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AN EXTENSION OF TELCI, TAS AND FISHER'S THEOREM

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I. Introduction

Let (X, d) be a metric space and let T be a mapping from X into itself. We say that a metric space (X, d) is *T-orbitally complete* if every Cauchy sequence of the form $\{T^{n_i}x\}_{i\in N}$ for $x \in X$ converges to a point in X.

Recently, in [24], Nesic proved the following:

THEOREM A. *Let (X, d) be a metric space and let T be a self-mapping of X satisfying the following condition:*

$$\begin{aligned} &[1+pd(x,y)]d(Tx,Ty) \\ (1.1)\qquad &\leq p[d(x,Tx)d(y,Ty)+d(x,Ty)d(y,Tx)] \\ &\quad + q\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\} \end{aligned}$$

for all $x, y \in X$, where $p \geq 0$ and $0 < q < 1$. If (X, d) is T-orbitally complete, then T has a unique fixed point in X.

In fact, if we put $p = 0$, then Theorem A is an extension of the following:

THEOREM B. *Let (X, d) be a complete metric space and let T be a mapping from X into itself satisfying any one of the following conditions:*

$$(1.2)\qquad d(Tx,Ty) \leq \alpha d(x,y)$$

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for all $x, y \in X$, *where* $0 \leq \alpha < 1$, (: Banach contraction),

$$d(Tx, Ty) \leq \alpha \max\{d(x, Tx), d(y, Ty)\} \tag{1.3}$$

for all $x, y \in X$, *where* $0 \leq \alpha < 1$, (: Fisher contraction [5]),

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \tag{1.4}$$

for all $x, y \in X$, *where* $0 \leq \alpha < \frac{1}{2}$, (: Kannan contraction [15]),

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx) \tag{1.5}$$

for all $x, y \in X$, *where* $0 \leq \alpha < \frac{1}{2}$, (: Fisher contraction [6]),

$$\begin{aligned} d(Tx, Ty) \leq \alpha \max\{&d(x, y), d(x, Tx), d(y, Ty), \\ &\frac{1}{2}[d(x, Ty) + d(y, Tx)] \end{aligned} \tag{1.6}$$

for all $x, y \in X$, *where* $0 \leq \alpha < 1$, (: Ciric contraction [2]). *Then* T *has a unique fixed point in* X.

Recently, Fisher, Telci and Tas [8] extended Theorem A. In fact, from Theorem 2 of Fisher et al. with $S = I_X$(: the identity mapping on X), we have Theorem A.

In this paper, motivated by the results of Nesic and Fisher et al., we define the new concept of weakly compatible mappings of type (A) and prove some fixed point theorems for these mappings. Our results extend, generalize and improve Theorems A and B, the results of Fisher et al. [8], Necic [24], Tas et al. [31], Telci et al. [33] and many others.

II. Weakly Compatible Mappings of Type (A)

In [10] and [13], Jungck and Jungck et al. introduced the concepts of compatible mappings and compatible mappings of type (A), respectively, and gave some fixed point theorems for these mappings. Under some conditions, these two concepts are equivalent. Of course, commuting mappings ([9]) are weakly commuting mappings ([28]) and weakly commuting mappings are compatible, but the converses are not true. We can find some examples on commuting, weakly commuting, compatible mappings and compatible mappings of type (A) in [4], [10]-[13] and [24]-[26].

Now, we introduce the concept of weakly compatible mappings of type (A) and give some properties of these mappings for our main results.

Let S and T be mappings of a metric space (X, d) into itself.

DEFINITION 2.1. ([10]) The mappings S and T are said to be *compatible* if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

DEFINITION 2.2. ([13]) The mappings S and T are said to be *compatible of type* (A) if

$$\lim_{n\to\infty} d(STx_n, TTx_n) = 0, \quad \lim_{n\to\infty} d(TSx_n, SSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

DEFINITION 2.3. The mappings S and T are said to be *weakly compatible of type* (A) if

$$\lim_{n\to\infty} d(STx_n, TTx_n) \le d(Sz, Tz) \le \lim_{n\to\infty} d(Tz, TTx_n),$$
$$\lim_{n\to\infty} d(TSx_n, SSx_n) \le d(Sz, Tz) \le \lim_{n\to\infty} d(Sz, SSx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

We give some properties of these mappings and relations between these kinds of compatible mappings in metric spaces.

PROPOSITION 2.1. ([13]) *Let S and T be continuous mappings from a metric space (X, d) into itself. Then S and T are compatible if and only if they are compatible of type (A).*

PROPOSITION 2.2. *Let S and T be mappings of a metric space (X, d) into itself such that*

$$d(Sz, Tz) \leq \lim_{n\to\infty} d(Tz, TTx_n),$$
$$d(Sz, Tz) \leq \lim_{n\to\infty} d(Sz, SSx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. If S and T are compatible of type (A), then they are weakly compatible of type (A). But the converse is not true.

Proof. It follows from Definition 2.3.

PROPOSITION 2.3. *Let S and T be continuous mappings from a metric space (X, d) into itself. If S and T are weakly compatible of type (A), then they are compatible of type (A).*

Proof. Suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since S and T are continuous, we have

$$\lim_{n\to\infty} d(Tz, TTx_n) = 0, \quad \lim_{n\to\infty} d(Sz, SSx_n) = 0.$$

From the definition of weakly compatible mappings of type (A), it follows that $d(Sz, Tz) = 0$ and so

$$\lim_{n\to\infty} d(STx_n, TTx_n) = 0, \quad \lim_{n\to\infty} d(TSx_n, SSx_n) = 0.$$

Therefore, S and T are compatible of type (A). This completes the proof.

Now, we give some examples related to Propositions 2.1~2.3 as follows:

EXAMPLE 2.1. Let $X = R$ be the set of real numbers with the usual metric $d(x, y) = |x - y|$. Define two mappings $S, T : X \to X$ by

$$Sx = \begin{cases} \frac{1}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{1}{x^2} & \text{for } x \neq 0 \\ 2 & \text{for } x = 0, \end{cases}$$

respectively. Note that S and T are not continuous at $z = 0$. Consider a sequence $\{x_n\}$ in X defined by $x_n = n^2$ for $n = 1, 2, \cdots$. Then we have, as $n \to \infty$,

$$Sx_n = \frac{1}{n^2} \to z = 0, \quad Tx_n = \frac{1}{n^4} \to z = 0,$$
$$\lim_{n\to\infty} d(STx_n, TSx_n) = \lim_{n\to\infty} d(n^4, n^4) = |n^4 - n^4| = 0,$$

but

$$\lim_{n\to\infty} d(STx_n, TTx_n) = \lim_{n\to\infty} d(n^8, n^4) = \lim_{n\to\infty} |n^8 - n^4| = \infty,$$
$$\lim_{n\to\infty} d(TSx_n, SSx_n) = \lim_{n\to\infty} d(n^2, n^4) = \lim_{n\to\infty} |n^2 - n^4| = \infty,$$
$$d(Sz, Tz) = d(S(0), T(0)) = |1 - 2| = 1.$$

Therefore, S and T are compatible, but they are neither compatible of type (A) nor weakly compatible of type (A).

EXAMPLE 2.2. Let $X = [0, 1]$ be a metric space with the usual metric $d(x, y) = |x - y|$. Define two mappings $S, T : X \to X$ by

$$Sx = \begin{cases} x & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1 - x & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 1], \end{cases}$$

respectively. Then S and T are not continuous at $z = \frac{1}{2}$. Now, we assert that S and T are not compatible but they are both compatible of type (A) and weakly compatible of type (A).

Suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$. By the definition S and T, $z \in \{\frac{1}{2}, 1\}$. Since S and T agree on $[\frac{1}{2}, 1]$, we need to consider $z = \frac{1}{2}$ and so we can suppose that $x_n \to \frac{1}{2}$ as $n \to \infty$ and $x_n < \frac{1}{2}$ for $n = 1, 2, \cdots$. Then we have

$$Sx_n = x_n \to \frac{1}{2} \quad \text{from the left,}$$
$$Tx_n = 1 - x_n \to \frac{1}{2} \quad \text{from the right,}$$
$$STx_n = S(1 - x_n) = 1,$$
$$TSx_n = Tx_n = 1 - x_n \to \frac{1}{2}.$$

Thus, it follows that

$$\lim_{n\to\infty} d(STx_n, TSx_n) = \lim_{n\to\infty}\left|1-\frac{1}{2}\right| = \frac{1}{2} \neq 0,$$

but

$$\begin{aligned}
\lim_{n\to\infty} d(STx_n, TTx_n) &= \lim_{n\to\infty} |STx_n - TTx_n| \\
&= \lim_{n\to\infty} |1 - T(1-x_n)| \\
&= 1 - 1 = 0, \\
\lim_{n\to\infty} d(TSx_n, SSx_n) &= \lim_{n\to\infty} |TSx_n - SSx_n| \\
&= \lim_{n\to\infty} |(1-x_n) - x_n| \\
&= 1 - 1 = 0, \\
d(Sz, Tz) &= \left|S\left(\frac{1}{2}\right) - T\left(\frac{1}{2}\right)\right| = 1 - 1 = 0, \\
\lim_{n\to\infty} d(Tz, TTx_n) &= \lim_{n\to\infty} \left|T\left(\frac{1}{2}\right) - T(1-x_n)\right| \\
&= 1 - 1 = 0, \\
\lim_{n\to\infty} d(Sz, SSx_n) &= \lim_{n\to\infty} \left|S\left(\frac{1}{2}\right) - x_n\right| \\
&= \left|1-\frac{1}{2}\right| = \frac{1}{2}.
\end{aligned}$$

Therefore, S and T are both compatible of type (A) and weakly compatible of type (A), but they are not compatible.

EXAMPLE 2.3. Let $X = [0,\infty)$ be a metric space with the usual metric $d(x,y) = |x-y|$. Define two mappings $S, T : X \to X$ by

$$Sx = \begin{cases} \frac{1}{2} + x & \text{for } x \in \left[0, \frac{1}{2}\right) \\ 1 & \text{for } x = \frac{1}{2} \\ 0 & \text{for } x \in \left(\frac{1}{2}, \infty\right) \end{cases}$$

and

$$Tx = \begin{cases} \frac{1}{2} - x & \text{for } x \in \left[0, \frac{1}{2}\right) \\ 2 & \text{for } x = \frac{1}{2} \\ 1 & \text{for } x \in \left(\frac{1}{2}, \infty\right), \end{cases}$$

respectively. Note that S and T are not continuous at $z = \frac{1}{2}$. Now we prove that S and T are both compatible and weakly compatible of type (A), but they are not compatible of type (A). Suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$. Let $\{x_n\}$ be a sequence in X such that $x_n \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned} Sx_n &= \frac{1}{2} + x_n \to \frac{1}{2} \quad \text{from the right,} \\ Tx_n &= \frac{1}{2} - x_n \to \frac{1}{2} \quad \text{from the left.} \end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} d(STx_n, TSx_n) = \lim_{n\to\infty} |1 - x_n - 1| = 0,$$

and so S and T are compatible. On the other hand, we have

$$\begin{aligned} \lim_{n\to\infty} d(STx_n, TTx_n) &= \lim_{n\to\infty} |1 - x_n - 0| = 1 \neq 0, \\ \lim_{n\to\infty} d(TSx_n, SSx_n) &= \lim_{n\to\infty} |1 - 0| = 1 \neq 0, \end{aligned}$$

which implies that S and T are not compatible of type (A). Further, we have

$$\begin{aligned} d(Sz, Tz) &= d\Big(S\Big(\frac{1}{2}\Big) - T\Big(\frac{1}{2}\Big)\Big) = |1 - 2| = 1, \\ \lim_{n\to\infty} d(Tz, TTx_n) &= \lim_{n\to\infty} |2 - 0| = 2, \\ \lim_{n\to\infty} d(Sz, SSx_n) &= \lim_{n\to\infty} |1 - 0| = 1 \end{aligned}$$

and so S and T are weakly compatible of type (A).

PROPOSITION 2.4. *Let S and T be weakly compatible mappings of type (A) from a metric space (X, d) into itself. If $Sz = Tz$ for some $z \in X$, then $STz = TTz = TSz = SSz$.*

Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = z$ for $n = 1, 2, \cdots$ and $Sz = Tz$. Then we have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sz = Tz.$$

Since S and T are weakly compatible of type (A), we have

$$d(STz, TTz) = \lim_{n\to\infty} d(STx_n, TTx_n) \le d(Sz, Tz) = 0,$$

which implies that $STz = TTz$. Similarly, $TSz = SSz$. On the other hand, $Sz = Tz$ implies $TTz = TSz$. Therefore, we have $STz = TTz = TSz = SSz$. This completes the proof.

PROPOSITION 2.5. *Let S and T be weakly compatible mappings of type (A) from a metric space (X, d) into itself and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Then we have the following:*

(1) $\lim_{n\to\infty} TSx_n = Sz$ if S is continuous.

(2) $\lim_{n\to\infty} STx_n = Tz$ if T is continuous.

(3) $STz = TSz$ if S and T are continuous.

Proof. (1) Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since S is continuous,

$$\lim_{n\to\infty} d(SSx_n, Sz) = 0$$

and so, from the definition of weakly compatible mappings of type (A), it follows that

$$\lim_{n\to\infty} d(TSx_n, SSx_n) = 0.$$

Thus, since $d(TSx_n, Sz) \le d(TSx_n, SSx_n) + d(SS_n, Sz)$, we have

$$\lim_{n\to\infty} d(TSx_n, Sz) = 0.$$

(2) Similarly, if T is continuous, then $\lim_{n\to\infty} STx_n = Tz$.

(3) Since T is continuous, $\lim_{n\to\infty} TSx_n = Tz$. On the other hand, by (1), since S is continuous, we have

$$\lim_{n\to\infty} TSx_n = Sz.$$

Hence, by the uniqueness of the limit, it follows that $Sz = Tz$ and so, by Proposition 2.4, we have $STz = TSz$. This completes the proof.

III. Main Theorems

Let R^+ denote the set of all non-negative real numbers and $\mathcal{F}$ be the family of mappings ϕ from $(R^+)^5$ into R^+ such that

(i) ϕ is non-decreasing,

(ii) ϕ is upper semi-continuous in each coordinate variable,

(iii) $\Gamma(t) = \phi(t, t, a_1 t, a_2 t, t) < t$, where $\Gamma : R^+ \to R^+$ is a mapping with $\Gamma(0) = 0$ and $a_1 + a_2 = 2$.

LEMMA 3.1. ([30]) *For all $t > 0, \Gamma(t) < t$ if and only if $\lim_{n\to\infty} \Gamma^n(t) = 0$, where Γ^n denotes the n-times composition of Γ.*

Now, we are ready to prove our main theorems:

THEOREM 3.2. *Let A, B, S and T be mappings of a complete metric space (X, d) into itself satisfying the following conditions:*
(3.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
(3.2) the pairs A, S and B, T are weakly compatible of type (A),
(3.3) one of A, B, S and T is continuous,

$$\begin{aligned} &[1 + pd(Sx, Ty)]d(Ax, By) \\ &\le p \max\{d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax)\} \\ &\quad + \phi(d(Sx, Ty), d(Sx, Ax), d(Ty, By), \\ &\qquad d(Sx, By), d(Ty, Ax)) \end{aligned} \tag{3.3}$$

for all $x, y \in X$, where $p \ge 0$ and $\phi \in \mathcal{F}$. Then A, B, S and T have a unique common fixed point in X.

Proof. Since $A(X) \subset T(X)$, for an arbitrary point $x_0 \in X$, we can choose a point $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $y_1 = Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \tag{3.5}$$

for $n = 0, 1, 2, \cdots$. For simplicity, let

$$\alpha_{2n} = d(Ax_{2n}, Bx_{2n+1}) \quad \text{and} \quad \alpha_{2n+1} = d(Bx_{2n+1}, Ax_{2n+2})$$

for $n = 0, 1, 2, \cdots$. Using (3.4), we have

$$\begin{aligned}
&(1 + p\alpha_{2n-1})\alpha_{2n} \\
&= [1 + pd(Sx_{2n}, Tx_{2n+1})]d(Ax_{2n}, Bx_{2n+1}) \\
&\leq p\max\{d(Sx_{2n}, Ax_{2n})d(Tx_{2n+1}, Bx_{2n+1}), \\
&\qquad d(Sx_{2n}, Bx_{2n+1})d(Tx_{2n+1}, Ax_{2n})\} \\
&\quad + \phi(d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}), \\
&\qquad d(Sx_{2n}, Bx_{2n+1}), d(Tx_{2n+1}, Ax_{2n})),
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1 + p\alpha_{2n-1})\alpha_{2n} \\
&\leq p\alpha_{2n-1}\alpha_{2n} + \phi(\alpha_{2n-1}, \alpha_{2n-1}, \alpha_{2n}, \alpha_{2n-1} + \alpha_{2n}, 0).
\end{aligned}$$

Thus, it follows that

$$\alpha_{2n} \leq \phi(\alpha_{2n-1}, \alpha_{2n-1}, \alpha_{2n}, \alpha_{2n-1} + \alpha_{2n}, 0).$$

If $\alpha_{2n} > \alpha_{2n-1}$ for some n, then $\alpha_{2n} \leq \Gamma(\alpha_{2n}) < \alpha_{2n}$, which is a contradiction. Thus, we have $\alpha_{2n} \leq \Gamma(\alpha_{2n-1})$ for $n = 1, 2, \cdots$. Similarly, we have $\alpha_{2n+1} \leq \Gamma(\alpha_{2n})$. Proceeding in this way, we have

$$\alpha_n \leq \Gamma(\alpha_{n-1}) \leq \Gamma^2(\alpha_{n-1}) \leq \cdots \leq \Gamma^n(\alpha_0)$$

and, by Lemma 3.1, since $\lim_{n\to\infty} \Gamma^n(\alpha_0) = 0$, $\lim_{n\to\infty} \alpha_n = 0$. If $\alpha_0 = 0$, we have

$$\lim_{n\to\infty} \alpha_n = 0 \tag{3.6}$$

since $\alpha_n = 0$ for $n = 1, 2, 3, \cdots$.

Now, we will show that the sequence $\{y_n\}$ defined by (3.5) is a Cauchy sequence in X. For this, it is sufficient to prove that $\{Ax_{2n}\}$ is a Cauchy sequence in X. Suppose that this is not true. Then there exist an $\epsilon > 0$ and a sequence $\{n(k)\}$ of even integers defined inductively with $n(1) = 2$ and $n(k+1)$, the smallest even integer greater than $n(k)$, such that

$$d(Ax_{n(k)}, Ax_{n(k+1)}) > \epsilon \tag{3.7}$$

so that

$$(3.8) \qquad d(Ax_{n(k)}, Ax_{n(k+1)-2}) \leq \epsilon.$$

It follows from (3.7) and (3.8) that

$$\epsilon < d(Ax_{n(k)}), Ax_{n(k+1)}) \leq \epsilon + \alpha_{n(k+1)-2} + \alpha_{n(k+1)-1}$$

for $k = 1, 2, \cdots$, which, by (3.6), implies that

$$(3.9) \qquad \lim_{n\to\infty} d(Ax_{n(k)}, Ax_{n(k+1)-2}) = \epsilon.$$

Now it follows from (i), (3.4) and (3.8) that

$$\begin{aligned}
&[1 + p|\alpha_{n(k+1)-1} - d(Ax_{n(k+1)}, Ax_{n(k)})|]d(Ax_{n(k)}, Ax_{n(k+1)}) \\
&= [1 + p|d(Bx_{n(k+1)-1}, Ax_{n(k+1)}) \\
&\quad - d(Ax_{n(k+1)}, Ax_{n(k)})|]d(Ax_{n(k)}, Ax_{n(k+1)}) \\
&\leq [1 + pd(Bx_{n(k+1)-1}, Ax_{n(k)})][\alpha_{n(k)} + d(Ax_{n(k+1)}, Bx_{n(k)+1})] \\
&= [1 + pd(Bx_{n(k+1)-1}, Ax_{n(k)})]\alpha_{n(k)} \\
&\quad + [1 + pd(Bx_{n(k+1)-1}, Ax_{n(k)})]d(Ax_{n(k+1)}, Bx_{n(k)+1}) \\
&\leq [1 + p(d(Bx_{n(k+1)-1}, Ax_{n(k+1)-2}) + d(Ax_{n(k+1)-2}, Ax_{n(k)}))]\alpha_{n(k)} \\
&\quad + [1 + pd(Sx_{n(k+1)}, Tx_{n(k)+1})]d(Ax_{n(k+1)}, Bx_{n(k)+1}) \\
&\leq [1 + p(\alpha_{n(k+1)-2} + \epsilon)]\alpha_{n(k)} + p\max\{\alpha_{n(k+1)-1}\alpha_{n(k)}, \\
&\qquad d(Bx_{(k+1)-1}, Bx_{n(k)+1})d(Ax_{n(k)}, Ax_{n(k+1)})\} \\
&\quad + \phi(d(Bx_{n(k+1)-1}, Ax_{n(k)}), \alpha_{n(k+1)-1}, \alpha_{n(k)}, \\
&\qquad d(Bx_{n(k+1)-1}, Ax_{n(k)}), d(Ax_{n(k)}, Ax_{n(k+1)})) \\
&\leq [1 + p(\alpha_{n(k+1)-2} + \epsilon)]\alpha_{n(k)} + p\max\{\alpha_{n(k+1)-2}\alpha_{n(k)}, \\
&\qquad (\alpha_{n(k+1)-2} + \epsilon + \alpha_{n(k)})d(Ax_{n(k)}, Ax_{n(k+1)})\} + \phi(\alpha_{n(k+1)-2} + \epsilon, \\
&\qquad \alpha_{n(k+1)-1}, \alpha_{n(k)}, \alpha_{n(k+1)-2} + \epsilon + \alpha_{n(k)}, d(Ax_{n(k)}, Ax_{n(k+1)})).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (ii), (iii), (3.6) and (3.9), we have

$$\epsilon + p\epsilon^2 \leq p\epsilon^2 + \phi(\epsilon, 0, 0, \epsilon, \epsilon)$$

and so it follows that

$$\epsilon \leq \phi(\epsilon, 0, 0, \epsilon, \epsilon) \leq \Gamma(\epsilon) < \epsilon,$$

which is a contradiction. Hence, $\{Ax_{2n}\}$ is a Cauchy sequence in X. Similarly, we can show that $\{Bx_{2n+1}\}$ is also a Cauchy sequence in X. Since (X, d) is complete, the sequence $\{y_n\}$ defined by (3.5) converges to a limit $z \in X$. Thus, the subsequences $\{Ax_{2n}\} = \{Tx_{2n+1}\}$, $\{Bx_{2n+1}\} = \{Sx_{2n+2}\}$ of $\{y_n\}$ also converge to z.

Now, suppose that S is continuous. Then $SAx_{2n} \to Sz$ as $n \to \infty$. Since A and S are weakly compatible mappings of type (A), by Proposition 2.5, $ASx_{2n} \to Sz$ as $n \to \infty$. Replacing x by Sx_{2n} and y by x_{2n+1} in (3.4), we have

$$\begin{aligned}
&[1 + pd(S^2x_{2n}, Tx_{2n+1})]d(ASx_{2n}, Bx_{2n+1}) \\
&\leq p\max\{d(S^2x_{2n}, ASx_{2n})d(Tx_{2n+1}, Tx_{2n+1}), \\
&\quad d(S^2x_{2n}, Bx_{2n+1})d(Tx_{2n+1}, ASx_{2n})\} \\
&\quad + \phi(d(S^2x_{2n}, Tx_{2n+1}), d(S^2x_{2n}, ASx_{2n}), \\
&\quad d(Tx_{2n+1}, Bx_{2n+1}), d(S^2x_{2n}, Bx_{2n+1}), \\
&\quad d(Tx_{2n+1}, ASx_{2n+1})).
\end{aligned} \tag{3.10}$$

Taking $n \to \infty$ in (3.10), we have

$$d(Sz, z) \leq \phi(d(Sz, z), 0, 0, d(Sz, z), d(Sz, z)) < d(Sz, z)$$

and so it follows that $Sz = z$. Again replacing x by z and y by x_{2n+1} in (3.4) and letting $n \to \infty$, we have

$$d(Az, z) \leq \phi(0, d(Az, z), 0, 0, d(Az, z)) < d(Az, z),$$

which means that $Az = z$. Since $A(X) \subset T(X)$, there exists a point u in X such that $Az = Tu = z$. Again from (3.4), it follows that

$$\begin{aligned}
&[1 + pd(Sz, Tu)]d(Az, Bu) \\
&\leq p\max\{d(Sz, Az)d(Tu, Bu), d(Sz, Bu)d(Tu, Az)\} \\
&\quad + \phi(d(Sz, Tu), d(Sz, Az), d(Tu, Bu), \\
&\quad d(Sz, Bu), d(Tu, Az)),
\end{aligned}$$

which implies $d(z, Bu) \leq \Gamma(d(z, Bu))$ and so we have $z = Bu = Tu$. But since B and T are weakly compatible of type (A), by Proposition 2.4, $Bz = BTu = TTu = Tz$. By (3.4) again,

$$\begin{aligned}
&[1 + pd(Sz, Tz)]d(Sz, Bz) \\
&\leq p \max\{d(Sz, Az)d(Tz, Bz), d(Sz, Bz)d(Tz, Az)\} \\
&\quad + \phi(d(Sz, Tz), d(Sz, Az), d(Tz, Bz), \\
&\quad\ d(Sz, Bz), d(Tz, Az)),
\end{aligned}$$

which implies that $d(z, Bz) \leq \Gamma(d,(z, Bz)) < d(z, Bz)$, which is a contradiction. Hence, $z = Bz = Tz$ and therefore z is a common fixed point of A, B, S and T.

For the uniqueness of the common fixed point z, we suppose that z and w $(z \neq w)$ are the common fixed points of A, B, S and T. Using (3.4) again, we have

$$d(z, w) \leq \Gamma(d(z, w)) < d(z, w),$$

which is a contradiction. Hence, z is a unique common fixed point of A, B, S and T. Similarly, we can complete the proof by assuming that T or A or B is continuous in lieu of S being continuous. This completes the proof.

As immediate consequences of Theorem 3.2 with $A = B$, we have the following:

Corollary 3.3. *Let A, S and T be mappings of a complete metric space (X, d) into itself satisfying* (3.3) *and the following conditions:*
(3.11) $A(X) \subset S(X) \cap T(X)$,
(3.12) *The pairs A, S and A, T are weakly compatible of type (A),*

$$\begin{aligned}
&[1 + pd(Sx, Ty)]d(Ax, Ay) \\
&\leq p \max\{d(Sx, Ax)d(Ty, Ax), d(Sx, Ay)d(Ty, Ax)\} \\
&\quad + \phi(d(Sx, Ty), d(Sx, Ax), d(Ty, Ay), \\
&\quad\ d(Sx, Ay), d(Ty, Ax))
\end{aligned} \tag{3.13}$$

for all $x, y \in X$, where $\phi \in \mathcal{F}$ and $p \geq 0$. Then A, S and T have a unique common fixed point in X.

COROLLARY 3.4. *Let A, B, S and T be self-mappings on a complete metric space (X, d) satisfying the conditions (3.1), (3.2), (3.3) and any one of the following conditions:*
(3.14) *for all $x, y \in X$,*

$$\text{(a)}\quad \begin{aligned}&[1+pd(Sx,Ty)]d(Ax,By)\\ &\le p\max\{d(Sx,Ax)d(Ty,By), d(Sx,By)d(Ty,Ax)\}\\ &\quad + q\max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By),\\ &\qquad d(Sx,By), d(Ty,Ax)\},\end{aligned}$$

where $0 < q < 1$ and $p \ge 0$,

$$\text{(b)}\quad \begin{aligned}&[1+pd(Sx,Ty)]d(Ax,By)\\ &\le p\max\{d(Sx,Ax)d(Ty,By), d(Sx,By)d(Ty,Ax)\}\\ &\quad + q\max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By),\\ &\qquad \frac{1}{2}[d(Sx,By)+d(Ty,Ax)]\},\end{aligned}$$

where $0 < q < 1$ and $p \ge 0$,

$$\text{(c)}\quad \begin{aligned}&[1+pd(Sx,Ty)]d(Ax,By)\\ &\le p\max\{d(Sx,Ax)d(Ty,By), d(Sx,By)d(Ty,Ax)\}\\ &\quad + \alpha d(Sx,Ty) + \beta[d(Sx,Ax)+d(Ty,By)]\\ &\qquad \delta[d(Sx,By)+d(Ty,Ax)],\end{aligned}$$

where α, β, δ are non-negative real numbers with $\alpha + 2\beta + 2\delta < 1$,

$$\text{(d)}\quad \begin{aligned}&[1+pd(Sx,Ty)]d(Ax,By)\\ &\le p\max\{d(Sx,Ax)d(Ty,By), d(Sx,By)d(Ty,Ax)\}\\ &\quad + f(\max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By),\\ &\qquad \frac{1}{2}[d(Sx,By)+d(Ty,Ax)]\}),\end{aligned}$$

where $p \ge 0$ and $f : R^+ \to R^+$ is a function satisfying the conditions (i), (ii) and $f(t) < t$ for all $t > 0$. Then A, B, S and T have a unique common fixed point in X.

Proof. If we define a mapping $\phi : (R^+)^5 \to R^+$ by

(a′) $$\phi(t_1, t_2, t_3, t_4, t_5) = q \max\{t_1, t_2, t_3, t_4, t_5\},$$

(b′) $$\phi(t_1, t_2, t_3, t_4, t_5) = q \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\},$$

(c′) $$\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta(t_2 + t_3) + \delta(t_4 + t_5),$$

(d′) $$\phi(t_1, t_2, t_3, t_4, t_5) = f(\max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\}),$$

respectively, where $0 < q < 1$. α, β, δ are non-negative real numbers with $\alpha + 2\beta + 2\delta < 1$ and $f : R^+ \to R^+$ is a function satisfying the conditions (i), (ii) and $f(t) < t$ for all $t > 0$, then $\phi \in \mathcal{F}$. Hence, by Theorem 3.2, this corollary follows.

If $p = 0$ in Corollary 3.4, then we have the following:

COROLLARY 3.5. *Let A, B, S and T be mappings of a complete metric space (X, d) into itself satisfying the conditions* (3.1), (3.2), (3.3) *and any one of the following conditions:*
(3.15) *for all $x, y \in X$,*

(a″) $$d(Ax, By) \le q \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\},$$

where $0 < q < 1$,

(b″) $$d(Ax, By) \le q \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\},$$

where $0 < q < 1$,

(c″) $$d(Ax, By) \le \alpha d(Sx, Ty) + \beta[d(Sx, Ax) + d(Ty, By)] + \delta[d(Sx, By) + d(Ty, Ax)],$$

where α, β, and δ are non-negative real numbers with $\alpha + 2\beta + 2\delta < 1$,

(d″) $$d(Ax, By) \le f(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\}),$$

where $f : R^+ \to R^+$ is a function satisfying the conditions (i), (ii) *and $f(t) < t$ for all $t > 0$. Then A, B, S and T have a unique common fixed point in X.*

REMARK 3.1. (a) If put $S = T = I_X$ in Theorem 3.2, define

$$\phi(t_1, t_2, t_3, t_4, t) = q \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\},$$

where $0 < q < 1$, relax the conditions (3.1), (3.2), (3.3) and the completeness of X is replaced by the AB-orbital completeness of X, then we have Theorem 2 of [8].

(b) Theorem 3.1 of Jungck [11] is a special case of Theorem 3.2 if put $p = 0$ and define the function ϕ as in (a) of this remark.

(c) We list here some of the references in which it is possible to replace one of commutativity, weak commutativity, compatibility, compatibility of type (A) by weak compatibility of type (A). Our main theorem includes the corresponding results due to Cho, Murthy and Jungck [1], Ding [3], Fisher [4]-[6], Fisher and Sessa [7], Jungck [7]-[12], Kang, Cho and Jungck [14], Murthy, Chang, Cho and Sharma [16], Murthy and Fisher [18], Murthy, Cho and Fisher [19], Murthy and Ding [20], Murthy [21], Naidu and Prasad [22], [23], Prasad [25] and Pathak [26], Sessa, Rhoades and Khan [29], Tas. Telci and Fisher [31], Telci, Tas and Fisher [32] and others.

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