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CONSTRUCTIVE WAVELET COEFFICIENTS MEASURING SMOOTHNESS THROUGH BOX SPLINES

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1. Introduction

In surface compression applications, one of the main issues is how to efficiently store and calculate the computer representation of certain surfaces. This leads us to consider a nonlinear approximation by box splines with free knots since, for instance, the nonlinear method based on wavelet decomposition gives efficient compression and recovery algorithms for such surfaces (cf. [12]). DeVore *et al.* [13] have established a characterization between such types of nonlinear approximations and the Besov space, $B^\alpha := B_q^\alpha(L_q(\mathbb{R}^d))$, $q = (\alpha/d + 1/p)^{-1}$, $0 < p \leq \infty$, $\alpha > 0$. Using this characterization, DeVore, Jawerth, and Lucier [12] have provided some optimal algorithms for surface compression via box splines.

The objective of this paper is to study a wavelet decomposition

$$f = \sum_{k\in\mathbb{Z},j\in\mathbb{Z}^d} c_{k,j} M_{k,j}, \tag{1.1}$$

and a characterization

$$\|f\|_{B^\alpha} \approx \left(\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}^d} |c_{k,j}|^q \|M_{k,j}\|_p^q\right)^{1/q} \tag{1.2}$$

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for the space B^α with box splines M in constructive way, where $M_{k,j}(x) := M(2^k x - j)$. While DeVore *et al.* (cf. [13], [14]) have shown the characterization (1.2) by the decomposition (1.1), they used bounded nonlinear projectors from L_q onto spline spaces for $0 < q < 1$. Their nonlinear projectors are then nonconstructive and complicated to compute. In this paper, we provide a construction of bounded linear projectors that are more constructive and easier to compute than those of [13], [14]. Roughly sketching, to calculate wavelet coefficients of (1.1), we find the local polynomial L_2-approximation and then apply quasi-interpolation techniques. This step initiates us into development of constructive wavelet coefficients with an explicit form. Then, our constructive wavelet coefficients can both be easily manipulated in numerical applications and characterize the space B^α.

In §2, the definition of box splines and their basic properties are reviewed from [4], [8] as well as some examples of box splines in $\mathbb{R}^2$ are presented for our purpose. In §3, a method to calculate the coefficients of (1.1) is developed. The idea of this method does first use a local L_2-projector from L_p space, $1 \le p \le \infty$, onto piecewise polynomial spaces and then project in a separate step from discontinuous piecewise polynomial spaces to the spline spaces using quasi-interpolants. This gives a bounded linear projector on L_p for all $1 \le p \le \infty$. In §4, the smoothness subspaces of L_p(the Besov spaces), $B_q^\alpha(L_p)$, defined by the modulus of smoothness are briefly reviewed from [19], [20]. Using the fact [15] that the spaces B^α are continuously embedded into $L_p(\mathbb{R}^d)$, the decomposition (1.1) of f in B^α is obtained with an explicit expression of the wavelet coefficients in §5. In addition, using a theorem of Frazier and Jawerth [17] and a sharp estimate for the modulus of smoothness of a box spline series, the characterization (1.2) of the space B^α is obtained. Here, our estimates cover a larger class of the spaces B^α; in other words, our range of α is larger than the one reported in [13].

We end this section with preliminary notations and definitions.

- Ω : a domain in $\mathbb{R}^d$ (in this paper, Ω is either the unit cube $[0,1]^d$ or $\mathbb{R}^d$).
- $\lfloor \alpha \rfloor$: the greatest integer less than or equal to $\alpha \in \mathbb{R}$.
- $\lceil \alpha \rceil$: the smallest integer greater than $\alpha \in \mathbb{R}$.
- $\mathcal{D}$: the collection of all dyadic cubes, $I_{k,j} := 2^{-k}j + 2^{-k}[0,1]^d$.

$j \in \mathbb{Z}^d$.

$\mathcal{D}_k$: the collection of all dyadic cubes I with the side length $\ell(I) = 2^{-k}$.

$\Delta_h^r(f,x) = \sum_{j=0}^r (-1)^{r-j}\binom{r}{j} f(x+hj)$: the rth differences of functions f in the direction $h \in \mathbb{R}^d$ with $\Delta_h^0(f,x) = f(x)$, where $k \in \mathbb{N}$ and $\binom{r}{j}$ are binomial coefficients.

$\mathcal{P}_r$: the collection of all polynomials with total degree less than r on $\mathbb{R}^d$.

$\approx$: the equivalence relation of two norms $\|\cdot\|_1$, $\|\cdot\|_2$ of a normed space X; that is, $\|f\|_1 \approx \|f\|_2$ means that there exist constants C_1 and C_2 such that $C_1\|f\|_2 \leq \|f\|_1 \leq C_2\|f\|_2$ for all $f \in X$.

Let $0 < p \leq \infty$, then $L_p(\Omega)$ denotes the collection of all complex-valued Lebesgue measurable functions f on Ω. For convenience, we write $\|f\|_p := \|f\|_{L_p(\mathbb{R}^d)}$ when $\Omega = \mathbb{R}^d$. Note that when $0 < p < 1$, the space $L_p(\Omega)$ is not a Banach space but a complete quasi-normed linear space because $\|\cdot\|_{L_p(\Omega)}$ is a quasi-norm, that is,

$$\|f+g\|_{L_p(\Omega)} \leq 2^{\frac{1}{p}-1}\{\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}\}, \quad \text{for } 0<p<1. \tag{1.3}$$

We shall use a useful fact: for any $\mu \leq \min(1,p)$ and sequence of functions (f_i),

$$\left\|\sum f_i\right\|_{L_p(\Omega)} \leq \left(\sum \|f_i\|_{L_p(\Omega)}^\mu\right)^{1/\mu}. \tag{1.4}$$

For notational brevity, we sometimes index the k, j term of (1.1) by the dyadic cube $I = 2^{-k}[0,1]^d + 2^{-k}j$ (where $x_I := j2^{-k}$ corresponds to I); that is,

$$f = \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_k} c_I M_I \tag{1.5}$$

where $M_I = M(2^k \cdot -j)$.

2. Box Splines

Let $X_n := \{t_i\}_{i=1}^n \subset \mathbb{Z}^d \setminus \{0\}$ be a *direction set* that spans $\mathbb{R}^d$. The box spline $M := M(\cdot \mid X_n)$ associated with X_n is defined as the distribution on $\mathbb{R}^d$ given by the equation

$$\int_{\mathbb{R}^d} M(x)\varphi(x)\,dx = \int_{Q_n} \varphi(X_n \cdot y)\,dy, \qquad \varphi \in C_0^\infty(\mathbb{R}^d) \tag{2.1}$$

with $Q_n := [-1/2, 1/2]^n$.

It is well known [4] that the distribution M is an $L_\infty(\mathbb{R}^d)$ function and this function is a piecewise polynomial of total order $m := n-d+1$ (exact total degree $n-d$) with compact support:

$$\operatorname{supp} M = \left\{ \sum_{i=1}^n y_i t_i \;\middle|\; -1/2 \le y_i \le 1/2, \quad t_i \in X_n \right\}. \tag{2.2}$$

Moreover, this box spline M is in $W_\infty^{r-1}(\mathbb{R}^d) \cap C^{r-2}(\mathbb{R}^d)$ where

$$r := r(X_n) := \min\{\#Z \mid Z \subset X_n,\ X_n \setminus Z \text{ does not span } \mathbb{R}^d\}. \tag{2.3}$$

Notice that $r \le m$.

By taking $\varphi(x) = e^{-\mathrm{i}x\cdot\xi}$ in (2.1), the Fourier transform of M is obtained as

$$\widehat{M}(\xi) = \prod_{i=1}^n \frac{\sin(t_i \cdot \xi/2)}{t_i \cdot \xi/2}. \tag{2.4}$$

Then, the Strang-Fix condition [16] follows immediately from (2.4) with $r = r(X_n)$; that is,

$$\begin{aligned} &\text{(i)} \quad \widehat{M}(0) = 1, \quad \widehat{M}(2\pi j) = 0, \quad j \in \mathbb{Z}^d \setminus \{0\} \\ &\text{(ii)} \quad D^\nu \widehat{M}(2\pi j) = 0, \quad j \in \mathbb{Z}^d \setminus \{0\}, \quad |\nu| < r, \end{aligned} \tag{2.5}$$

where D^ν is the differential operator of order ν.

One of the most important properties of the box spline is that the M satisfies the refinement equation:

$$M(x) = \sum_{j \in \Gamma_M} a_j M(2x - j), \tag{2.6}$$

where a_j are certain constants with finite support Γ_M depending only on the support of M. Some computational and theoretical consequences of this refinement equation have been developed in [10]. The equation (2.6) plays a crucial part in the study of subdivision algorithms of computer-aided design (see [7], for example).

There are some basic properties of the box spline M concerned in integer translates M_I. One is that the integer translates M_I, $I \in \mathcal{D}_0$ form a partition of unity (see [4]),

$$\sum_{I\in\mathcal{D}_0} M_I(x) = 1, \qquad x \in \mathbb{R}^d. \tag{2.7}$$

Another is that M_I are (algebraically) linearly independent if and only if (cf. [9], [18])

$$|\det(Y_d)| = 1 \tag{2.8}$$

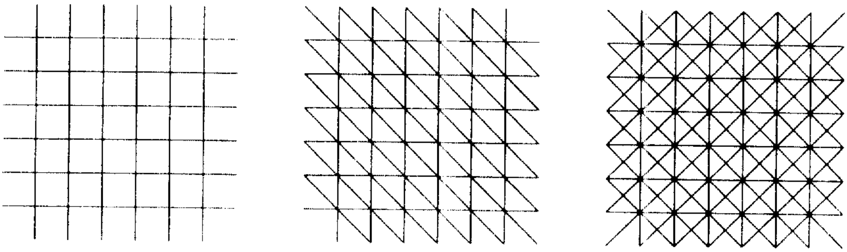
for each $d \times d$ matrix Y_d whose column vectors are from X_n and span $\mathbb{R}^d$. We are interested in only box splines M satisfying (2.8). From this assumption, for each $Q \in \mathcal{D}$, the functions $M(\cdot - j)$, $j \in \Lambda_Q$, are linearly independent over Q, where $\Lambda_Q = \{ j \in \mathbb{Z}^d \mid M(\cdot - j)$ does not vanish identically on $Q\}$. Further, the functions M_I, $I \in \mathcal{D}_0$, are (globally) linearly independent (see [4]).

We next give some examples of box splines on $\mathbb{R}^2$ for later reference. Let $\mathbf{e}_i := (\delta_{i,j})_{j=1}^2$ denote the unit coordinate vectors in $\mathbb{R}^2$ where $\delta_{i,j}$ is the Kronecker delta. Also, let us set $M_{suvw}(x) := M(x \mid X_{(s,u,v,w)})$ with

$$X_{(s,u,v,w)} = \{\mathbf{e}_1 : s,\ \mathbf{e}_2 : u,\ \mathbf{e}_2 - \mathbf{e}_1 : v,\ \mathbf{e}_2 + \mathbf{e}_1 : w\}. \tag{2.9}$$

Here, (2.9) means that the four directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_2 - \mathbf{e}_1$, $\mathbf{e}_2 + \mathbf{e}_1$ characterize the direction set $X_{(s,u,v,w)}$ with multiplicities s, u, v, w, respectively.

The first type of box spline is $M_{su} := M_{su00}$. It is clear that M_{su} is a tensor product of B-splines whose polynomial pieces are separated by a rectangular partition (see Figure 1(a), for example). The second type of box spline is $M_{suv} := M_{suv0}$. This box spline is a piecewise polynomial separated by a three-directional mesh (or type-1 triangulation, which is obtained by drawing all diagonals with negative slop in each rectangle) (see Figure 1(b), for example). Finally, M_{suvw} is also a piecewise polynomial separated by a four-directional mesh (or type-2 triangulation which is obtained by drawing both diagonal of each rectangle) (see Figure 1(c), for example).



(a) (b) (c)

FIGURE 1. The various meshes developed by box splines: (a) a rectangular partition(two-direction mesh); (b) a type-1 triangulation(three-direction mesh) (c) a type-2 triangulation (four-direction mesh)

To simplify our study, we are interested in the second type of box splines on $\mathbb{R}^2$. In particular, we focus on the type M_{sss}, $s \in \mathbb{N}$. Our analysis can be adapted to any types of box splines satisfying (2.8). It is clear that the first and second type of box splines always satisfy (2.8) (which is not always satisfied by box splines on type-2 triangulation). The box splines on type-1 triangulation have been extensively studied in [5] (see also [8]).

In the rest of this section, we consider the space of all spans of integer translates of the box spline $M := M_{sss}$, $n = s + s + s$ and describe its dilations. We shall employ the dilations of the space for approximation spaces in the next section. We recall that for $I \in \mathcal{D}_k$, $M_I(x) := M_{k,j}(x) := M(2^k(x - x_I))$, where $x_I := x_{k,j} = 2^{-k}j$. Then, M_I is supported on the set that is the dyadic dilate (by 2^k) and translate (by x_I) of the support of M (see Figure 2(a), for example). We associate both x_I and I with M_I.

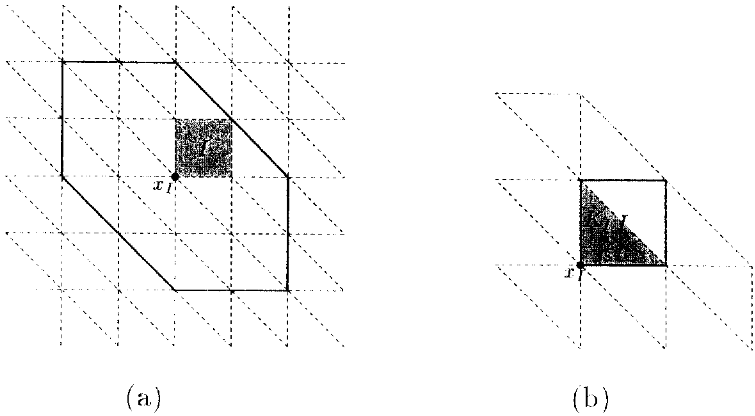


FIGURE 2 The support of the box spline, M_I: (a) the support and the center $x_I := 2^{-k}j$ of M_I; (b) the triangle K_I corresponding to x_I and the center ξ_I of K_I in the support of $M_I, I \in \mathcal{D}_k$, for $M = M_{222}$, $X_{(2,2,2)} = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$.

For given $k \in \mathbb{Z}$, let $\mathcal{S}^k = \text{span}\{M_I \mid I \in \mathcal{D}_k\}$. Our first observation about $\mathcal{S}^k$ is that any sum $\sum_{I \in \mathcal{D}_k} d_I M_I$ converges uniformly on compact sets since each M_I is compactly supported. Also, from the basic properties of box splines, for any $S \in \mathcal{S}^k$, S is in $W_\infty^{r-1}(\mathbb{R}^2) \cap C^{r-2}(\mathbb{R}^2)$ with $r = r(X_n)$. In addition, the M_I, $I \in \mathcal{D}_k$, form a partition of unity, that is, $\sum_{I \in \mathcal{D}_k} M_I \equiv 1$, $k \in \mathbb{Z}$. Further, due to the Strang-Fix condition (2.5), the spaces $\mathcal{S}^k$ locally contain the space $\mathcal{P}_r$ of polynomials of total degree less then $r = r(X_n)$ (see [16]). Our last observation about $\mathcal{S}^k$ is that M_I, $I \in \mathcal{D}_k$ are globally linearly independent for fixed k. This follows from (2.8) and dilations.

3. Wavelet Decompositions via Dyadic Spline Approximations

In this section, we develop a wavelet decomposition of $L_p(\mathbb{R}^2)$, $1 \le p \le \infty$ through the box splines. For this, we focus on the type-1 triangulation with side length 2^{-k} and let $\mathcal{T}_k$ denote the collection of all triangles K in the triangulation (see Figure 1(b)). In order to develop a constructive approximation method calculating the wavelet coefficients, we introduce two spline spaces on $\mathcal{T}_k$, and then combine the spline spaces.

The first spline space on $\mathcal{T}_k$ is the space $\Pi_k^r := \Pi_k^r(\mathcal{T}_k)$ that consists of all piecewise polynomials of total degree less than r on $\mathcal{T}_k$. To obtain an approximation from this space, we shall employ local near-best approximations on each $K \in \mathcal{T}_k$(cf. [15]).

DEFINITION 3.1. For given $K \in \mathcal{T}_k$, a polynomial P_K is a *near-best $L_p(K)$-approximation* to f in $L_p(K)$ from polynomials in $\mathcal{P}_r$ with constant A if

$$\|f - P_K\|_{L_p(K)} \le AE_r(f, K)_p \tag{3.1}$$

where $E_r(f, K)_p := \inf_{P\in\mathcal{P}_r} \|f - P\|_{L_p(K)}$ is the local error of approximation of $f \in L_p(K)$ by the elements from the $\mathcal{P}_r$.

A near-best approximation $R_k(f)$ of $f \in L_p(\mathbb{R}^2)$ from Π_k^r is now defined by simply taking $R_k(f)|_K = P_K$, $K \in \mathcal{T}_k$. Then, it follows from (3.1) that

$$\begin{aligned} \|P_K\|_{L_p(K)} &\le AE_r(f, K)_p + \|f\|_{L_p(K)} \\ &\le C\|f\|_{L_p(K)}, \qquad K \in \mathcal{T}_k \end{aligned} \tag{3.2}$$

where C depends only on r and A. Hence, $\|R_k(f)\|_p \le C\|f\|_p$.

REMARK. There are many ways to construct a near-best approximation R_k of $f \in L_p(\mathbb{R}^2)$ from Π_k^r. One way is due to DeVore and Popov [15]: for each $K \in \mathcal{T}_k$, R_k is chosen to be a best $L_\gamma(K)$-approximation to $f \in L_p(\mathbb{R}^2)$ for some fixed $0 < \gamma \le p$ (see [15, Lemma 3.2]). Here, we can choose R_k to be a best $L_1(K)$-approximation to $f \in L_p(\mathbb{R}^2)$ by virtue of [6]. However, one should note that the best $L_2(K)$ projection operator onto $\mathcal{P}_r$ is bounded on $L_p(K)$, for $1 \le p \le \infty$ and any $r > 0$. In §5, we shall employ the best $L_2(K)$-approximation from $\mathcal{P}_r$ for the R_k to obtain a certain constructive approximation.

Notice that an element in Π_k^r may be discontinuous along the mesh lines in $\mathcal{T}_k$ (see Figure 1(b)). We shall also need to construct good approximations (which have smoothness) from Π_k^r. For this, we use the second type of box spline $M = M_{sss}$ corresponding to $X_{(s,s,s)}$ with $n = s + s + s$ (see (2.9)). All arguments of this section apply equally well to other types of box splines satisfying the condition (2.8). In what follows, we set $r = r(X_{(s,s,s)})$ and $m = r - 2$.

The second spline space on $\mathcal{T}_k$ is the space $\mathcal{S}^k$, $k \in \mathbb{Z}$, consisting of the span of the dyadic dilations and translations of the box spline M. By the observations for $\mathcal{S}^k$ in the previous section, $\mathcal{S}^k$ is a smooth spline space in Π_k^m locally containing $\mathcal{P}_r$ (notice that $r \leq m$). Also, since the box spline M satisfies the condition (2.8), the functions $M(\cdot - j)$, $j \in \mathbb{Z}^2$ form a basis for $\mathcal{S}^0$. Further, by dilations, M_I, $I \in \mathcal{D}_k$ form a basis for $\mathcal{S}^k$.

To provide an explicit formula calculating spline approximations from the space $\mathcal{S}^k$, we shall construct for given $k \in \mathbb{Z}$ a set of dual functionals $d_{k,j}$ to the basis $M_{k,j}$ in $\mathcal{S}^k$; that is, for fixed $k \in \mathbb{Z}$, $d_{k,j}(M_{k,i}) = \delta_{i,j}$. Then, each spline S in $\mathcal{S}^k$ can be written as

$$S = \sum_{j \in \mathbb{Z}^2} d_{k,j}(S) M_{k,j}. \tag{3.3}$$

To simplify matters, by dilations and translations, we can characterize $d_{k,j}$ by $d_{0,0}$ (cf. [13]) as

$$d_{k,j}(S) = d_{0,0}(S(2^{-k}(\cdot + j))). \tag{3.4}$$

Notice that there are many representations for the functionals $d_{k,j}$ (see [1] and [8], for example). In order to find a constructive formula for $d_{k,j}$, we utilize the de Boor-Fix formula [3] which was originally given for the series of univariate B-splines $\mathcal{N}_r$ of order r (and for tensor products of $\mathcal{N}_r$ in the multivariate case). The de Boor-Fix formula says that for any point ξ_j in the support of $\mathcal{N}_r(\cdot - j)$, we can write

$$d_{0,j}(S) = \sum_{0 \leq \nu \leq r-1} \theta_\nu D^\nu(S)(\xi_j), \qquad j \in \mathbb{Z}, \quad S \in \mathcal{S}^0 \tag{3.5}$$

with certain coefficients θ_ν depending on ξ_j and r. We have not seen this formula for box splines. However, we only need an analog of (3.5) for $P \in \mathcal{P}_r$.

The analogous formula is given by de Boor and DeVore [2]: for any point $\xi_{0,j}$ in the support of $M_{0,j}$,

$$(3.6) \qquad d_{0,j}(P) = \sum_{|\nu|\le r-1} \Theta_\nu(j - \xi_{0,j}) D^\nu(P)(\xi_{0,j})$$

where Θ_ν is a polynomial of degree ν such that

$$(3.7) \qquad \sum_{j\in\mathbb{Z}^2} \Theta_\nu(j) M(x-j) = x^\nu/\nu!, \qquad |\nu| \le r-1.$$

Here, the existence of such polynomials Θ_ν is guaranteed by the Strang-Fix condition (2.5). Further, by dilations we obtain

$$(3.8) \qquad d_{k,j}(P) := \sum_{|\nu|\le r-1} \Theta_\nu(x_{k,j} - \xi_{k,j}) D^\nu(P)(\xi_{k,j})$$

where $\xi_{k,j}$ is any point in the support of $M_{k,j}$ and $x_{k,j} = 2^{-k}j$. Therefore, each $P \in \mathcal{P}_r$ has the representation

$$(3.9) \qquad P = \sum_{j\in\mathbb{Z}^2} d_{k,j}(P) M_{k,j}$$

with the expression (3.8) for $d_{k,j}$.

In order to combine two spline spaces Π_k^r, $\mathcal{S}^k$, we next introduce quasi-interpolant operators using the representation (3.9). For this, we choose the point $\xi_{k,j}$ as the center of the lower triangle K_I corresponding to the center x_I of M_I, $I \in \mathcal{D}_k$ (see Figure 2(b)). With this choice, the functionals $d_{k,j}$ are well defined for any function f that is $r-1$ times continuously differentiable at each of the points $\xi_{k,j}$. Then, for fixed $k \in \mathbb{Z}$,

$$(3.10) \qquad \mathbf{Q}_k f := \sum_{j\in\mathbb{Z}^2} d_{k,j}(f) M_{k,j}$$

is well defined for such f. This $\mathbf{Q}_k$ is called a *quasi-interpolant* operator [3] . Moreover, $\mathbf{Q}_k$ is a bounded operator from $\Pi_k^r \cap L_p(\mathbb{R}^2)$ to $\mathcal{S}^k \cap L_p(\mathbb{R}^2)$ by the lemma below.

LEMMA 3.2. *If* $1 \le p \le \infty$, *then* $\|\mathbf{Q}_k S\|_p \le C\|S\|_p$ *for all* $S \in \Pi_k^r$ *where* C *depends only on* r.

Proof. For a proof see [15 Corollary 4.4]. □

We now provide a constructive approximation to $f \in L_p(\mathbb{R}^2)$ from $\mathcal{S}^k$. For $f \in L_p(\mathbb{R}^2)$, we define

$$\begin{aligned}\mathbf{P}_k f &:= \mathbf{Q}_k R_k(f), \qquad k \in \mathbb{Z}\\ &= \sum_{j\in\mathbb{Z}^2} d_{k,j}(R_k(f))M_{k,j}\end{aligned} \tag{3.11}$$

where $R_k(f) := R_k$ is a near-best approximation to f from Π_k^r with a constant A. Then, it follows from Lemma 3.2 and (3.2) that

$$\|\mathbf{P}_k f\|_p \le C\|f\|_p \tag{3.12}$$

with a constant C depending only on r and A. By taking R_k to be the best $L_2(K)$-approximation from $\mathcal{P}_r$ (see the remark above), $\mathbf{P}_k$ is a well-defined and bounded operator from $L_p(\mathbb{R}^2)$ onto $\mathcal{S}^k \cap L_p(\mathbb{R}^2)$. So, the $\mathbf{P}_k$ are constructive linear projectors. Further, since the constant C in (3.12) is independent of k, the $\mathbf{P}_k$ are uniformly bounded operators.

Using a relation between $E_r(f,K)_p$ and the averaged modulus of smoothness (cf. [15]), one can prove for each $f \in L_p(\mathbb{R}^2)$, $1 \le p < \infty$,

$$\|f - \mathbf{P}_k f\|_p \le C\omega_r(f, 2^{-k})_p, \qquad k \in \mathbb{Z} \tag{3.13}$$

with a constant C depending only on r, p and A (see [15, §4], for example). Here, the *modulus of smoothness* of order r in $L_p(\Omega)$ is defined by

$$\omega_r(f,t)_p := \omega_r(f,t,\Omega)_p := \sup_{|h|<t} \|\Delta_h^r(f,\cdot)\|_{L_p(\Omega(rh))} \qquad t > 0,$$

on the set $\Omega(rh) := \{x \,|\, x, \cdots, x + rh \in \Omega\}$. This implies that for $f \in L_p(\mathbb{R}^2)$, $1 \le p < \infty$,

$$\|f - \mathbf{P}_k f\|_p \to 0 \quad \text{as} \quad k \to \infty. \tag{3.14}$$

Also, using (3.12) and certain characterizations of the functional $d_I := d_{k,j}$ (see [15] or [13]), one can prove that

$$\|\mathbf{P}_k f\|_p \to 0 \qquad \text{as} \quad k \to -\infty. \tag{3.15}$$

Thus, each $f \in L_p(\mathbb{R}^2)$, $1 \le p < \infty$, can be represented by the series

$$f = \sum_{k \in \mathbb{Z}} (\mathbf{P}_k f - \mathbf{P}_{k-1} f) = \sum_{k \in \mathbb{Z}} \sum_{I \in \mathcal{D}_k} c_I M_I \tag{3.16}$$

with convergence in the L_p norm. Here, the last equality uses the refinement equation (2.6) to rewrite $\mathbf{P}_k f - \mathbf{P}_{k-1} f$ in terms of M_I, $I \in \mathcal{D}_k$. Thus, we obtain a constructive wavelet decomposition (1.5) for $L_p(\mathbb{R}^2)$, $1 \le p < \infty$. When $p = \infty$, the wavelet decomposition (1.5) is also valid for $\mathbf{C}_0$ and all functions in $L_\infty(\mathbb{R}^d)$ with compact support. In §5, we shall provide an explicit formula for the coefficients c_I.

4. Besov Spaces

In this section, we introduce Besov spaces defined by the modulus of smoothness. The Besov spaces $B_q^\alpha(L_p(\Omega))$, $0 < \alpha < \infty$, $0 < p, q \le \infty$ are smoothness subspaces of $L_p(\Omega)$. The parameter α gives the smoothness order in $L_p(\Omega)$, much like the order of differentiation, while the parameter q gives a finer scaling that measures subtle gradations in smoothness of fixed order α.

DEFINITION 4.1. Let $0 < \alpha < \infty$, $0 < p, q \le \infty$ and r be a positive integer with $\alpha < r$. The Besov space $B_q^\alpha(L_p(\Omega))$ is defined as the set of all functions $f \in L_p(\Omega)$ for which

(4.1)

$$|f|_{B_q^\alpha(L_p(\Omega))} := \begin{cases} \left(\displaystyle\int_0^\infty \left[t^{-\alpha} \omega_r(f,t)_p \right]^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{t>0} t^{-\alpha} \omega_r(f,t)_p, & q = \infty, \end{cases}$$

is finite. A (quasi-)norm for $B_q^\alpha(L_p(\Omega))$ is defined by

$$\|f\|_{B_q^\alpha(L_p(\Omega))} := \|f\|_{L_p(\Omega)} + |f|_{B_q^\alpha(L_p(\Omega))}. \tag{4.2}$$

REMARK. Because we allow values of p and q less than one, this norm does not always satisfy the triangle inequality, but it is always a quasi-norm; that is, there exists a constant C such that for all f and g in $B_q^\alpha(L_p(\Omega))$,

$$\|f+g\|_{B_q^\alpha(L_p(\Omega))} \le C\big(\|f\|_{B_q^\alpha(L_p(\Omega))} + \|g\|_{B_q^\alpha(L_p(\Omega))}\big).$$

In addition, it is well known that different values of $r > \alpha$ give equivalent norms (see [15], for example).

Notice that we could replace the integral and the supremum in (4.1) by the integral and supremum over $(0,1)$ respectively, because $\omega_r(f,t)_p \le C\|f\|_p$. Further, by dividing $(0,1)$ into $[2^{-\nu-1}, 2^{-\nu})$ for $k \in \mathbb{N}$, it follows that

$$\int_0^1 \big[t^{-\alpha}\omega_r(f,t)_p\big]^q \frac{dt}{t} \approx \sum_{\nu\ge 1} \big[2^{\nu\alpha}\omega_r(f,\frac{1}{2^\nu})_p\big]^q.$$

We shall sometimes use the following discretized version of the (quasi-)norm for $B_q^\alpha(L_p(\Omega))$:
(4.3)

$$\|f\|_{B_q^\alpha(L_p(\Omega))} \approx \|f\|_{L_p(\Omega)} + \begin{cases} \left(\sum_{\nu\ge 1}\big[2^{\nu\alpha}\omega_r(f,\frac{1}{2^\nu})_p\big]^q\right)^{1/q}, & 0<q<\infty, \\ \sup_{\nu\ge 1} 2^{\nu\alpha}\omega_r(f,\frac{1}{2^\nu})_p, & q=\infty, \end{cases}$$

because it is easier to compute upper and lower estimates for the discretized sum than the integral.

The Besov space, $B_q^\alpha(L_p)$, includes several classical function spaces. For instance, we have the followings (cf. [20]): for $\alpha \ge 0$,

$$B_2^\alpha(L_2) \approx L_2^\alpha, \qquad B_\infty^\alpha(L_\infty) \approx \mathcal{C}^\alpha,$$

where L_p^α is the fractional order Sobolev space defined by the Bessel potential and $\mathcal{C}^\alpha$ is the Zygmund spaces.

Especially, the special space $B^\alpha = B_q^\alpha(L_q(\Omega))$ $(q = (\alpha/d + 1/p)^{-1})$ is closely related to nonlinear approximation theory. For instance, f

is in the spaces B^α if and only if nonlinear approximations by linear combinations with N terms in the decomposition (1.1) for a function f have order $\mathcal{O}(N^{-\alpha/d})$ in L_p approximately (cf. [13]). In this paper, we shall concentrate on this space.

We end this section by stating an important embedding property of Besov spaces. Let $0 < \alpha < \infty$, $0 < p \leq \infty$. Then, the scale of Besov spaces, $B^\alpha := B_q^\alpha(L_q(\mathbb{R}^d))$, $q = (\alpha/d + 1/p)^{-1}$, is continuously embedded into $L_p(\mathbb{R}^d)$; that is, if $f \in B^\alpha$, then

$$\|f\|_p \leq C\|f\|_{B^\alpha} \tag{4.4}$$

with a constant C independent of f (see [15], for a proof).

5. Constructive Wavelet Coefficients Measuring Smoothness of L_p

In this section, we shall establish the constructive decomposition (3.16) for the space $B^\alpha = B_q^\alpha(L_q(\mathbb{R}^d))$ $(q = (\alpha/d+1/p)^{-1}, 1 < p \leq \infty)$ and provide its Littlewood-Paley type characterization (1.2):

$$\|f\|_{B^\alpha} \approx \left(\sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_k} |c_I|^q \|M_I\|_p^q\right)^{1/q}$$

with the coefficients c_I of (3.16). Notice that when $0 < q < 1$, we use for R_k the L_2-local polynomial approximations of B^α; this allows us to use a computationally effective linear projector onto $\mathcal{S}^k$. On the other hand, DeVore *et al.* [13], [14] used nonlinear operators bounded on L_q to obtain the characterization (1.2) of the space B^α. Nonlinear operators bounded on L_q are more difficult to implement than the simple local L_2 projector that we use.

Throughout this section, let us employ the second type of box splines $M = M_{sss}$, $s \in \mathbb{N}$ (see §2). Also, let us fix $r = r(X_{sss})$ and $d = 2$. The argument of this section can apply equally to other types of box splines satisfying (2.8). To reach our goal, we need a couple of preliminary lemmas. First of these is a theorem of Frazier and Jawerth for the homogeneous Besov spaces, $\dot{B}_q^{\alpha,q}$ (see [17, Theorem 3.7]).

LEMMA 5.1. *Let $\alpha \in \mathbb{R}$ and $0 < q < \infty$. Also, let $N = \max(\lfloor J - 2 - \alpha \rfloor, -1)$ and $J^* > J$ where $J = 2/\min(1, q)$. Assume that $\{b_I\}_{I \in \mathcal{D}}$ is a family of functions satisfying the followings: for some δ with $J - \alpha - \lfloor J - \alpha \rfloor < \delta \leq 1$,*

(5.1)

$$\text{(i)} \quad \int_{\mathbb{R}^d} x^\gamma b_I(x)dx = 0 \qquad \text{if } |\gamma| \leq \lfloor \alpha \rfloor,$$

$$\text{(ii)} \quad |b_I(x)| \leq 2^k(1 + 2^k|x - x_I|)^{-\max(J^*, J^*+2+\alpha-J)},$$

$$\text{(iii)} \quad |D^\gamma b_I(x)| \leq 2^{k(1+|\gamma|)}|x - x_I|)^{-J^*} \qquad \text{if } |\gamma| \leq N,$$

$$\text{(iv)} \quad |D^\gamma b_I(x) - D^\gamma b_I(y)| \leq 2^{k(1+|\gamma|+\delta)}|x - y|^\delta \sup_{|z| \leq |x-y|} (1 + 2^k|x - z - x_I|)^{-J^*} \qquad \text{if } |\gamma| = N.$$

Then, if $f \in \dot{B}_q^{\alpha,q}(\mathbb{R}^2)$ we have

$$\left(\sum_{\nu \in \mathbb{Z}} \left\| \sum_{I \in \mathcal{D}_\nu} |I|^{-\alpha/d} |\langle f, b_I \rangle| \tilde{\chi}_I \right\|_p^q \right)^{1/q} \leq C \|f\|_{\dot{B}_q^{\alpha,q}(\mathbb{R}^2)} \tag{5.2}$$

with a constant C independent of f, where $\tilde{\chi}_I(\cdot) := |Q|^{-1/2}\chi_I(\cdot)$ is the L^2-normalized characteristic function of I.

Proof. For a proof, we refer to [17, Theorem 3.7]. □

REMARK. In our case, $d = 2$, $\alpha > 0$, $1 < p \leq \infty$, and $q := (\alpha/d + 1/p)^{-1}$. Then $J = 2/\min(1, q)$ is either 2 or $2/q$. If $J = 2$, then $N = \max(\lfloor -\alpha \rfloor, -1) = -1$; otherwise, $N = \max(\lfloor 2/q - 2 - \alpha \rfloor, -1) = \max(\lfloor 2(1/p - 1) \rfloor, -1) = -1$. Thus, either way, the conditions (5.1)(iii) and (iv) are vacuous.

The lemma above will prove one direction ($\geq$) of (1.2). The next lemma is an analog of a corollary of DeVore and Popov (cf. [15, Corollary 5.2]), for the box spline M. This will prove the other direction ($\leq$) of (1.2).

LEMMA 5.2. *Let r be the integer associated with the box spline M. Assume that $1 < p \leq \infty$, $0 < \alpha < \infty$, and $q = (\alpha/2 + 1/p)^{-1}$. If $0 < \alpha < \lambda$ with $\lambda = r - 1 + 1/p$, then for each $S \in \mathcal{S}^k$, $k \in \mathbb{Z}$, we have*

$$|S|_{B^\alpha} \leq C 2^{\alpha k} \|S\|_q \tag{5.3}$$

with a constant C independent of S and k.

Proof. We may assume that $S \in L_q(\mathbb{R}^2)$. Let $e_k(f)_p$ be the error of approximation of f by the elements of $\mathcal{S}^k$; that is,

$$e_k(f)_p := \inf_{S \in \mathcal{S}^k} \|f - S\|_p \qquad k \in \mathbb{Z}. \tag{5.4}$$

Since $S \in \mathcal{S}^k$, we have $e_\nu(S)_q = 0$ for $\nu \geq k$ and $e_\nu(S)_q \leq \|S\|_q$ for $\nu < k$. Then, we obtain

$$\begin{aligned}\left(\sum_{\nu \in \mathbb{Z}} [2^{\nu\alpha} e_\nu(S)_q]^q\right)^{1/q} &\leq \left(\sum_{\nu=-\infty}^{k} [2^{\nu\alpha} e_\nu(S)_q]^q\right)^{1/q} \\ &\leq C 2^{\alpha k} \|S\|_q.\end{aligned} \tag{5.5}$$

On the other hand, by the lemma below and the discrete Hardy inequality (cf. [11]), we obtain for $0 < \alpha < \lambda$,

$$\left(\sum_{\nu \in \mathbb{Z}} [2^{\nu\alpha} \omega_r(S, 2^{-k})_q]^q\right)^{1/q} \leq C \left(\sum_{\nu \in \mathbb{Z}} [2^{\nu\alpha} e_\nu(S)_q]^q\right)^{1/q}. \tag{5.6}$$

Therefore, combining (5.5) and (5.6) completes the proof of the lemma. □

The lemma below is also an analog of a theorem of DeVore and Popov (cf. 15, Theorem 4.8]) for the box splines M. In the lemma below, we improve the range of α $(0 < \alpha < r - 2 + 1/p)$ reported in [13, §7] to $0 < \alpha < r - 1 + 1/p$ for the box spline M even though $M \in C^{r-2}(\mathbb{R}^2)$. Here, the proof of the lemma below is almost the same as that of Theorem 4.8 of [15].

LEMMA 5.3. *Let p, q, r and λ be as in Theorem 5.2. If $f \in L_q(\mathbb{R}^2)$, then*

$$\omega_r(f, 2^{-k})_q \leq C 2^{-k\lambda} \left(\sum_{\nu=-\infty}^{k} [2^{\nu\lambda} e_\nu(f)_q]^\mu\right)^{1/\mu} \tag{5.7}$$

where $\mu \leq \min(1, q)$.

Proof. For a proof see §A.1 of Appendix. □

The following theorem is the main result of this section.

THEOREM 5.4. *Let r be the integer associated with M. Assume that $1 < p \le \infty$ and $0 < \alpha < \infty$. If $f \in B^\alpha$, $q = (\alpha/2 + 1/p)^{-1}$ and $\alpha < \lambda$ with $\lambda = r - 1 + 1/p$, then f admits the constructive decomposition* (3.16):

$$f = \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_k} c_I M_I.$$

Furthermore, the decomposition above provides the following characterization:

$$|f|_{B^\alpha} \approx \left(\sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_k} |c_I|^q \|M_I\|_p^q\right)^{1/q}. \tag{5.8}$$

Proof. To begin with, we recall that the space B^α is continuously embedded into $L_p(\mathbb{R}^d)$; that is

$$\|f\|_p \le C\|f\|_{B^\alpha} \tag{5.9}$$

with a constant C independent of f (see §4). Then, every $f \in B^\alpha$ admits the decomposition (3.16) in $L_p(\mathbb{R}^2)$, $1 < p \le \infty$. So, it remains to show the characterization (5.8) with the coefficients c_I of (3.16). Let us first establish the direction

$$\left(\sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_k} |c_I|^q \|M_I\|_p^q\right)^{1/q} \le C|f|_{B^\alpha} \tag{5.10}$$

with C independent of f. For this, in §A.2 of Appendix, we shall show that there exists a family $\{b_I\}_{I\in\mathcal{D}}$ satisfying the conditions (5.1)(i) and (ii) of Lemma 5.1 such that for $I \in \mathcal{D}_k$,

$$c_I = 2^k \langle f, b_I\rangle. \tag{5.11}$$

Once such a family $\{b_I\}_{I\in\mathcal{D}}$ is found, we obtain by Lemma 5.1 (see the remark to this lemma)

$$\left(\sum_{k\in\mathbb{Z}} \left\| \sum_{I\in\mathcal{D}_k} 2^{k\alpha} |\langle f, b_I\rangle| \tilde{\chi}_I \right\|_q^q\right)^{1/q} \le C\|f\|_{\dot{B}_q^{\alpha,q}(\mathbb{R}^2)}. \tag{5.12}$$

On the other hand, we have
(5.13)

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\Big\|\sum_{I\in\mathcal{D}_k} 2^{k\alpha}|\langle f,b_I\rangle|\tilde{\chi}_I\Big\|_q^q &= \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k} 2^{k\alpha q}|\langle f,b_I\rangle|^q 2^{kq}\int_{\mathbb{R}^2}\chi_I(x)dx\\ &= \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k} 2^{kq(2/q-2/p)}2^{kq}2^{-2k}|\langle f,b_I\rangle|^q\\ &= \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k} 2^{kq(1-2/p)}|\langle f,b_I\rangle|^q,\end{aligned}$$

where the second equality has used the fact that $1/q = \alpha/2 + 1/p$. In addition, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k}|c_I|^q\|M_I\|_p^q &= \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k}|2^k\langle f,b_I\rangle|^q\|M(2^k\cdot -j)\|_p^q\\ &= \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k}2^{kq(1-2/p)}|\langle f,b_I\rangle|^q\|M\|_p^q \qquad (5.14)\\ &\le C\sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k}2^{kq(1-2/p)}|\langle f,b_I\rangle|^q.\end{aligned}$$

Then, substituting (5.13) and (5.14) into (5.12) leads to (5.10). Here, we have used the fact that $\|f\|_{\dot{B}_q^{\alpha,q}(\mathbb{R}^d)} \le C|f|_{B_q^\alpha(L_q(\mathbb{R}^d))}$ for every $\alpha > d(\frac{1}{\min(q,1)} - 1)$ and $0 < p \le \infty$ (cf. [19]). In our case, $d = 2$, $\alpha > 0$, $1 < p \le \infty$, and $1/q = \alpha/d + 1/p$. Then, we have $\alpha > 2(\frac{1}{\min(q,1)} - 1)$ because $1/q < \alpha/2 + 1$. Let us next establish the other direction

$$|f|_{B^\alpha} \le C\Big(\sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k}|c_I|^q\|M_I\|_p^q\Big)^{1/q} \qquad (5.15)$$

with a constant C independent of f. Here, we shall use Lemma 5.2. So, we start by rewriting (3.16) as

$$f = \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k} c_I M_I = \sum_{k\in\mathbb{Z}} S_k, \qquad (5.16)$$

where $S_k = \sum_{I\in\mathcal{D}_k} c_I M_I \in \mathcal{S}^k$. When $1 < q < \infty$, we just follow Lemma 4.2 of [13] with the discrete Hardy inequality and Lemma 5.3

to establish (5.15). To establish (5.15) for the case $0 < q \le 1$, it is enough to show that

$$\sum_{k\in\mathbb{Z}} |S_k|_{B^\alpha}^q \le C\Big(\sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k} |c_I|^q \|M_I\|_p^q\Big). \tag{5.17}$$

Once we show the estimate (5.17), the sum $\sum_{k\in\mathbb{Z}} S_k$ converges in the $\|\cdot\|_{B^\alpha}^q$-norm because of (5.10). Therefore, since B^α is a complete quasi-normed space, the decomposition (5.16) is valid in B^α. Hence, the subadditivity of $|\cdot|_{B^\alpha}^q$ implies clearly the estimate (5.15).

Now, it remains to show (5.17). Using the subadditivity of $\|\cdot\|_q^q$ and Lemma 5.2, we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |S_k|_{B^\alpha}^q &\le C\sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k} 2^{\alpha kq}|c_I|^q\|M_I\|_q^q \\
&\le C\sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k} 2^{\alpha kq}|c_I|^q\left(\int_{\mathbb{R}^2}|M_I|^p\right)^{q/p}\left(\int_{\mathbb{R}^2}\chi_{\operatorname{supp} M_I}\right)^{\alpha q/2} \\
&\le C\sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k} 2^{\alpha kq}|c_I|^q\|M_I\|_p^q\, 2^{-\alpha kq} \\
&= C\sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{D}_k} |c_I|^q\|M_I\|_p^q
\end{aligned}$$

with a constant C independent of f. Here, the second inequality uses Hölder's inequality with $\frac{1}{p/q}+\frac{1}{p/(p-q)} = 1$ and $1/q = \alpha/2 + 1/p$. Hence, the proof of the theorem has been completed. □

Appendix A.1: Proofs of Lemma 5.3

Since $f \in L_q(\mathbb{R}^2)$ can be rewritten as $f = f - S_k + \sum_{\nu=-\infty}^{k}(S_\nu - S_{\nu-1})$ with $S_\nu \in \mathcal{S}^\nu$ satisfying $\|f - S_\nu\|_q = e_\nu(f)_q$, $\nu \in \mathbb{Z}$, it follows from (1.3) that

$$\omega_r(f,2^{-k})_q \le C\Big(\|f - S_k\|_q + \sum_{\nu=-\infty}^{k} \omega_r(S_\nu - S_{\nu-1}, 2^{-k})_q^\mu\Big)^{1/\mu}$$

where $\mu = \min(1, q)$, and C is independent of f and k.

So, once we show that for each $S \in \mathcal{S}^\nu$, $\nu \in \mathbb{Z}$, and $0 < q < \infty$,

$$\omega_r(S, t)_q \le C \min(1, 2^{\nu\lambda} t^\lambda) \|S\|_q \tag{A.1.1}$$

with C depending only on the box spline M and q, and $\lambda = r - 1 + 1/p$, then (5.7) follows immediately from $\|f - S_\nu\|_q = e_\nu(f)_q$, $\nu \in \mathbb{Z}$ and $\|S_\nu - S_{\nu-1}\|_q \le C\big(\|f - S_\nu\|_q + \|f - S_{\nu-1}\|_q\big)$ with C depending only on q.

To show (A.1.1), we shall estimate $\Delta_h^r(M_{\nu,j}, \cdot)$, for given $\nu \in \mathbb{Z}$. Let Γ denote the set of all $x \in \mathbb{R}^2$ such that the segment x and $x + rh$ are in the same triangle $K \in \mathcal{T}_\nu$ and $M_{\nu,j}$ does not vanish identically on K. Also, let Γ' denote the set of all $x \in \mathbb{R}^2$ such that x and $x + rh$ are in different triangles from $K \in \mathcal{T}_\nu$ and $M_{\nu,j}$ does not vanish identically on both of those triangles.

First, note that since $M_{\nu,j}\big|_{K \in \mathcal{T}_\nu}$ is a polynomial of total order m (recall that $r \le m$), we have for any $x \in \Gamma$, $|D^\gamma M_{\nu,j}(x)| \le C 2^{\nu r}$, $|\gamma| = r$. Then, we obtain

$$|\Delta_h^r(M_{\nu,j}, x)| \le C(2^\nu |h|)^r, \qquad \text{for} \quad x \in \Gamma.$$

In addition, we have $|\Gamma| \le C 2^{-2\nu}$ since $|\operatorname{supp} M_{\nu,j}| \le C 2^{-2\nu}$. Here, the C's are all independent of ν and K. Next, note that $\|M_{\nu,j}\|_{W_\infty^{r-1}} \le C 2^{\nu(r-1)}$ since $M_{\nu,j} = M(2^\nu \cdot - j)$. Then, we obtain

$$|\Delta_h^r(M_{\nu,j}, x)| \le C(2^\nu |h|)^{r-1}, \qquad \text{for} \quad x \in \Gamma'.$$

In addition, we have $|\Gamma'| \le C|h| 2^{-\nu}$. To see this, notice that for each $x \in \Gamma'$, the distance between x and the boundary of K is less than $r|h|$, where K is the triangle containing x. Then, the measure of all such $x \in K$ is less than $C|h|2^{-\nu}$ with a constant C depending only on r. So, since $M_{\nu,j}$ vanishes on all but C triangles with C depending only on M, the claim for the measure of Γ' follows immediately.

By combining the results above, we obtain

(A.1.2)
$$\begin{aligned} \int |\Delta_h^r(M_{\nu,j}, x)|^q dx &\le C\big(2^{\nu r q}|h|^{rq} 2^{-2\nu} + 2^{\nu(r-1)q}|h|^{(r-1)q}|h| 2^{-\nu}\big) \\ &\le C \min(1, 2^{\nu\lambda q}|h|^{\lambda q}) 2^{-2\nu}, \end{aligned}$$

where for the last inequality, we used the fact that $|\Delta_h^r(M_{\nu,j},\cdot)| \le C\|M\|_\infty$ with C independent of ν, j and h, when $2^\nu|h| \ge 1$; otherwise, we used the fact that $\lambda = r - 1 + 1/p \le \min(r, r-1+1/q)$ because $1/p \le 1/q$ and $1 < p < \infty$.

To complete the proof, let us recall from [13] that for $S = \sum_{j\in\mathbb{Z}^d} d_{\nu,j}(S)M_{\nu,j} \in \mathcal{S}^\nu$,

$$\text{(A.1.3)} \qquad \|S\|_q \approx \begin{cases} \left(\sum_{j\in\mathbb{Z}^d} 2^{-\nu d}|d_{\nu,j}(S)|^q\right)^{1/q}, & 0 < q < \infty, \\ \sup_{j\in\mathbb{Z}^d} |d_{\nu,j}(S)|, & q = \infty. \end{cases}$$

Also, note that for fixed $x \in \mathbb{R}^d$,

$$\text{(A.1.4)} \qquad |\Delta_h^r(S,x)|^q \le C \sum_{j\in\mathbb{Z}^d} |d_{\nu,j}(S)|^q |\Delta_h^r(M_{\nu,j},x)|^q,$$

with C depending only on M, d, and q when q is small. Here, we have used the fact that for fixed $x \in \mathbb{R}^d$, the sum $S(x) = \sum_{j\in\mathbb{Z}^d} d_{\nu,j}(S)M_{\nu,j}(x)$ has at most C terms with C depending only on M. Hence, combining the (A.1.2), (A.1.3), and (A.1.4) yields the (A.1.1). $\square$

Appendix A.2: Completing the proof of Theorem 5.4

In this section, we shall show that there exists a family $\{b_I\}_{I\in\mathcal{D}}$ satisfying the conditions (5.1)(i) and (ii) of Lemma 5.1 such that for $I \in \mathcal{D}_k$,

$$\text{(A.2.1)} \qquad c_I = 2^k\langle f, b_I\rangle.$$

Further, we shall give an explicit formula for b_I.

To begin with, let us recall from §3 that every $f \in L_p(\mathbb{R}^2)$, $1 \le p < \infty$ admits the decomposition

$$\text{(A.2.2)} \qquad f = \sum_{k\in\mathbb{Z}} (\mathbf{P}_k f - \mathbf{P}_{k-1} f) = \sum_{k\in\mathbb{Z}} \sum_{I\in\mathcal{D}_k} c_I M_I$$

with the operators $\mathbf{P}_k f = \mathbf{Q}_k R_k(f) = \sum_{I\in\mathcal{D}_k} d_I(f) M_I$, where

$$(\text{A.2.3})\qquad d_I(f) := \sum_{|\nu|\le r-1} \Theta_\nu(x_I - \xi_I) D^\nu(\tilde{f}_{K_I})(\xi_I).$$

Here, we have chosen the lower triangle K_I and its center ξ_I corresponding to $I \in \mathcal{D}_k$ (see Figure 2 (b)) to provide the quasi-interpolation operator $\mathbf{Q}_k$. Also, we have set up $\tilde{f}_K := R_k(f)\big|_K$ to be the local best $L_2(K)$-approximation to f from $\mathcal{P}_r$ on each $K \in \mathcal{T}_k$.

Let us first provide an explicit form for the $d_I(f)$. For this, we may start by considering the functional $d_{0,0}$ (cf. (3.4)). Let us fix the triangle $K_O := K_{0,0}$ as a reference triangle (cf. Figure 2)(b)). For a local basis for the space $\mathcal{P}_r\big|_{K_O}$, we employ the monomials on K_O, $\mathbf{m}^\gamma(x) := x^\gamma\big|_{K_O}$, $|\gamma| < r$, $\gamma \in \mathbb{Z}^2$. Then, we can derive a local dual basis $\eta^\gamma(x)$ associated with $\mathbf{m}^\gamma$ on K_O with the conditions

$$(\text{A.2.4})\qquad \langle \mathbf{m}^\beta, \eta^\gamma\rangle = \delta_{\beta,\gamma}, \qquad |\beta|,\ |\gamma| < r,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the $L_2(K_O)$ inner product. So, each η^γ is a linear combination of $\mathbf{m}^\beta$, $|\beta| < r$ with $\operatorname{supp}\mathbf{m}^\gamma \subseteq K_O$. By dilations and translations, we obtain a local basis $\{\mathbf{m}_I^\gamma\}$ and its local dual basis $\{\eta_I^\gamma\}$ for the space $\Pi_k^r\big|_{K_I}$ on K_I, $I \in \mathcal{D}$ by setting $\mathbf{m}_I^\gamma(x) := 2^k\mathbf{m}^\gamma(2^k x - j)$ and $\eta_I^\gamma(x) := 2^k\eta^\gamma(2^k x - j)$, $|\gamma| < r$. Indeed, for each $I \in \mathcal{D}_k$, the $\mathbf{m}_I^\gamma$, η_I^γ, $|\gamma| < r$ are supported in K_I and satisfy

$$(\text{A.2.5})\qquad \langle \mathbf{m}_I^\beta, \eta_I^\gamma\rangle = \delta_{\beta,\gamma}, \qquad |\beta|,\ |\gamma| \le r-1.$$

Therefore, for given K_I, $\tilde{f}_{K_I}$ has the following representation:

$$(\text{A.2.6})\qquad \tilde{f}_{K_I}(x) = \sum_{|\gamma|\le r-1} \langle f, \eta_I^\gamma\rangle \mathbf{m}_I^\gamma(x), \qquad x \in K_I.$$

Now, combining (A.2.3) and (A.2.6) leads to an explicit form for $d_I(f)$:

$$(\text{A.2.7})\qquad \begin{aligned} d_I(f) &= \sum_{|\nu|\le r-1}\ \sum_{|\gamma|\le r-1} \Theta_\nu(x_I - \xi_I)\langle f, \eta_I^\gamma\rangle D^\nu(\mathbf{m}_I^\gamma)(\xi_I) \\ &= \langle f, \tau_I\rangle, \end{aligned}$$

where we set for $x \in K_I$,

$$\tau_I(x) := \sum_{|\gamma| \le r-1} \sum_{|\nu| \le r-1} \Theta_\nu(x_I - \xi_I) D^\nu(\mathbf{m}_I^\gamma)(\xi_I) \eta_I^\gamma(x) = \sum_{|\gamma| \le r-1} d_I(\mathbf{m}_I^\gamma) \eta_I^\gamma. \tag{A.2.8}$$

Here, for later reference, we note that $\|\tau_I\|_{L_\infty(K_I)} \le C 2^{2k}$ with C depending only on r. To see this, we recall from [13] (cf. [15]) that for any $S \in \Pi_k^r$,

$$|d_I(S)| \le C \|S\|_{L_\infty(K_I)}. \tag{A.2.9}$$

Then, we obtain

$$\begin{aligned} \|\tau_I\|_{L_\infty(K_I)} &= \Big\| \sum_{|\gamma| \le r-1} d_I(\mathbf{m}_I^\gamma) \eta_I^\gamma \Big\|_{L_\infty(K_I)} \\ &\le C \sum_{|\gamma| \le r-1} \|\mathbf{m}_I^\gamma\|_{L_\infty(K_I)} \|\eta_I^\gamma\|_{L_\infty(K_I)} \\ &\le C 2^{2k} \sum_{|\gamma| \le r-1} \|\mathbf{m}^\gamma\|_{L_\infty(K_O)} \|\eta^\gamma\|_{L_\infty(K_O)} \\ &\le C 2^{2k} \end{aligned} \tag{A.2.10}$$

with C depending only on r, where the second inequality uses (A.2.9).

To construct the family $\{b_I\}_{I \in \mathcal{D}}$ satisfying (5.1)(i), (ii) and (A.2.1), let us rewrite $\mathbf{P}_{k-1} f$ in terms of $M_{k,j}$ by using the refinement equation (2.6) as follows:

$$\begin{aligned} \mathbf{P}_{k-1} f &= \sum_{l \in \mathbb{Z}^2} \langle f, \tau_{k-1,l} \rangle M_{k-1,l} \\ &= \sum_{l \in \mathbb{Z}^2} \sum_{i \in \Gamma_M} a_i \langle f, \tau_{k-1,l} \rangle M(2^k x - 2l - i) \\ &= \sum_{l \in \mathbb{Z}^2} \sum_{j \in \Gamma_M + 2l} a_{j-2l} \langle f, \tau_{k-1,l} \rangle M_{k,j}, \end{aligned} \tag{A.2.11}$$

where $\tau_{k-1,l} := \tau_I$ for $I = I_{k-1,l} \in \mathcal{D}_{k-1}$ and Γ_M is the support of the coefficients of the refinement equation (2.6) for the box spline M. To be explicit, let us define some index subsets of $\mathbb{Z}^2$ as follows:

$$\Gamma_1 := \{ j \in \mathbb{Z}^2 \mid j = (2j'_1, 2j'_2),\ j' := (j'_1, j'_2) \in \mathbb{Z}^2 \},$$
$$\Gamma_2 := \{ j \in \mathbb{Z}^2 \mid j = (2j'_1 + 1, 2j'_2),\ j' := (j'_1, j'_2) \in \mathbb{Z}^2 \},$$
$$\Gamma_3 := \{ j \in \mathbb{Z}^2 \mid j = (2j'_1, 2j'_2 + 1),\ j' := (j'_1, j'_2) \in \mathbb{Z}^2 \},$$
$$\Gamma_4 := \{ j \in \mathbb{Z}^2 \mid j = (2j'_1 + 1, 2j'_2 + 1),\ j' := (j'_1, j'_2) \in \mathbb{Z}^2 \}.$$

Then, from (A.2.11), we have

$$\begin{aligned} \mathbf{P}_{k-1} f = &\sum_{j \in \Gamma_1} \sum_{i \in \Gamma_M \cap \Gamma_1} a_i \langle f, \tau_{k-1,j'-i'} \rangle M_{k,j} \\ &+ \sum_{j \in \Gamma_2} \sum_{i \in \Gamma_M \cap \Gamma_2} a_i \langle f, \tau_{k-1,j'-i'} \rangle M_{k,j} \\ &+ \sum_{j \in \Gamma_3} \sum_{i \in \Gamma_M \cap \Gamma_3} a_i \langle f, \tau_{k-1,j'-i'} \rangle M_{k,j} \\ &+ \sum_{j \in \Gamma_4} \sum_{i \in \Gamma_M \cap \Gamma_4} a_i \langle f, \tau_{k-1,j'-i'} \rangle M_{k,j}. \end{aligned} \tag{A.2.12}$$

Taking account of $\mathbf{P}_k f - \mathbf{P}_{k-1} f$, we define for given $I = I_{k,j} \in \mathcal{D}$, $b_I := b_{k,j}$ as follows:

$$b_{k,j} := 2^{-k} \begin{cases} \tau_{k,j} - \sum_{i \in \Gamma_M \cap \Gamma_1} a_i \, \tau_{k-1,j'-i'}, & \text{for } j \in \Gamma_1 \\ \tau_{k,j} - \sum_{i \in \Gamma_M \cap \Gamma_2} a_i \, \tau_{k-1,j'-i'}, & \text{for } j \in \Gamma_2 \\ \tau_{k,j} - \sum_{i \in \Gamma_M \cap \Gamma_3} a_i \, \tau_{k-1,j'-i'}, & \text{for } j \in \Gamma_3 \\ \tau_{k,j} - \sum_{i \in \Gamma_M \cap \Gamma_4} a_i \, \tau_{k-1,j'-i'}, & \text{for } j \in \Gamma_4. \end{cases} \tag{A.2.13}$$

Then, by (A.2.10), b_I, $I \in \mathcal{D}_k$ are well-defined functions in $L_\infty(\mathbb{R}^2)$ with $\|b_I\|_\infty \le C 2^k$ where C is independent of k. Further, it follow from (A.2.8) and (2.6) that the $b_{k,j}$ is supported on the set

$$B_{k,j} := \Big(\bigcup \{ K_{k-1,l} \in \mathcal{T}_{k-1} \mid l \in \widetilde{\Lambda}_{k,j} \}\Big) \cup K_{k,j} \tag{A.2.14}$$

where $\widetilde{\Lambda}_{k,j} = \{\, l \in \mathbb{Z}^2 \mid j - 2l \in \Gamma_M \,\}$ (see Figure 3, for example). Thus, we have $\mathbf{P}_k f - \mathbf{P}_{k-1} f = \sum_{I \in \mathcal{D}_k} 2^k \langle f, b_I \rangle M_I$.

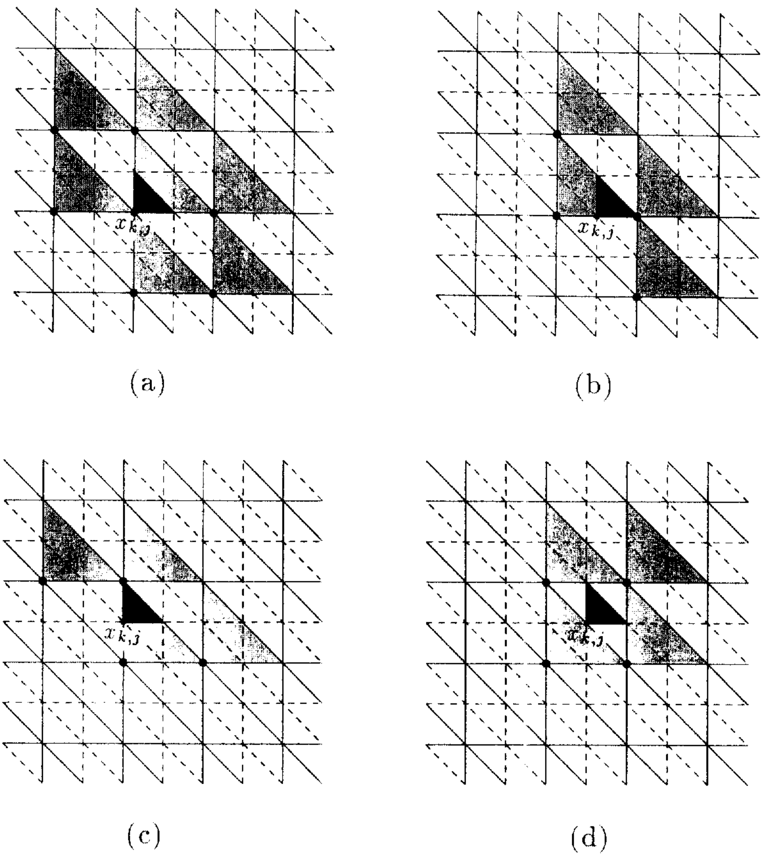


FIGURE 3. The supports of $b_{k,j}$ for the case $M = M_{222}$ with $X_{(2,2,2)} = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$: (a) $B_{k,j}$, $j \in \Gamma_1$; (b) $B_{k,j}$, $j \in \Gamma_2$; (c) $B_{k,j}$, $j \in \Gamma_3$; (d) $B_{k,j}$, $j \in \Gamma_4$, where the big shaded triangles show $K_{k-1,l}$ and the small shaded triangles shows $K_{k,j}$ corresponding to $x_{k,j}$.

Now, we have $c_I = 2^k \langle f, b_I \rangle$ because the functions M_I, $I \in \mathcal{D}_k$ are globally linearly independent (cf. (2.8)). Also, $b_{k,j}$, $j \in \mathbb{Z}^2$, $k \in \mathbb{Z}$

satisfy the condition (5.1)(ii) up to constants C independent of k and j, because of their support (A.2.14) and $\|b_{k,j}\|_\infty \le C2^k$. So, it remains to show that the $b_{k,j}$ satisfy the condition (5.1)(i). Let us fix $k \in \mathbb{Z}$ and define for each $j \in \mathbb{Z}^2$,

$$E_{k,j} := \bigcup_{l\in\mathbb{Z}^2} \{\operatorname{supp} M_{k-1,l} \mid M_{k-1,l} \text{ does not vanish identically on } B_{k,j}\},$$
$$\widetilde{E}_{k,j} := \bigcup_{l\in\mathbb{Z}^2} \{\operatorname{supp} M_{k-1,l} \mid M_{k-1,l} \text{ does not vanish identically on } E_{k,j}\}.$$

Then, the Strang-Fix condition (2.5) allows us to define $g^\gamma_{k,j} \in \mathcal{S}^{k-1}$ such that

$$\text{(A.2.15)} \qquad g^\gamma_{k,j}(x) := \begin{cases} x^\gamma, & \text{for } x \in E_{k,j}, \\ 0, & \text{for } x \notin \widetilde{E}_{k,j}, \end{cases}$$

where $|\gamma| \le r-1$. Using the fact that for given $k \in \mathbb{Z}$, the $\mathbf{P}_k$, reproduce all polynomials $P \in \mathcal{P}_r \subset \mathcal{S}^k$ and $\mathcal{S}^{k-1} \subset \mathcal{S}^k$, we obtain for each $j \in \mathbb{Z}^2$,

$$(\mathbf{P}_k g^\gamma_{k,j} - \mathbf{P}_{k-1} g^\gamma_{k,j})\big|_{E_{k,j}} = 0,$$

which implies that

$$\text{(A.2.16)} \qquad \sum_{l\in\mathbb{Z}^2} \langle g^\gamma_{k,j}, b_{k,l}\rangle M_{k,l}\big|_{E_{k,j}} = 0, \qquad |\gamma| \le r-1, \quad j \in \mathbb{Z}^2.$$

for fixed k. Therefore, for given $j \in \mathbb{Z}^2$, $\langle g^\gamma_{k,j}, b_{k,j}\rangle = 0$, $|\gamma| \le r-1$ because $M_{k,l}$ are locally independent. Thus, we obtain for given $I \in \mathcal{D}_k$,

$$\text{(A.2.17)} \qquad \int x^\gamma b_I(x)\,dx = \langle g^\gamma_I, b_I\rangle = 0, \qquad |\gamma| \le r-1,$$

where the first equality uses the fact that $\operatorname{supp} b_I \subset E_I \subset \operatorname{supp} g^\gamma_I$. This proves (5.1)(i) because $\alpha < r$. ❐

References

1. C. de Boor, *Quasi-interpolants and approximation power of multivariate splines*, Computation of Curves and Surfaces, W. Dahmen, M. Gasca, and C. Micchelli, eds., Kluwer, 1990, pp. 313–345
2. C. de Boor and R. DeVore, *Approximation by smooth multivariate splines*, Trans. Amer. Math. Soc. **276** (1983), 775–788.
3. C. de Boor and G. Fix, *Spline approximation by quasi-interpolants*, J. Approx. Theory **8** (1973), 19–45.
4. C. de Boor and K. Höllig, *B-splines from parallelepipeds*, J. Analyse Math. **42** (1982), 95–115.
5. C. de Boor, K. Höllig, and S. Riemenschneider, *Bivariate cardinal interpolation by splines on a three-direction mesh*, Illinois J. Math. **29** (1985), 533–566.
6. L. Brown and B. J. Lucier, *Best approximations in L^1 are near best in L^p, $p < 1$*, Proc. Amer. Math. Soc. **No. 1** (1994), 97–100.
7. A. Cavaretta, W. Dahmen, and C. Micchelli, *Stationary subdivision*, Mem. Amer. Math. Soc. **93 No. 453.** (1991).
8. C. K. Chui, *Multivariate Splines* CBMS-NSF Reg. Conf. Series in Appl. Math., Vol. 54, SIAM, Philadelphia, 1988.
9. W. Dahmen and C. Micchelli, *Local linear independence of translates of a box spline*, Studia Math. **82** (1985), 243–262.
10. ———, *Subdivision algorithms for the generation of box-splines surfaces*, Comput. Aided Geom. Design **1** (1984), 115–129.
11. R. DeVore and G. Lorentz, *Constructive Approximation*, Springer-Verlag, New York, 1993.
12. ———, *Surface compression*, Comput. Aided. Geom. Design **9(3)** (1992), 219–237.
13. R. DeVore, B. Jawerth, and V. Popov, *Compression of wavelet decomposition*, Amer. J. Math. **114** (1992), 737–785.
14. R. DeVore, G. Kyriazis, D. Leviatan, and V. Tikhomirov, *Wavelet compression and nonlinear n-widths*, Adv. Comput. Math. **1(2)** (1993), 197–214.
15. R. DeVore and V. Popov, *Interpolation of Besov spaces*, Trans. Amer. Math. Soc. **305** (1988), 397–414.
16. G. Fix and G. Strang, *A Fourier analysis of the finite element variational method*, Constructive Aspects of Functional Analysis, G. Geymonant, ed., C. I. M. E. II Cilo, 1971, pp. 793–840.
17. ———, *A discrete transform and decompositions of distribution spaces*, J. of Functional Analysis **93** (1990), 34–170.
18. R.-Q. Jia, *Local linear independence of the translates of a box spline*, Constructive Approximation **1** (1985), 175–182.
19. J. Peetre, *New Thoughts on Besov Spaces*, Duke University Mathematics Series 1, Duke University Mathematics Department, Durham, North Carolina, 1976.
20. H. Triebel, *Theory of function spaces* II, Monographs in Mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.

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