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THE INVARIANCE PRINCIPLE FOR LINEARLY POSITIVE QUADRANT DEPENDENT RANDOM FIELDS

TAE-SUNG KIM AND HYE-YOUNG SEO

1. Introduction

Let Z^d denote the set of all d-tuples of integers($d \geq 1$, a positive integer). The points in Z^d will be denoted by $\underline{m}, \underline{n}$, etc., or sometime, when necessary, more explicitly by $(m_1, m_2, \cdots, m_d)$, $(n_1, n_2, \cdots, n_d)$ etc. Z^d is partially ordered by stipulating $\underline{m} \leq \underline{n}$ iff $m_i \leq n_i$ for each i, $1 \leq i \leq d$. We write $\underline{0}, \underline{1}$ and $n\underline{1}$ respectively for points $(0, 0, \cdots, 0)$, $(1, 1, \cdots, 1)$ and $(n, n, \cdots, n)$ in Z^d. For $\underline{n} = (n_1, \cdots, n_d)$, let $|\underline{n}|$ stand for the product $n_1 \times n_2 \times \cdots \times n_d$. Define $|\underline{t}|$ similarly for $\underline{t} \in [\underline{0}, \underline{1}]$.

Let $\{X_{\underline{j}} : \underline{j} = (j_1, j_2) \in Z^d\}$ be a random field, i.e., a collection of random variables indexed by time set Z^d, on some probability space $(\Omega, \mathcal{F}, P)$ with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. For $n \in N$ put

$$S_{n\underline{1}} = \sum_{\underline{1} \leq \underline{j} \leq n\underline{1}} X_{\underline{j}}, \tag{1.1}$$

assume

$$n^{-d} E(S_{n\underline{1}}^2) \longrightarrow_n \sigma^2 \in (0, \infty). \tag{1.2}$$

Define

$$W_n(\underline{t}) = W_n((\underline{0}, \underline{t}]) = (\sigma^2 n^{\frac{d}{2}})^{-1} \sum_{j_1=1}^{[nt_1]} \cdots \sum_{j_d=1}^{[nt_d]} X_{\underline{j}}, \tag{1.3}$$

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where $W_n(\underline{t}) = 0$ for some $t_i = 0$ and for $B = (\underline{s}, \underline{t}] \subset [\underline{0}, \underline{1}]$

$$W_n(B) = (\sigma^2 n^{\frac{d}{2}})^{-1} \sum_{[n\underline{s}] < \underline{j} \leq [n\underline{t}]} X_{\underline{j}}. \tag{1.3)'$$

Then W_n is a measurable map from $(\Omega, \mathcal{F})$ into $(D_d, \mathcal{B}(D_d))$, where D_d is the set of all functions on $[0,1]^d$ which have left limits and are continuous from the right, and $\mathcal{B}(D_d)$ is the Borel σ -field induced by the Skorohod topology. $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ fulfills the invariance principle (functional central limit theorem) if W_n converges weakly to the $d-$ parameter Wiener process W on D_d.

A random field $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ is said to be pairwise positive quadrant dependent (pairwise PQD) if for any real $r_{\underline{i}}, r_{\underline{j}}$ and $\underline{i} \neq \underline{j}$

$$P\{X_{\underline{i}} > r_{\underline{i}}, X_{\underline{j}} > r_{\underline{j}}\} \geq P\{X_{\underline{i}} > r_{\underline{i}}\} P\{X_{\underline{j}} > r_{\underline{j}}\}. \tag{1.4}$$

A much stronger concept than PQD was considered by Esary, Proschan and Walkup[6]; A random field $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ is said to be associated if for any finite collection $\{X_{\underline{j}(1)}, \cdots, X_{\underline{j}(m)}\}$ and any real coordinatewise increasing functions f, g on R^m

$$Cov[f(X_{\underline{j}(1)}, \cdots, X_{\underline{j}(m)}), g(X_{\underline{j}(1)}, \cdots, X_{\underline{j}(m)})] \geq 0, \tag{1.5}$$

whenever the covariance is defined. Newman[11] first introduced the concept of linearly positive quadrant dependent notion. We extend this notion to the random field, that is, we say that a random field $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ is linearly positive quadrant dependent(LPQD) if for any disjoint subsets A, B of $\mathcal{Z}^d$ and positive $r_{\underline{j}}' s$

$$\sum_{\underline{i} \in A} r_{\underline{i}} X_{\underline{i}} \quad \text{and} \quad \sum_{\underline{j} \in B} r_{\underline{j}} X_{\underline{j}} \text{ are PQD.} \tag{1.6}$$

Linearly positive quadrant dependence implies, in particular, nonnegative correaltions of the random variables $X_{\underline{j}}$. The following theorem is an extension of the central limit theorem for associated random fields of Cox and Grimmett[4] to linearly positive quadrant dependent random fields by using conditions on the coefficient of maximal covariance

$$u(r) = \sup_{\underline{1} \leq \underline{k} \leq n\underline{1}} \sum_{\underline{j} : \|\underline{j} - \underline{k}\| \geq r} \operatorname{Cov}(X_{\underline{j}}, X_{\underline{k}})$$

where $\|\underline{j}\| = \max(|j_1|, \cdots, |j_d|)$.

Theorem A(Kim, 1995). *Let $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ be a linearly positive quadrant dependent random field with $EX_{\underline{j}} = 0$. Assume*

(i) $inf_{\underline{1} \le \underline{j} \le n\underline{1}} Var(X_{\underline{j}}) \ge 0$,

(ii) $sup_{\underline{1} \le \underline{j} \le n\underline{1}} E|X_{\underline{j}}|^3 < \infty$,

(iii) $u(0) < \infty, u(r) \to 0$, as $r \to \infty$.

Then $(S_{n\underline{1}} - E(S_{n\underline{1}}))/(Var(S_{n\underline{1}}))^{\frac{1}{2}}$ is asymptotically normally distributed as $n \to \infty$.

Theorem A can be extended to the invariance principle if instead of LPQD the stronger concept of association is required:

Theorem B(Kim 1995). *Let $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ be an associated random field with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$ and define $W_n(\cdot)$ as in (1.3). Assume*

$$E\{W_n(\underline{s})W_n(\underline{t})\} \to_n |\underline{s}| \text{ for } \underline{0} \le \underline{s} \le \underline{t} \le \underline{1}, \tag{1.7}$$

$$E|W_n(B)|^{2+\delta} \le C|B| \text{ for some finite } C \tag{1.8}$$

where, $B = (\underline{s}, \underline{t}] \subset [\underline{0}, \underline{1}]$ and $W_n(B) = (\sigma^2 n^{\frac{d}{2}})^{-1} \sum_{[n\underline{s}] < \underline{j} \le [n\underline{t}]} X_{\underline{j}}$. Then $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ fulfills the invariance principle.

In this note we show that Theorem B still holds for LPQD random fields and present that the LPQD random field with Lebowitz inequality property satisfies the invariance principle.
All result are stated in Section 2. The proofs of our theorems as well as some lemmas are given in Section 3. In Section 4 we also apply it to the random measures.

2. Results

Theorem 2.1. *Let $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ be an LPQD random field with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Assume*

$$E\{W_n(\underline{s})W_n(\underline{t})\} \to_n |\underline{s}| \quad \text{for} \quad \underline{0} \le \underline{s} \le \underline{t} \le \underline{1}, \tag{2.1}$$

$$\{(W_n(\underline{t}) - W_n(\underline{s}))^2 : \underline{t} \in [0,1]^d, n \ge 1\} \text{ is uniformly integrable,} \tag{2.2}$$

and for every $\varepsilon > 0$

$$\text{(2.3)} \qquad lim\ sup_{n\geq 1} P\{w(W_n, \delta) > \varepsilon\} \to_n 0 \text{ as } \delta \downarrow 0$$

where $w(W_n, \delta) = sup\{|W_n(\underline{s}) - W_n(\underline{t})| : ||\underline{t} - \underline{s}|| < \delta\}$.
Then $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *fulfills the invariance principle.*
The following theorem shows that condition (2.1) *is necessary for the invariance principle:*

THEOREM 2.2. *Let* $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *be an LPQD random field with* $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$ *and define* $W_n(\cdot)$ *as in* (1.3). *If* $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *fulfills the invariance principle, then condition* (2.1) *holds.*

THEOREM 2.3. *Let* $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *be an LPQD random field with* $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. *Assume*

$$\text{(2.4)} \qquad E\{W_n(\underline{s})W_n(\underline{t})\} \to_n |\underline{s}| \quad \text{for} \quad \underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{1}$$

$$\text{(2.5)} \qquad E|W_n(B)|^{2+\delta} \leq C|B|^{1+\frac{\delta}{2}} \quad \text{for some } C$$

where $B = (\underline{s}, \underline{t}], \underline{0} \leq \underline{s}, \underline{t} \leq \underline{1}$ *and* $W_n(B) = (\sigma^2 n^{\frac{d}{2}})^{-1} \sum_{[n\underline{s}] < \underline{j} \leq [n\underline{t}]} X_{\underline{j}}$.
Then $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *fulfills the invariance principle.*

We say that the random field $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ satisfies the Lebowitz Inequality if

$$\text{(2.6)} \qquad \begin{aligned} E(X_{\underline{i}}X_{\underline{j}}X_{\underline{k}}X_{\underline{l}}) &\leq E(X_{\underline{i}}X_{\underline{j}})E(X_{\underline{k}}X_{\underline{l}}) \\ &+ E(X_{\underline{i}}X_{\underline{k}})E(X_{\underline{j}}X_{\underline{l}}) + E(X_{\underline{i}}X_{\underline{l}})E(X_{\underline{j}}X_{\underline{k}}) \end{aligned}$$

THEOREM 2.4. *Let* $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *be an LPQD random field with* $EX_{\underline{j}} = 0, EX_{\underline{j}}^4 < \infty$ *and satisfy the Lebowitz Inequality and define* $W_n(\cdot)$ *as in* (1.3). *Assume* (1.2), (iii) *of Theorem A, and* (2.4). *Then* $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ *fulfills the invariance principle.*

3. Proof

LEMMA 3.1(LEHMANN, 1966). *Let $X_{\underline{i}}, X_{\underline{j}}$ be PQD random variables with finite variance. Then*

(i) $Cov(X_{\underline{i}}, X_{\underline{j}}) \geq 0$,

(ii) $Cov(X_{\underline{i}}, X_{\underline{j}}) = 0$ *if and only if* $X_{\underline{i}}, X_{\underline{j}}$ *are independent.*

LEMMA 3.2(BIRKEL, 1993). *For each $k \geq 1$ let $X_{\underline{i}}^{(k)}, X_{\underline{j}}^{(k)}$ be PQD random variables such that*

$$(X_{\underline{i}}^{(k)}, X_{\underline{j}}^{(k)}) \longrightarrow_k (X_{\underline{i}}, X_{\underline{j}}) \text{ in distribution.} \tag{3.1}$$

Then $X_{\underline{i}}, X_{\underline{j}}$ are PQD.

It is easy to see that Theorem 2.1 of Kim[9] still holds for random variables which are nonnegatively correlated. Hence we obtain :

LEMMA 3.3. *Let $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ be an LPQD random field with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Assume*

$$E\{W_n^2(\underline{t})\} \rightarrow_n |\underline{t}| \text{ for } \underline{t} \in [\underline{0}, \underline{1}]. \tag{3.2}$$

Then the following conditions are equivalent:

(i) $E\{W_n(\underline{s})W_n(\underline{t})\} \rightarrow_n |\underline{s}|$ *for* $\underline{o} \leq \underline{s} \leq \underline{t} \leq \underline{1}$,

(ii) $E\{(W_n(\underline{t}) - W_n(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u}))\} \rightarrow_n 0$,

$$\text{for } \underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{u} \leq \underline{v} \leq \underline{1}.$$

A subset B of $[\underline{0}, \underline{1}]$ is called a block if it is of the form $(\underline{s}, \underline{t}] = \Pi_{j=1}^d (s_j, t_j]$ where $\underline{s} = (s_1, \cdots, s_d)$, $\underline{t} = (t_1, \cdots, t_d)$, and (s_j, t_j)'s are half closed subintervals of $[\underline{0}, \underline{1}]$. For each i, $1 \leq i \leq d$, let $0 = a_1^{(i)} < b_1^{(i)} < a_2^{(i)} < b_2^{(i)} < \cdots < a_{n_i}^{(i)} < b_{n_i}^{(i)} = 1$ be real numbers. Call a collection of blocks in $[\underline{0}, \underline{1}]$ 'strongly separated' if it is of the form $\{\Pi_{i=1}^d (a_{k_i}^{(i)}, b_{k_i}^{(i)}]; 1 \leq k_i \leq n_i, 1 \leq i \leq d\}$, or if it is a subfamily of such a family of blocks.

The following lemma is obtained by Deo[5](see Lemma 3 of Deo[5]):

LEMMA 3.4(DEO, 1976). *Let $\{W_n\}$ be a collection stochastic processes in $\mathcal{D}_d$ such that,*

(i) $EW_n(\underline{t}) \longrightarrow_n 0$, $EW_n^2(\underline{t}) \longrightarrow |\underline{t}|$ *for each* $\underline{t} \in (\underline{0}, \underline{1}]$,

(ii) $\{W_n^2(\underline{t})\}$ *is uniformly integrable for each* $\underline{t} \in (\underline{0}, \underline{1}]$,

(iii) *if* $B_1, \cdots, B_k$ *are a collection of strongly separated blocks then the increments* $W_n(B_1), W_n(B_2), \cdots . W_n(B_k)$ *are asymptotically independent,*

(iv) *for each* $\varepsilon > 0$, $\eta > 0$ *we can find a* $\delta > 0$ *such that* $P\{w(W_n, \delta) > \varepsilon\} < \eta$ *for all sufficiently large* n. *Then* $\{W_n\}$ *converges weakly in* $\mathcal{D}_d$, *to the Wiener process.*

Proof of Theorem 2.1. Condition (2.1) implies (3.2) and hence, we have (ii) of Lemma 3.3. We will apply Theorem 19.1 of Billingsley [2]. Since $W_n(\underline{0}) = 0$, (2.3) here and Theorem 15.5 of Billingsley [2] yield the tightness of the sequence $\{W_n : n \geq 1\}$. Let X be a limit in distribution of a subsequence of $\{W_n : n \geq 1\}$. Then $P\{X \in C[\underline{0}, \underline{1}]\} = 1$ by Theorem 15.5 of [2]. It suffices to show that X is distributed like W. By (2.2) and (3.2) $\{W_n(\underline{t}) : n \geq 1\}$ and $\{W_n^2(\underline{t}) : n \geq 1\}$ are uniformly integrable for every $\underline{t} \in [\underline{0}, \underline{1}]$. As

$$W_n(\underline{t}) \to_n X(\underline{t}), \quad W_n^2(\underline{t}) \to_n X^2(\underline{t})$$

in distribution (for a subsequence), Theorem 5.4 of Billingsley [2] and (3.2) imply

$$EX(\underline{t}) = 0, \quad EX^2(\underline{t}) = |\underline{t}|.$$

According to Theorem 19.1 of Billingsley [2], X is distributed like W if X has independent increments, that is

(3.3)

$$X(\underline{t}_1) - X(\underline{t}_0), \cdots, X(\underline{t}_k) - X(\underline{t}_{k-1}) \text{ are independent for all } k \geq 1,$$

$$\underline{0} \leq \underline{t}_0 \leq \underline{t}_1 \leq \cdots \leq \underline{t}_k \leq \underline{1}.$$

To show (3.3) put

$$U_{ni} = W_n(\underline{t}_i) - W_n(\underline{t}_{i-1}), \ 1 \leq i \leq k.$$

Since $(U_{n1}, \cdots, U_{nk}) \to_n (X(\underline{t}_1) - X(\underline{t}_0), \cdots, X(\underline{t}_k) - X(\underline{t}_{k-1}))$ in distribution(for a subsequence), and since the U_{ni} are pairwise PQD,

$$X(\underline{t}_1) - X(\underline{t}_0), \cdots, X(\underline{t}_k) - X(\underline{t}_{k-1})$$

are pairwise PQD, according to Lemma 3.2. A similar argument as above(using Theorem 5.4 of Billingsley[2]) yields, for $i \neq j$,

$$\mathrm{Cov}(X(\underline{t}_i) - X(\underline{t}_{i-1}), X(\underline{t}_j) - X(\underline{t}_{j-1})) = \lim_{n\geq 1}\mathrm{Cov}(U_{ni}, U_{nj}) = 0,$$

according to (ii) of Lemma 3.3. Hence the $X(\underline{t}_i) - X(\underline{t}_{i-1})$ are pairwise PQD and uncorrelated random variables and thus independent by Lemma 3.1. This proves (3.3). This completes the proof.

Proof of Theorem 2.2. Since the invariance principle is fulfilled $\{W_n^2(\underline{t}) : n \geq 1\}$ is uniformly integrable and hence

$$E\{W_n^2(\underline{t})\} \to_n E\{W^2(\underline{t})\} = |\underline{t}| \text{ for } \underline{t} \in [\underline{0}, \underline{1}], \tag{3.4}$$

according to Theorem 5.4 of Billingsley[2]. By Lemma 3.3 it remains to show

$$E\{(W_n(\underline{t}) - W_n(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u}))\} \to_n 0 \tag{3.5}$$

for $\underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{u} \leq \underline{v} \leq \underline{1}$.

To prove (3.5), let $\underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{u} \leq \underline{v} \leq \underline{1}$ be given. Since the invariance principle is fulfilled, $\{W_n^2(\underline{t}) : n \geq 1\}$ is uniformly integrable. Hence

$$\{(W_n(\underline{t}) - W_n(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u})) : n \geq 1\} \tag{3.6}$$

is uniformly integrable by (3.4). As

$$(W_n(\underline{t}) - W_n(\underline{s}), W_n(\underline{v}) - W_n(\underline{u})) \to_n (W(\underline{t}) - W(\underline{s}), W(\underline{v}) - W(\underline{u}))$$

in distribution, according to Theorem 5.4 of Billingsley[2] and (3.6)

$$\begin{aligned} &E\{(W_n(\underline{t}) - W(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u}))\} \\ &\qquad \to_n E\{(W(\underline{t}) - W(\underline{s}))(W(\underline{v}) - W(\underline{u}))\}. \end{aligned}$$

But

$$\begin{aligned} &E\{(W(\underline{t}) - W(\underline{s}))(W(\underline{v}) - W(\underline{u}))\} \\ &\quad = E\{W(\underline{t}) - W(\underline{s})\}E\{W(\underline{v}) - W(\underline{u})\} = 0, \end{aligned}$$

which proves (3.5). Thus by Lemma 3.3 we obtain

$$E(W_n(\underline{t})W_n(\underline{s})) \to_n |\underline{s}| \text{ for } \underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{1}.$$

Proof of Theorem 2.3. The proof of Theorem 2.3 can now be completed by applying Lemma 3.4(Lemma 3 of Deo[5]) to the sequence $\{W_n\}$: That the conditions (i) and (ii) of this lemma are satisfied by our sequence $\{W_n\}$ here is a straightforward verification from assumptions $EX_{\underline{j}} = 0$ (2.4) and (2.5). The condition (iii) of Lemma 3.4 is also satisfied by $\{W_n\}$ from Lemmas 3.1 and 3.3 and condition (2.4) here. Finally condition (iv) of this lemma is satisfied by $\{W_n\}$ because of assumption (2.5) of Theorem 2.3 here and the equation (1) and Theorem 1 of Bickel and Wichura[1]. This completes the proof.

Proof of Theorem 2.4. Let $B = (\underline{s}, \underline{t}], \underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{1}$. First note that

$$E|W_n(B)|^3 \leq (E(W_n(B))^4)^{\frac{3}{4}}.$$

where $W_n(B) = (\sigma^2 n^{\frac{d}{2}})^{-1} \sum_{[n\underline{s}]<\underline{j}\leq[n\underline{t}]} X_{\underline{j}}$ for $B = (\underline{s}, \underline{t}] \subset [\underline{0}, \underline{t}]$. The Lebowitz Inequality and the linear positive quadrant dependence give

$$\begin{aligned}
&E(W_n(B))^4 \\
&= E((\sigma^2 n^{\frac{d}{2}})^{-1} \sum_{[n\underline{s}]<\underline{j}\leq[n\underline{t}]} X_{\underline{j}})^4 \\
&= (\frac{1}{\sigma^8})(\frac{1}{n^{2d}}) \sum_{[n\underline{s}]<\underline{i},\ \underline{j},\ \underline{k},\ \underline{l}\leq[n\underline{t}]}\sum\sum\sum E(X_{\underline{i}}X_{\underline{j}}X_{\underline{k}}X_{\underline{l}}) \\
&\leq (\frac{1}{\sigma^8})(\frac{3}{n^{2d}}) \sum_{[n\underline{s}]<\underline{i},\ \underline{j},\ \underline{k},\ \underline{l}\leq[n\underline{t}]}\sum\sum\sum [\sup\{E(X_{\underline{i}}X_{\underline{j}})E(X_{\underline{k}}X_{\underline{l}})\}] \\
&\leq (\frac{3}{\sigma^8})(\frac{1}{n^{2d}}) \sum_{[n\underline{s}]<\underline{i},\underline{k}\leq[n\underline{t}]}\sum \\
&\qquad [\sup_{\underline{i}}(\sum_{\underline{j}:|\underline{j}-\underline{i}|\geq 0} \mathrm{Cov}(X_{\underline{i}}, X_{\underline{j}}))\sup_{\underline{k}}(\sum_{\underline{l}:|\underline{k}-\underline{l}|\geq 0} \mathrm{Cov}(X_{\underline{k}}, X_{\underline{l}}))] \\
&\leq (\frac{3}{\sigma^8})(\frac{1}{n^{2d}}) n^{2d} |\underline{t} - \underline{s}|^2 u^2(0) \\
&= C|B|^2.
\end{aligned}$$

Thus we have $E|W_n(B)|^3 \le C|B|^{\frac{3}{2}}$. This completes the proof according to Theorem 2.3.

4. Applications

In this section we will apply the notions of LPQD random fields to the random measures, that is, a simple argument using Chebyshev's inequality allows us to extend the invariance principle for LPQD random fields to random measure. $\mathcal{B}^d$ denotes the collection of Borel subsets of $d-$dimensional Euclidean space R^d. The space M of all nonnegative measure μ defined on $(R^d, \mathcal{B}^d)$ and finite on bounded sets will be equipped with the smallest $\sigma-$field containing basic sets of the form $\{\mu \in M : \mu(A) \le r\}$ for $A \in \mathcal{B}^d$, $0 \le r < \infty$. A random measure X is a measurable map from a probability space $(\Omega, \mathcal{F}, P)$ into $(M, \mathcal{M})$, the induced measure $P_X = P \circ X^{-1}$ on $(M, \mathcal{M})$ is the distribution of X and if X is a random measure and $\mathcal{B}^d$ is a Borel subset of R^d then $X(B)$ represents the random mass of the region B. (see Kallenberg[7]).

For the random measure X define the $K-$renormalization of X to be the signed random measure X_K where

$$X_K(B) = \frac{X(KB) - EX(KB)}{\sigma K^{\frac{d}{2}}} \tag{4.1}$$

and let $X_K(\underline{t}) = X_K((t_1, \cdots, t_d))$ be defined by

$$X_K(\underline{t}) = X_K((0, t_1] \times \cdots \times (0, t_d]) \tag{4.2}$$

for $\underline{t} \in [0, \infty)^d$. Let $\{X_K\}$ be a sequence of random measures on R^d. A set function X_K satisfies the central limit theorem if for any bounded $B \in \mathcal{B}^d, X_K(B)$ converges in distribution to $N(0, |B|)$ as $K \to \infty$ where $X_K(B)$ is difined in (4.1) and $|B|$ denotes the Lebesgue measure of B and the random measure X satisfies the invariance principle if X_K converges weakly to the $d-$dimensional Wiener measure W.

DEFINITION 4.1. A random measure X is linearly positive quadrant dependent if and only if the family of random variables $\mathcal{F} = \{X(B) :$ B a Borel set $\}$ is LPQD.

THEOREM 4.2. *Let X be an LPQD random measure with $EX(B) = 0$,*
$EX^2(B) < \infty$ and define $X_K(\underline{t})$ as in (4.2). Assume

$$E\{X_K(\underline{s})X_K(\underline{t})\} \to_K |\underline{s}| \quad \text{for } \underline{0} \le \underline{s} \le \underline{t} \le \underline{1}. \tag{4.3}$$

For $A \in \mathcal{B}^d, A$ bounded, $|A| > 1$, there exists constants $C < \infty$, and $\delta > 0$ such that

$$E(|X(A) - EX(A)|^{2+\delta}) \le C(\sigma^2|A|)^{(1+\frac{\delta}{2})}. \tag{4.4}$$

Then X satisfies the invariance principle.

Proof. Note that for a block $B \subset [0,1]^d$

$$X_K(B) = \frac{X(KB) - EX(KB)}{\sigma K^{\frac{d}{2}}} \tag{4.5}$$

where, if $B = \Pi_{i=1}^d(s_i, t_i]$, then $KB = \Pi_{i=1}^d(Ks_i, Kt_i]$.
As the similar arguments to the proof of Theorem 2.3 the proof of Theorem 4.2 can be now completed by applying Lemma 3.4 to the sequence $\{X_K\}$.
From (4.5) and condition (4.3) it is easily seen

$$E(X_K(\underline{t})) = 0, \quad EX_K^2(\underline{t}) \to_K |\underline{t}| \tag{4.6}$$

which satisfies condition (i) of this lemma. By condition (4.4), for K large enough,

$$E(|X_K(\underline{t})|^{2+\delta}) \le \frac{1}{(\sigma K^{\frac{d}{2}})^{2+\delta}} C(\sigma^2 K^d |\underline{t}|)^{1+\frac{\delta}{2}} = C|\underline{t}|^{1+\frac{\delta}{2}} \tag{4.7}$$

and so $\{X_K(\underline{t})\}$ and $\{X_K^2(\underline{t})\}$ are uniformly integrable for every $\underline{t} \in [0,1]^d$, that is, condition (ii) of this lemma is satisfied. To prove that $\{X_K\}$ satisfies condition (iii) of this lemma let $B_1, \cdots, B_m \subset [0,1]^d$ be strongly separated blocks, and let $B_i = (\underline{s}, \underline{t}], B_j = (\underline{u}, \underline{v}]$, where $\underline{0} \le \underline{s} \le \underline{t} \le \underline{u} \le \underline{v} \le \underline{1}$. Since the random variables $X(\underline{I_j})'s$ are nonnegative correlated it follows from (4.3) that
(4.8)
$$\mathrm{Cov}(X_K(B_i), X_K(B_j)) \le \mathrm{Cov}(X_K(\underline{t}) - X_K(\underline{s}), X_K(\underline{v}) - X_K(\underline{u})) \to_K 0$$

according to Lemma 3.3, where $I_{\underline{j}} = (\underline{j} - \underline{1}, \underline{j}]$ for $\underline{1} \leq \underline{j} \in Z^d$. Since $X_K(B_{\underline{j}})'s$ are LPQD, by Theorem 6 of Newman[11] and (4.8) the $X_K(B_{\underline{j}})$'s are independent as $K \to \infty$. Finally condition (iv) of this lemma is satisfied by $\{X_K\}$ because of (4.7) here and the equation (1) and Theorem 1 of Bickel and Wichura[1]. This completes the proof.

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Department of Statistics
Won-Kwang University
Iksan 570-749, Korea