{n=0}^{\infty} n!|f_n|_0^2$, where $|\cdot|_0$ denotes the norm on $H_{\mathbb{C}}^{\otimes n}$ induced by the norm $|\cdot|_0$ on H.

Let $\Gamma(A)$ be the second quantization operator of A defined by

$$\Gamma(A)\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, A^{\otimes n} f_n \rangle,$$

where $\phi \in (L^2)$ is given by the expansion (2.1). Then we note that $\Gamma(A)$ is a positive self-adjoint operator with $\|\Gamma(A)^{-1}\|_{\mathrm{OP}} < 1$ and $\|\Gamma(A)^{-1}\|_{\mathrm{HS}} < \infty$. From (L^2) and $\Gamma(A)$, we can construct a Gel'fand triple $(E) \subset (L^2) \subset (E)^*$ as above.

Elements $\phi \in (E)$ and $\Phi \in (E)^*$ are called a test (white noise) functional and a generalized (white noise) functional (or a Hida distribution), respectively. We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical bilinear form on $(E)^* \times (E)$.

It is obvious that $\phi \in (L^2)$ belongs to (E) if and only if for each n, $f_n \in E_{\mathbb{C}}^{\otimes n}$ and for each $p \geq 0$, $\|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|f_n|_p^2 < \infty$.

For each $\xi \in E_{\mathbb{C}}$, *an exponential vector* φ_ξ is defined by $\varphi_\xi = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :\cdot^{\otimes n}:, \xi^{\otimes n} \rangle$. Then it is well-known that $\{\varphi_\xi ; \xi \in E_{\mathbb{C}}\}$ spans a dense subspace of (E). The S-transform of a generalized functional $\Phi \in (E)^*$ is a function on $E_{\mathbb{C}}$ defined by $S\Phi(\xi) = \langle\langle \Phi, \varphi_\xi \rangle\rangle$ for $\xi \in E_{\mathbb{C}}$. The following result is the characterization theorem for white noise functionals due to Potthoff Streit [15] with norm estimate due to Kubo Kuo [6].

THEOREM 2.1. *The S-transform $F = S\Phi$ of $\Phi \in (E)^*$ satisfies the following conditions:*

(F1) *For any $\xi, \eta \in E_{\mathbb{C}}$, the function $z \mapsto F(z\xi + \eta)$ is entire on $\mathbb{C}$.*

(F2) *There exist $K > 0, a > 0$ and $p \geq 0$ such that*

$$|F(\xi)| \leq K e^{a|\xi|_p^2}, \qquad \xi \in E_{\mathbb{C}}.$$

Conversely, assume that a $\mathbb{C}$-valued function F defined on $E_{\mathbb{C}}$ satisfies the above two conditions. Then there exists a unique $\Phi \in (E)^*$ such

that $F = S\Phi$. *Moreover, for any* $q > p$ *with* $2ae^2\|A^{-(q-p)}\|_{HS}^2 < 1$, *we have the following norm estimate:*

$$\|\Phi\|_{-q} \le K\big(1 - 2ae^2\|A^{-(q-p)}\|_{HS}^2\big)^{-1/2}.$$

Throughout this paper, for topological vector spaces X and Y, $L(X,Y)$ denotes the space of all continuous linear operator from X into Y equipped with the topology of uniform convergence on bounded subsets of X.

For $\Xi \in L((E),(E)^*)$ (resp. $L((E),(E))$, $L((E)^*,(E)^*)$), we define a mapping $G : E_{\mathbb{C}} \mapsto (E)^*$ by

$$G(\xi) = \Xi\varphi_\xi, \qquad \xi \in E_{\mathbb{C}}. \tag{2.2}$$

Then G satisfies properties (G1) and (G2) (resp. (G2′), (G2″)):

(G1) For $\xi, \xi', \eta \in E_{\mathbb{C}}$, the map $z \mapsto \langle\langle G(z\xi + \xi'), \varphi_\eta\rangle\rangle$ is entire on $\mathbb{C}$.

(G2) There exist $p \ge 0, q \ge 0, K > 0$ and $a > 0$ such that

$$\|G(\xi)\|_{-q} \le Ke^{a|\xi|_p^2}.$$

(G2′) For any $p \ge 0$, there exist $q \ge 0, K > 0$ and $a > 0$ such that

$$\|G(\xi)\|_p \le Ke^{a|\xi|_q^2}.$$

(G2″) For any $p \ge 0$ and $a > 0$, there exist $q \ge 0$ and $K > 0$ such that

$$\|G(\xi)\|_{-q} \le Ke^{a|\xi|_{-p}^2}.$$

The following is the characterization theorem for operators on white noise functionals (see [1]).

THEOREM 2.2. *Let* G *be an* $(E)^*$*-valued function defined on* $E_{\mathbb{C}}$ *satisfying (G1) and (G2). Then there exists a unique* $\Xi \in L((E),(E)^*)$ *such that (2.2) holds. If* G *satisfies (G1) and (G2′) (resp. (G2″)), it holds that* $\Xi \in L((E),(E))$ *(resp.* $L((E)^*,(E)^*)$*).*

For any $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$, *the integral kernel operator* $\Xi_{l,m}(\kappa_{l,m})$ is defined by

$$\langle\langle \Xi_{l,m}(\kappa_{l,m})\phi, \psi\rangle\rangle = \langle \kappa_{l,m}, \langle\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m}\phi, \psi\rangle\rangle\rangle,$$

which can be written in a formal integral expression as

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{T^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial^*_{s_1} \cdots \partial^*_{s_l} \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

It is known (see [11]) that each $\Xi \in L((E), (E)^*)$ has a unique Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}),$$

where $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ and the right hand side converges in $L((E), (E)^*)$.

For $\Xi \in L((E), (E)^*)$, a function on $E_{\mathbb{C}} \times E_{\mathbb{C}}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi \varphi_\xi, \varphi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}$$

is called *the symbol* of Ξ. It is known (see [11]) that $\widehat{\Xi}_1 = \widehat{\Xi}_2$ implies $\Xi_1 = \Xi_2$.

The *Wick product* of two generalized functionals Φ and Ψ, denoted by $\Phi \diamond \Psi$, is the unique generalized functional in $(E)^*$ such that $S(\Phi \diamond \Psi) = S\Phi \cdot S\Psi$. The next lemma is due to [7].

LEMMA 2.3.

(1) *For any $p \geq 0$, there exists $q > p$ such that*

$$\|\Phi \diamond \Psi\|_{-q} \leq \|\Phi\|_{-p} \|\Psi\|_{-p}, \qquad \Phi, \Psi \in (E)^*.$$

(2) *For any $p \geq 0$, there exists $q > p$ such that*

$$\|\phi \diamond \psi\|_p \leq \|\phi\|_q \|\psi\|_q, \qquad \phi, \psi \in (E).$$

3. First order Wick differential operators

We first need some notational conventions on $E_{\mathbb{C}} \hat{\otimes} (E)$, the space of $E_{\mathbb{C}}$-valued white noise test functionals. We use the same symbol $\|\cdot\|_0$ as the norm on $H_{\mathbb{C}} \otimes (L^2)$. It is well-known [12] that the topology of $E_{\mathbb{C}} \otimes (E)$ is given by the norms

$$\|\omega\|_p = \|(A \otimes \Gamma(A))^p \omega\|_0, \qquad \omega \in E_{\mathbb{C}} \otimes (E),\ p \in \mathbb{R}.$$

The canonical bilinear form on $(E_{\mathbb{C}} \hat{\otimes} (E))^* \times E_{\mathbb{C}} \otimes (E)$ is also denoted by $\langle\langle \cdot, \cdot \rangle\rangle$.

LEMMA 3.1. *For $\Phi \in (E_{\mathbb{C}} \hat{\otimes} (E))^*$ and $\xi \in E_{\mathbb{C}}$, there exists a unique $\langle \Phi, \xi \rangle \in (E)^*$ such that*

$$\langle\langle \langle \Phi, \xi \rangle, \phi \rangle\rangle = \langle\langle \Phi, \xi \otimes \phi \rangle\rangle, \qquad \phi \in (E).$$

Moreover, the map $\langle \cdot, \cdot \rangle$ is an $(E)^*$-valued continuous bilinear form on $(E_{\mathbb{C}} \hat{\otimes} (E))^* \times E_{\mathbb{C}}$ and for $\Phi \in (E_{\mathbb{C}} \hat{\otimes} (E))^*, \xi \in E_{\mathbb{C}}$ and $p \in \mathbb{R}$ we obtain

$$\|\langle \Phi, \xi \rangle\|_{-p} \leq \|\Phi\|_{-p} |\xi|_p.$$

Proof. Let $\Phi \in (E_{\mathbb{C}} \hat{\otimes} (E))^*, \xi \in E_{\mathbb{C}}$ and $\phi \in (E)$. Then we have the norm estimate:

$$|\langle\langle \Phi, \xi \otimes \phi \rangle\rangle| \leq \|\Phi\|_{-p} |\xi|_p \|\phi\|_p. \tag{3.1}$$

So, the map $\phi \mapsto \langle\langle \Phi, \xi \otimes \phi \rangle\rangle$ is continuous linear and hence there exists a unique Hida distribution, denoted by $\langle \Phi, \xi \rangle$, such that

$$\langle\langle \langle \Phi, \xi \rangle, \phi \rangle\rangle = \langle\langle \Phi, \xi \otimes \phi \rangle\rangle, \qquad \phi \in (E).$$

The second assertion is clear from (3.1) □

The next lemma can be found in [7] and [11].

LEMMA 3.2. *For $\phi, \psi \in (E)$, we put $h_{\phi,\psi}(t) = \langle\langle \partial_t \phi, \psi \rangle\rangle$. Then for each $p \geq 0$, there exists $K > 0$ such that $|h_{\phi,\psi}|_p \leq K \|\phi\|_p \|\psi\|_p$. In particular, $h_{\phi,\psi} \in E_{\mathbb{C}}$.*

THEOREM 3.3. *For $\Phi \in (E_{\mathbb{C}} \otimes (E))^*$, there exists a unique $\Xi \in L((E), (E)^*)$ such that*

$$\Xi(\langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle) = n\langle :\cdot^{\otimes(n-1)}:, \xi^{\otimes(n-1)}\rangle \diamond \langle \Phi, \xi\rangle, \tag{3.2}$$

for any $\xi \in E_{\mathbb{C}}$ and for any $n \geq 0$. Moreover, its symbol is given by

$$\widehat{\Xi}(\xi, \eta) = e^{\langle \xi, \eta\rangle} \langle\langle \Phi, \xi \otimes \varphi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

Proof. For $\Phi \in (E_{\mathbb{C}} \otimes (E))^*$ and $\phi \in (E)$, define a $\mathbb{C}$-valued function $F_{\Phi,\phi}$ on $E_{\mathbb{C}}$ by

$$F_{\Phi,\phi}(\xi) = \langle\langle \Phi, h_{\phi,\varphi_\xi} \otimes \varphi_\xi \rangle\rangle, \qquad \xi \in E_{\mathbb{C}},$$

where h_{ϕ,φ_ξ} is defined as in Lemma 3.2. Then by Lemma 3.2, for any $p \geq 0$ with $\|\Phi\|_{-p} < \infty$, there exists $K > 0$ such that

$$|F_{\Phi,\phi}(\xi)| \leq K\|\Phi\|_{-p}\|\phi\|_p e^{|\xi|_p^2}. \tag{3.3}$$

Hence $F_{\Phi,\phi}$ satisfies (F2) in Theorem 2.1. Now we will prove that $F_{\Phi,\phi}$ also satisfies (F1) in Theorem 2.1. First observe that if Φ is of the form $\Phi = y \otimes \Psi$, $y \in E_{\mathbb{C}}^*$, $\Psi \in (E)^*$ then

$$F_{\Phi,\phi}(\xi) = \langle y, h_{\phi,\varphi_\xi}\rangle \langle\langle \Psi, \varphi_\xi \rangle\rangle.$$

Since the map $\xi \mapsto \langle y, h_{\phi,\varphi_\xi}\rangle$ is the S-transform of a Hida distribution $\psi \mapsto \langle y, h_{\phi,\psi}\rangle$, it holds that $F_{\Phi,\phi}$ satisfies (F1) in Theorem 2.1. For an arbitrary $\Phi \in (E_{\mathbb{C}} \otimes (E))^*$ we can choose a sequence $\{\Phi_k\}$ in the linear span of the set $\{y \otimes \Psi \ ; \ y \in E_{\mathbb{C}}^*, \Psi \in (E)^*\}$ such that Φ_k converges to Φ in $(E_{\mathbb{C}} \otimes (E))^*$. Then by (3.3) we obtain

$$|F_{\Phi_k,\phi}(z\xi + \xi') - F_{\Phi,\phi}(z\xi + \xi')| \leq K\|\Phi_k - \Phi\|_{-p}\|\phi\|_p e^{|z\xi+\xi'|^2}.$$

Therefore $F_{\Phi_k,\phi}(z\xi + \xi')$ converges to $F_{\Phi,\phi}(z\xi + \xi')$ uniformly on every compact subset of $\mathbb{C}$. Thus the function $z \mapsto F_{\Phi,\phi}(z\xi + \xi')$ is entire on $\mathbb{C}$. Consequently, by Theorem 2.1 there exists a unique $\Omega_\phi \in (E)^*$ such that $F_{\Phi,\phi}(\xi) = S\Omega_\phi(\xi)$. Define an operator Ξ by $\Xi\phi = \Omega_\phi$ for $\phi \in (E)$. The linearity of Ξ is clear from the linearity of the mapping

$\phi \mapsto F_{\Phi,\phi}(\xi)$. And by Theorem 2.1, for $q > p$ with $2e^2\|A^{-(q-p)}\|_{\mathrm{HS}}^2 < 1$, we obtain the norm estimate

$$\|\Xi\phi\|_{-q} \le K\|\Phi\|_{-p}\|\phi\|_p\big(1 - 2e^2\|A^{-(q-p)}\|_{\mathrm{HS}}^2\big)^{-1/2},$$

from which we have $\Xi \in L((E),(E)^*)$.

Now we will verify that Ξ satisfies (3.2). Fix $\xi \in E_{\mathbb{C}}$ and put $\psi_n(x) = \langle : x^{\otimes n} :, \xi^{\otimes n}\rangle$ for $n \ge 0$. Then

$$h_{\psi_n,\varphi_\eta}(t) = \langle\!\langle \partial_t\psi_n, \varphi_\eta\rangle\!\rangle = n\langle\!\langle \xi(t)\psi_{n-1}, \varphi_\eta\rangle\!\rangle = n\langle\xi,\eta\rangle^{n-1}\xi(t).$$

Thus we have $h_{\psi_n,\varphi_\eta} = n\langle\xi,\eta\rangle^{n-1}\xi$ and hence

$$S\Xi\psi_n(\eta) = F_{\Phi,\psi_n}(\eta) = \langle\!\langle \Phi, h_{\psi_n,\varphi_\eta} \otimes \varphi_\eta\rangle\!\rangle = n\langle\xi,\eta\rangle^{n-1}\langle\!\langle \Phi, \xi\otimes\varphi_\eta\rangle\!\rangle.$$

On the other hand,

$$S(n\psi_{n-1}\diamond\langle\Phi,\xi\rangle)(\eta) = nS\psi_{n-1}(\eta)\langle\!\langle\langle\Phi,\xi\rangle,\varphi_\eta\rangle\!\rangle = n\langle\xi,\eta\rangle^{n-1}\langle\!\langle \Phi,\xi\otimes\varphi_\eta\rangle\!\rangle.$$

Hence (3.2) holds. Further, we have

$$\Xi\varphi_\xi = \sum_{n=0}^{\infty}\frac{1}{n!}\Xi\psi_n = \sum_{n=1}^{\infty}\frac{1}{n!}n\psi_{n-1}\diamond\langle\Phi,\xi\rangle = \varphi_\xi\diamond\langle\Phi,\xi\rangle \tag{3.4}$$

and

$$\widehat{\Xi}(\xi,\eta) = \langle\!\langle \varphi_\xi\diamond\langle\Phi,\xi\rangle,\varphi_\eta\rangle\!\rangle = e^{\langle\xi,\eta\rangle}\langle\!\langle\langle\Phi,\xi\rangle,\varphi_\eta\rangle\!\rangle = e^{\langle\xi,\eta\rangle}\langle\!\langle \Phi,\xi\otimes\varphi_\eta\rangle\!\rangle.$$

This completes the proof. □

DEFINITION 3.4. The operator Ξ given in Theorem 3.3 is called *a first order Wick differential operator with coefficient* $\Phi \in (E_{\mathbb{C}} \otimes (E))^*$ and is denoted by

$$\Xi = \int_T \Phi(t)\diamond\partial_t\,dt.$$

We note that the map $t \mapsto \Phi(t)$ is an $(E)^*$-valued distribution on T, i.e., an element in $E_{\mathbb{C}}^* \otimes (E)^* \cong (E_{\mathbb{C}} \otimes (E))^*$.

We are now interested in the first order Wick differential operators acting on (E) into itself and acting on $(E)^*$ into itself.

THEOREM 3.5. *Let $\Xi \in L((E),(E)^*)$ be a first order Wick differential operator with coefficient $\Phi \in (E_{\mathbb{C}} \otimes (E))^*$. Then*

(1) *$\Xi \in L((E),(E))$ if and only if $\Phi \in E_{\mathbb{C}}^* \otimes (E)$.*
(2) *$\Xi \in L((E)^*,(E)^*)$ if and only if $\Phi \in E_{\mathbb{C}} \otimes (E)^*$.*

Proof. (1) We first note that $\Phi \in E_{\mathbb{C}}^* \otimes (E)$ if and only if $\langle \Phi, \cdot \rangle \in L(E_{\mathbb{C}},(E))$.

Suppose $\Xi \in L((E),(E))$. Then for any $p \geq 0$, there exist $q \geq p$ and $K > 0$ such that

$$\|\Xi\phi\|_p \leq K\|\phi\|_q, \qquad \phi \in (E).$$

Since $\Xi(\langle \cdot, \xi \rangle) = \langle \Phi, \xi \rangle$ by (3.2), we have for $\xi \in E_{\mathbb{C}}$

$$\|\langle \Phi, \xi \rangle\|_p = \|\Xi(\langle \cdot, \xi \rangle)\|_p \leq K\|\langle \cdot, \xi \rangle\|_q = K|\xi|_q.$$

So $\langle \Phi, \cdot \rangle \in L(E_{\mathbb{C}},(E))$ and hence $\Phi \in E_{\mathbb{C}}^* \otimes (E)$.

Conversely, let $\Phi \in E_{\mathbb{C}}^* \otimes (E)$. We will show that the function $G(\xi) = \varphi_\xi \diamond \langle \Phi, \xi \rangle$, $\xi \in E_{\mathbb{C}}$ satisfies (G2′) in Section 2. Since $\langle \Phi, \cdot \rangle \in L(E_{\mathbb{C}},(E))$, for any $q \geq 0$ there exist $K > 0$ and $r \geq q$ such that

$$\|\langle \Phi, \xi \rangle\|_q \leq K|\xi|_r, \qquad \xi \in E_{\mathbb{C}}.$$

Hence for $p \geq 0$, by Lemma 2.3, there exists $q > p$ such that

$$\|G(\xi)\|_p \leq \|\varphi_\xi\|_q \|\langle \Phi, \xi \rangle\|_q \leq Ke^{(1+\frac{1}{2}\rho^{2(r-q)})|\xi|_r^2}.$$

Thus (G2′) holds. In view of (3.4), by Theorem 2.2, we conclude that $\Xi \in L((E),(E))$.

(2) We observe that $\Phi \in E_{\mathbb{C}} \otimes (E)^*$ if and only if $\langle \Phi, \cdot \rangle \in L(E_{\mathbb{C}}^*,(E)^*)$. Suppose $\Xi \in L((E)^*,(E)^*)$. Then for any $p \geq 0$, there exist $q \geq 0$ and $K > 0$ such that

$$\|\Xi\phi\|_{-q} \leq K\|\phi\|_{-p}.$$

In view of (3.2), we obtain

$$\|\langle \Phi, \xi \rangle\|_{-q} = \|\Xi(\langle \cdot, \xi \rangle)\|_{-q} \leq K\|\langle \cdot, \xi \rangle\|_{-p} = K|\xi|_{-p}.$$

Therefore $\langle \Phi, \cdot \rangle \in L(E_{\mathbb{C}}^*,(E)^*)$, and hence we obtain $\Phi \in E_{\mathbb{C}} \otimes (E)^*$.

Conversely, let $\Phi \in E_{\mathbb{C}} \otimes (E)^*$. As in the proof of (1), we need only to show that $G(\xi) = \varphi_\xi \diamond \langle \Phi, \xi \rangle$ satisfies (G2″) in Section 2. Take $p \geq 0$ and $a > 0$. And choose $p' > p$ with $\frac{3}{2}\rho^{2(p'-p)} \leq a$. Since $\langle \Phi, \cdot \rangle \in L(E_{\mathbb{C}}^*, (E)^*)$, there exist $q \geq p'$ and $K > 0$ such that

$$\|\langle \Phi, \xi \rangle\|_{-q} \leq K|\xi|_{-p'}.$$

For this q, there exists $r > q$ such that

$$\|G(\xi)\|_{-r} \leq \|\varphi_\xi\|_{-q}\|\langle \Phi, \xi \rangle\|_{-q} \leq Ke^{a|\xi|_{-p}^2}.$$

This completes the proof. □

EXAMPLE 3.6. (1) The differential operator D_y ($y \in E_{\mathbb{C}}^*$) is a first order Wick differential operator with coefficient $\Phi = y \otimes 1 \in E_{\mathbb{C}}^* \otimes (E)$.

(2) The number operator N has the integral representation

$$N = \int_T \partial_t^* \partial_t dt = \int_T x(t) \diamond \partial_t dt.$$

Let $\Phi_t(x) = x(t)$. Then we see that Φ satisfies for $\xi, \eta \in E_{\mathbb{C}}$

$$\langle\langle \Phi, \xi \otimes \varphi_\eta \rangle\rangle = \langle\langle \langle \Phi, \xi \rangle, \varphi_\eta \rangle\rangle = \langle \langle\langle \Phi, \varphi_\eta \rangle\rangle, \xi \rangle = \langle \xi, \eta \rangle.$$

So N is a first order Wick differential operator with coefficient $\Phi \in E_{\mathbb{C}}^* \otimes (E)$ and with $\Phi \in E_{\mathbb{C}} \otimes (E)^*$.

4. Wick derivations on white noise functionals

We begin with the definition of Wick derivation.

DEFINITION 4.1. An operator $\Xi \in L((E), (E)^*)$ is called *a Wick derivation* if

$$\Xi(\phi \diamond \psi) = \Xi\phi \diamond \psi + \phi \diamond \Xi\psi, \qquad \phi, \psi \in (E).$$

Now we shall give two criteria for checking whether or not a continuous linear operator becomes a Wick derivation.

PROPOSITION 4.2. *Let $\Xi \in L((E),(E)^*)$. Then it is a Wick derivation if and only if*

$$\widehat{\Xi}(\xi+\eta,\zeta) = e^{\langle \eta,\zeta\rangle}\widehat{\Xi}(\xi,\zeta) + e^{\langle \xi,\zeta\rangle}\widehat{\Xi}(\eta,\zeta), \quad \xi,\eta,\zeta \in E_{\mathbb{C}}. \tag{4.1}$$

Proof. Since the set $\{\varphi_\xi\ ;\ \xi \in E_{\mathbb{C}}\}$ spans a dense subspace of (E), Ξ is a Wick derivation if and only if

$$\Xi(\varphi_\xi \diamond \varphi_\eta) = \Xi\varphi_\xi \diamond \varphi_\eta + \varphi_\xi \diamond \Xi\varphi_\eta, \qquad \xi,\eta \in E_{\mathbb{C}}^*.$$

But (4.1) follows from the obvious facts:

$$\varphi_\xi \diamond \varphi_\eta = \varphi_{\xi+\eta} \qquad \xi,\eta \in E_{\mathbb{C}}$$

and

$$\langle\!\langle \varphi_\xi \diamond \Psi, \varphi_\eta\rangle\!\rangle = e^{\langle \xi,\eta\rangle}\langle\!\langle \Psi, \varphi_\eta\rangle\!\rangle, \qquad \Psi \in (E)^*,\ \xi,\eta \in E_{\mathbb{C}}.$$

Hence we complete the proof. □

PROPOSITION 4.3. *Let $\Xi \in L((E),(E)^*)$. Then it is a Wick derivation if and only if for all $\xi \in E_{\mathbb{C}}$ and $n \geq 0$*

$$\Xi(\langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle) = n\langle :\cdot^{\otimes(n-1)}:, \xi^{\otimes(n-1)}\rangle \diamond \Xi(\langle \cdot, \xi\rangle). \tag{4.2}$$

Proof. Immediate from the fact that the set $\{\langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle\ ;\ \xi \in E_{\mathbb{C}}, n \geq 0\}$ spans a dense subspace of (E). □

LEMMA 4.4. *Any first order Wick differential operator with coefficient $\Phi \in (E_{\mathbb{C}} \otimes (E))^*$ is a Wick-derivation.*

Proof. We easily verify that the symbol of a first order Wick differential operator with coefficient $\Phi \in (E_{\mathbb{C}} \otimes (E))^*$ satisfies (4.1). Hence by Proposition 4.2, the proof follows. □

THEOREM 4.5. (1) *Let $\Xi \in L((E),(E)^*)$. Then Ξ is a Wick derivation if and only if it is a first order Wick differential operator with coefficient $\Phi \in (E_{\mathbb{C}} \hat{\otimes} (E))^*$.*

(2) *Let $\Xi \in L((E),(E))$ (resp. $L((E)^*,(E)^*)$). Then Ξ is a Wick derivation if and only if it is a first order Wick differential operator with coefficient $\Phi \in E_{\mathbb{C}}^* \hat{\otimes} (E)$ (resp. $E_{\mathbb{C}} \hat{\otimes} (E)^*$).*

Proof. Suppose that $\Xi \in L((E),(E)^*)$ is a Wick derivation. By the continuity of Ξ, we can choose $p \geq 0$ and $K > 0$ such that

$$\|\Xi\phi\|_{-p} \leq K\|\phi\|_p, \qquad \phi \in (E).$$

In particular, for $\xi \in E_{\mathbb{C}}$,

$$\|\Xi(\langle \cdot, \xi \rangle)\|_{-p} \leq K\|\langle \cdot, \xi \rangle\|_p = K|\xi|_p.$$

This implies that the mapping $\xi \mapsto \Xi(\langle \cdot, \xi \rangle)$ is a continuous linear function from $E_{\mathbb{C}}$ into $(E)^*$. Since $L(E_{\mathbb{C}},(E)^*) \cong (E_{\mathbb{C}} \hat{\otimes} (E))^*$, there exists a unique $\Phi \in (E_{\mathbb{C}} \hat{\otimes} (E))^*$ such that

$$\langle\langle \Phi, \xi \otimes \phi \rangle\rangle = \langle\langle \Xi(\langle \cdot, \xi \rangle), \phi \rangle\rangle. \qquad \xi \in E_{\mathbb{C}}, \phi \in (E).$$

Hence we obtain $\Xi(\langle \cdot, \xi \rangle) = \langle \Phi, \xi \rangle$ for $\xi \in E_{\mathbb{C}}$.

It is enough to show that Ξ satisfies (3.2). By the definition of Wick derivation, we can compute

$$\begin{aligned} \Xi(\langle :\cdot^{\otimes n}:, \xi^{\otimes n} \rangle) &= \Xi(\langle \cdot, \xi \rangle^{\diamond n}) \\ &= n\langle \cdot, \xi \rangle^{\diamond (n-1)} \diamond \Xi(\langle \cdot, \xi \rangle) \\ &= n\langle :\cdot^{\otimes (n-1)}:, \xi^{\otimes n} \rangle \diamond \langle \Phi, \xi \rangle. \end{aligned}$$

Hence by Theorem 3.3, Ξ is the first order Wick differential operator with coefficient $\Phi \in (E_{\mathbb{C}} \hat{\otimes} (E))^*$.

The rest of the assertion is immediate from Lemma 4.4 and Theorem 3.5. □

The following theorem is an another characterization for the Wick derivation in terms of its Fock expansion.

THEOREM 4.6. *Let $\Xi \in L((E),(E)^*)$ be a Wick derivation. Then it has the Fock expansion of the form:*

$$\Xi = \sum_{n=0}^{\infty} \Xi_{n,1}(\kappa_{n,1}) \tag{4.3}$$

for some sequence of distributions $\{\kappa_{n,1}\}$ with $\kappa_{n,1} \in (E_{\mathbb{C}}^{\otimes(n+1)})^*_{\mathrm{sym}(n,1)}$.

Conversely, if the Fock expansion of Ξ is given as in (4.3), then it is a Wick derivation.

Proof. Let $\Xi \in L((E),(E)^*)$ be a Wick derivation. Then, by Theorem 4.5, Ξ is the first order Wick differential operator with some coefficient $\Phi \in (E_{\mathbb{C}} \otimes (E))^*$. We note that $\Phi \in (E_{\mathbb{C}} \otimes (E))^* \cong E_{\mathbb{C}}^* \otimes (E)^*$ is an $E_{\mathbb{C}}^*$-valued generalized functional. Hence in view of Proposition 2.3 in [12], there exists a unique sequence $\{\kappa_{n,1}\}_{n=0}^{\infty}$ with $\kappa_{n,1} \in (E_{\mathbb{C}}^{\otimes(n+1)})^*_{\mathrm{sym}(n,1)}$ such that

$$\|\Phi\|_{-p}^2 = \sum_{n=0}^{\infty} n!|\kappa_{n,1}|_{-p}^2 < \infty \qquad \text{for some } p \geq 0$$

and

$$\langle\langle \Phi, \xi \otimes \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle \kappa_{n,1}, f_n \otimes \xi \rangle \tag{4.4}$$

for any $\xi \in E_{\mathbb{C}}$ and $\phi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, f_n \rangle$, $f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}$. Using (4.4), we have

$$\widehat{\Xi}(\xi,\eta) = e^{\langle \xi,\eta \rangle} \langle\langle \Phi, \xi \otimes \varphi_\eta \rangle\rangle = e^{\langle \xi,\eta \rangle} \sum_{n=0}^{\infty} \langle \kappa_{n,1}, \eta^{\otimes n} \otimes \xi \rangle.$$

Hence Ξ admits the Fock expansion of the form (4.3).

Conversely, assume that $\Xi \in L((E),(E)^*)$ has its Fock expansion of the form (4.3). So we have

$$\widehat{\Xi}(\xi,\eta) = e^{\langle \xi,\eta \rangle} \sum_{n=0}^{\infty} \langle \kappa_{n,1}, \eta^{\otimes n} \otimes \xi \rangle, \qquad \xi,\ n \in E_{\mathbb{C}}$$

Hence it is easy to see that $\widehat{\Xi}$ satisfies (4.1), so that Ξ is a Wick derivation. □

5. Lie algebras of Wick derivations acting on $(E)^*$

Let $\mathrm{Der}((E)^*)$ (resp. $\mathrm{Der}((E))$) be the set of Wick derivations in $L((E)^*,(E)^*)$ (resp. $L((E),(E))$). Then $\mathrm{Der}((E)^*)$ (resp. $\mathrm{Der}((E))$) forms a Lie algebra under the operation of the commutator $[\cdot,\cdot]$. In this section, we will study only Lie subalgebras of $\mathrm{Der}((E)^*)$, since we can prove the corresponding results for $\mathrm{Der}((E))$ by similar argument.

The following lemma is taken from [14].

LEMMA 5.1. *Let $\eta \in E_{\mathbb{C}}$ and $\kappa_{n,1} \in (E_{\mathbb{C}}^{\otimes n})^*_{\mathrm{sym}} \otimes E_{\mathbb{C}}$. Then*

(1) $[D_\eta, \Xi_{n,1}(\kappa_{n,1})] = n\Xi_{n-1,1}(\kappa_{n,1} \otimes^1 \eta)$, *where $\otimes^1$ denotes the left contraction (see [11]).*
(2) $[N, \Xi_{n,1}(\kappa_{n,1})] = (n-1)\Xi_{n,1}(\kappa_{n,1})$.

The following theorem is a simple modification of Theorem 4.3 proved in [14].

THEOREM 5.2. *Let $\mathfrak{g} \subset \mathrm{Der}((E)^*)$ be a 2-dimensional Lie subalgebra containing the number operator N. Then there exists $\Xi \in \mathfrak{g}$ of the form $\Xi = \Xi_{m,1}(\kappa_{m,1})$ such that*

$$\mathfrak{g} = \mathbb{C}N + \mathbb{C}\Xi, \qquad [N, \Xi] = (m-1)\Xi.$$

For $\eta \in E_{\mathbb{C}} - \{0\}$, $\mathfrak{g} = \mathbb{C}D_\eta + \mathbb{C}N$ is a 2-dimensional Lie subalgebra of $\mathrm{Der}((E)^*)$. In the next theorem, we determine all possible 3-dimensional Lie subalgebras of $\mathrm{Der}((E)^*)$ containing $\mathbb{C}D_\eta + \mathbb{C}N$.

THEOREM 5.3. *Let $\mathfrak{g} \subset \mathrm{Der}((E)^*)$ be a 3-dimensional Lie subalgebra containing the differential operator D_η with $\eta \in E_{\mathbb{C}} - \{0\}$ and the number operator N. Then there exists $\Xi \in \mathfrak{g}$ which is one of the forms (a)–(d) below such that $\mathfrak{g} = \mathbb{C}D_\eta + \mathbb{C}N + \mathbb{C}\Xi$:*

(a) $\Xi = \Xi_{0,1}(\kappa_{0,1})$ *with $\kappa_{0,1}$ being linearly independent of η.*
(b) $\Xi = \Xi_{1,1}(\kappa_{1,1})$ *with $\kappa_{1,1} \otimes^1 \eta = \eta$.*
(c) $\Xi = \Xi_{2,1}(\kappa_{2,1})$ *with $\kappa_{2,1} \otimes^1 \eta = \tau$.*
(d) $\Xi = \Xi_{m,1}(\kappa_{m,1})$ *with $\kappa_{m,1} \otimes^1 \eta = 0$ for some $m \geq 1$.*

In particular, if Ξ is of the form (b), then $\mathfrak{g}$ is solvable, and if Ξ is of the form (c), then $\mathfrak{g}$ is simple.

Proof. Let $\Xi \in \mathfrak{g}$ be a linearly independent element of D_η and N. Suppose that $\Xi = \sum_{n=0}^{\infty} \Xi_{n,1}(\kappa_{n,1})$ is its Fock expansion. Without

loss of generality, we may assume that $\kappa_{0,1}$ and $\kappa_{1,1}$ are not non-zero constant multiples of η and τ, respectively.

Using Lemma 5.1, we have $[N, \Xi] = \sum_{n=0}^{\infty}(n-1)\Xi_{n,1}(\kappa_{n,1}) = aD_\eta + bN + c\Xi$ for some $a, b, c \in \mathbb{C}$. For this relation, we obtain

(i) $a\eta + (c+1)\kappa_{0,1} = 0$.
(ii) $b\tau + c\kappa_{1,1} = 0$.
(iii) $c\kappa_{n,1} = (n-1)\kappa_{n,1}$ for all $n \geq 2$.

First, suppose that $\kappa_{n,1} \neq 0$ for some $n \geq 2$. Then by (iii), c must be an integer ≥ 1 and $\kappa_{n,1} = 0$ for $n \neq c+1$. By (i) and (ii), we obtain $\kappa_{0,1} = 0$ and $\kappa_{1,1} = 0$. Put $m = c+1$. Then $\Xi = \Xi_{m,1}$. On the other hand, $[D_\eta, \Xi] = m\Xi_{m-1,1}(\kappa_{m,1} \otimes^1 \eta) = a'D_\eta + b'N + c'\Xi$ for some $a', b', c' \in \mathbb{C}$. So we have $\kappa_{m,1} \otimes^1 \eta = 0$ or $m = 2$, $\kappa_{2,1} \otimes^1 \eta = b'\tau$. Hence we can take Ξ of the form (c) or (d).

Next, suppose that $\Xi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,1}(\kappa_{1,1})$. Then by (i) and (ii), $\kappa_{0,1} = 0$ or $\kappa_{1,1} = 0$. If $\kappa_{0,1} \neq 0$ and $\kappa_{1,1} = 0$, then $\Xi = \Xi_{0,1}(\kappa_{0,1})$ is of the form (a). If $\kappa_{0,1} = 0$ and $\kappa_{1,1} \neq 0$, then $[D_\eta, \Xi] = \Xi_{0,1}(\kappa_{1,1} \otimes^1 \eta) = a'D_\eta + b'N + c'\Xi$ for some $a', b', c' \in \mathbb{C}$. We then have $b' = c' = 0$ and $\kappa_{1,1} \otimes^1 \eta = a'\eta$. Hence we can take Ξ of the form (b) or (d).

The rest of the theorem is easily verified. □

Note added in Proof. Our Theorem 4.6 is found in a recent paper by S. W. He, R. Q. Yao and J. G. Wang (*The Characterizations of Laplacians in White Noise Analysis*, Nagoya Math. J. Vol. 143 (1996), 93-109)

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