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PROTTER'S CONJUGATE BOUNDARY VALUE PROBLEMS FOR THE TWO DIMENSIONAL WAVE EQUATION

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1. Introduction

In 1954 M. H. Protter [1] formulated the following boundary value problem as an analogue of the plane Darboux problem.

PROBLEM D_0. Find a solution $u(x,y,\tau)$ of the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial \tau^2}\right)u(x,y,\tau)=0 \tag{1}$$

in the domain $Q: 0<\tau<\rho\equiv\sqrt{x^2+y^2}<1-\tau$ such that $u\in C(\bar{Q})\cap C^2(Q)$ and

$$u|_{K_i}=\varphi_i, \qquad i=0,1. \tag{2}$$

where $\varphi_i,\ \ i=0,1$ are given functions, the conic surface $K_0: 0<\rho=\tau<1/2$ and the circle without the center $K_1:\tau=0,\ \ 0<\rho<1$.

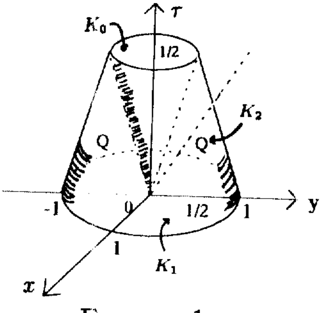

FIGURE 1.

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Let $K_2 = \partial Q \setminus (\bar{K}_0 \cup \bar{K}_1)$ be the surface of frustum of cone.

In 1957 Tong Kwang-chang [2] noted that the linear space of solutions of the homogeneous Problem D_0 is infinite-dimensional. Examples similar to [2] are presented in [3]-[5] :functions

$$
\begin{aligned}
&u_{i,k,n}(x,y,\tau) = u_{i,k,n}(\rho\cos\theta, \rho\sin\theta, \tau) \\
&\quad = \tau\rho^{n-3-2i}(1-\tau^2/\rho^2)^{n-2i-3/2}F(n-i,-i;3/2;\tau^2/\rho^2)Y_{k,n}(\theta), \\
&\qquad n \geq 3;\ \ k = 0,1;\ \ i = 0,1,\cdots,[(n-3)/2], \\
&\qquad\quad \rho = \sqrt{x^2+y^2} \geq 0,\ \ 0 \leq \theta = \mathrm{Arctg} y/x < 2\pi,
\end{aligned}
$$

where $Y_{0,n}(\theta) = \cos n\theta, Y_{1,n}(\theta) = \sin n\theta, F(a,b,c;t)$-hypergeometric functions, are nontrivial solutions of the equations (1) in the domain Q and

$$u_{i,k,n}(x,y,\tau) \in C(\bar{Q}) \cup C^2(Q)$$

and they are satisfied to the homogeneous boundary conditions: $u|_{Ki} \equiv 0,\ \ i = 0,1$. Therefore, a well-posed formulation of boundary value problems for equation (1) in Q has attracted the attention of many authors (see [6]-[8] and the papers cited in their references). In [9]-[11] sufficient conditions were given that are essential for uniqueness of solution of problem D_0 and of the following problem.

Problem D_1. Find a solution $u(x,y,\tau)$ of equation (1) in Q such that $u \in C(\bar{Q}) \cap C^1(Q \cup K_1) \cup C^2(Q)$ and

$$u|_{\bar{K}_0} = \varphi_0,\ \ \partial u/\partial\tau|_{K_1} = \varphi_1.$$

The existence of a classical solution satisfying the uniqueness conditions was proved in [10] for problems $D_i,\ \ i = 0,1$.

In this note we shall consider the following conjugate boundary value problems.

Problem D_0^*. Find a solution $u(x,y,\tau)$ of the equation (1) in Q such that $u \in C(\bar{Q})\setminus O(0,0,0)) \cap C^2(Q)$ and

$$u|_{K_1} = \varphi_1,\ \ u|_{\bar{K}_2} = \varphi_2 \tag{3}$$

where the $\varphi_i,\ \ i = 1,2$, are given functions.

PROBLEM D_1^*. Find a solution $u(x,y,\tau)$ of the equation (1) in Q such that $u \in C(\bar{Q} \setminus O(0,0,0)) \cap C^1(Q \cup K_1) \cap C^2(Q)$ and

$$\frac{\partial u}{\partial \tau}|_{K_1} = \varphi_1, \; u|_{\bar{K}_2} = \varphi_2. \tag{4}$$

We note the conjugate boundary value problems (1), (3) and (1), (4) are overdetermined in the class of functions $C(\bar{Q}) \cap C^2(Q)$ because the homogeneous Problems D_i have the infinite-dimensional linear space of solutions. We denote by U the class of unbounded functions

$$u(x,y,\tau) \in C((\bar{Q}) \setminus O(0,0,0)) \cap C^1(Q \cup \cup_{i=0}^2 K_i) \cap C^2(Q) \tag{5}$$

and which are represented in the form of

$$\begin{aligned} u(x,y,\tau) &= u(\rho\cos\theta, \rho\sin\theta, \tau) \\ &= \sum_{n=0}^{\infty} \rho^{-n} \sum_{k=0}^{1} \nu_{n,k}(\rho,\tau) Y_{k,n}(\theta) \end{aligned} \tag{6}$$

Where the functions

$$\nu_{n,k}(\rho,\tau) \in C(\bar{G}) \cap C^1(G \cup \cup_{i=0}^2 \Gamma_i) \cap C^2(G) \tag{7}$$

the domain $G: 0 < \tau < \rho < 1 - \tau$; the lines

$$\begin{aligned} &\Gamma_0 : 0 < \tau = \rho < 1/2, \\ &\Gamma_1 : \tau = 0, 0 < \rho < 1, \\ &\Gamma_2 : 0 < \tau = 1 - \rho < 1/2. \end{aligned}$$

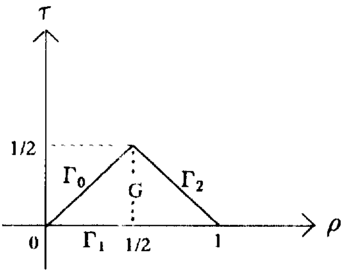

FIGURE 2.

THEOREM 1. (A UNIQUENESS OF SOLUTIONS). *Let $l = 0$ or $l = 1$ and the functions $u_i(x, y, \tau), i = 1, 2$, are the solutions of the equation (1) in Q with the boundary conditions (3) if $l = 0$ or (4) if $l = 1$ and $u_i(x, y, \tau) \in U, \quad i = 1, 2$. Then $u_1(x, y, \tau) \equiv u_2(x, y, \tau)$ in Q.*

2. Proof of Theorem 1

Evidently the function $u = u_1 - u_2$ is a solution of the equation (1) in Q and belongs to the class U of functions (5) and (6) and it satisfies to homogeneous boundary conditions

$$\frac{\partial^l u}{\partial \tau^l}|_{K_1=0}, u|_{K_2} = 0 \tag{8}$$

Then we may show that functions $\nu_{n,k}(\rho, \tau)$ in (6) are solutions of equation (respectively)

$$L_n \nu = \Big(\frac{\partial^2}{\partial \rho^2} + \frac{1-2n}{\rho}\frac{\partial}{\partial \rho} - \frac{\partial^2}{\partial \tau^2}\Big)\, \nu(\rho, \tau) = 0 \text{ in } G,\ n = 0, 1, 2, \cdots \tag{9}$$

and satisfy to the homogeneous boundary conditions

$$\frac{\partial^l \nu}{\partial \tau^l}|_{\Gamma_1} = 0, \quad \nu|_{\Gamma_2} = 0 \tag{10}$$

because the functions

$$\nu_{n,k}(\rho, \tau) = \frac{\rho^n}{\pi}\int_{-\pi}^{\pi} u(\rho\cos\theta, \rho\sin\theta, \tau) Y_{k,n}(\theta) d\theta$$

Let the domain $G_\varepsilon : 0 < \tau < \rho - \varepsilon < 1 - \tau - \varepsilon$; the lines

$$\Gamma_0^\varepsilon : 0 < \tau = \rho - \varepsilon < (1-\varepsilon)/2,$$
$$\Gamma_1^\varepsilon : \tau = 0, \varepsilon < \rho < 1,$$
$$\Gamma_2^\varepsilon : (1+\varepsilon)/2 < \rho = 1 - \tau < 1,$$

where arbitrary number $\varepsilon : 0 < \varepsilon < 1/2$.

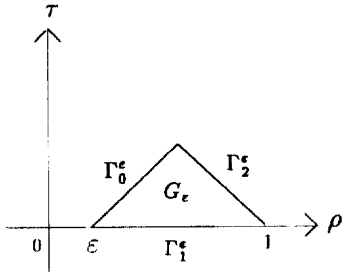

FIGURE 3.

If $n \geq 1$ then by integrationg on parts $\frac{\partial\nu}{\partial\rho} L_n \nu$ in G_ε we get

(11)

$$\int_{\Gamma_1^\varepsilon} \frac{\partial\nu}{\partial\tau}\frac{\partial\nu}{\partial\rho} d\Gamma_1^\varepsilon + \frac{\sqrt{2}}{4}\int_{\Gamma_2^\varepsilon}\left(\frac{\partial\nu}{\partial\rho} - \frac{\partial\nu}{\partial\tau}\right)^2 d\Gamma_2^\varepsilon - \frac{\sqrt{2}}{4}\int_{\Gamma_2^\varepsilon}\left(\frac{\partial\nu}{\partial\tau} + \frac{\partial\nu}{\partial\rho}\right)^2 d\Gamma_0^\varepsilon$$
$$- (2n-1)\int\int_{G\varepsilon} \rho^{-1}\left(\frac{\partial\nu}{\partial\rho}\right)^2 dG_\varepsilon = 0$$

So from (7) and (10) we have

$$\int_{\Gamma_1^\varepsilon} \frac{\partial\nu}{\partial\tau}\frac{\partial\nu}{\partial\rho} d\Gamma_1^\varepsilon = 0$$

$$\int_{\Gamma_2^\varepsilon}\left(\frac{\partial\nu}{\partial\rho} - \frac{\partial\nu}{\partial\tau}\right)^2 d\Gamma_2^\varepsilon = 0.$$

Therefore from (11) we get $\frac{\partial\nu}{\partial\rho} \equiv 0$ in G_ε and $\frac{\partial\nu}{\partial\tau} + \frac{\partial\nu}{\partial\rho} \equiv 0$ in Γ_0^ε. Then from the boundary conditions (10) we have $\nu(\rho,\tau) \equiv 0$ in G_ε. When $\varepsilon \to 0$ we get $\nu \equiv 0$ in $\bar{G}$.

Therefore in (6) we have got the functions $\nu_{n,k}(\rho,\tau) \equiv 0 \quad \forall_n \geq 1, \quad k = 0, 1$. Then the function $u(\rho,\theta,\tau) = \nu_{0,0}(\rho,\tau)$.

Now we consider the function

$$W(\rho,\tau) = \int_{1-\tau}^{\rho} \sigma\nu(0,0)(\sigma,\tau)d\sigma, \quad (\rho,\tau) \in G.$$

It may show

$$L_1 W = 0 \text{ in } G,$$

$$\frac{\partial^l W}{\partial\tau^l}|_{\Gamma_1} = 0, \quad W|_{\Gamma_2} = 0.$$

As proved above there we may proved $W \equiv 0$ in $\bar{G}$.
Then

$$\nu_{0,0}(\rho,\tau) = \rho^{-1}\frac{\partial W(\rho,\tau)}{\partial \rho} \equiv 0 \text{ in } G.$$

Thus $u(x,y,\tau) = u_1(x,y,\tau) - u_2(x,y,\tau) \equiv 0$ in $\bar{Q}$.
So $u_1(x,y,\tau) \equiv u_2(x,y,\tau)$ in $\bar{Q}$. Theorem 1 is proved.

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