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A RELATIVE NIELSEN COINCIDENCE NUMBER FOR THE COMPLEMENT, I

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1. Introduction

Nielsen coincidence theory is concerned with the determination of a lower bound of the minimal number $MC[f,g]$ of coincidence points for all maps in the homotopy class of a given map $(f,g) : X \to Y$. The Nielsen number $N(f,g)$ is always a lower bound for $MC[f,g]$. The relative Nielsen number $N_R(f,g)$ (similar to [9]) is introduced in [3], which is a lower bound for the number of coincidence points in the relative homotopy class of (f,g) and $N_R(f,g) \geq N(f,g)$.

It is the purpose of this paper to determine the minimal number $MC[f,g;X-A]$ of coincidence points on the complement $X-A$. The Nielsen number on the complementary space, $N(f,g;X-A)$ is defined, which is a lower bound for $MC[f,g;X-A]$, and has the basic properties. The method used here follows that of Zhao[10].

2. Weakly common coincidence classes

Let $f,g : (X,A) \to (Y,B)$ be a pair of maps between pairs of compact polyhedra with X, Y connected. We denote the set of coincidence points of the pair (f,g) by

$$\Gamma(f,g) = \{x \in X | f(x) = g(x)\}.$$

We shall write $(\bar{f},\bar{g}) : A \to B$ for the restriction of $(f,g) : (X,A) \to (Y,B)$ to A and write $(f,g) : X \to Y$ if the condition that $(f,g)(A) \subset B$ is immaterial. The homotopies of (f,g) are maps of the form (F,G) :

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$(f_0, g_0) \simeq (f_1, g_1) : (X \times I, A \times I) \to (Y, B)$, i.e. F is a homotopy from f_0 to f_1 and G is a homotopy from g_0 to g_1. For this map (f, g), let $\hat{A} = \cup_1^n A_k$ be the disjoint union of all components of A such that for each k, A_k is mapped by f and g into some component B_k of B. Then we shall write $(f_k, g_k) : A_k \to B_k$ for the restriction of (f, g) to A_k. Then we have a morphism of maps

$$\begin{array}{ccc} A_k & \xrightarrow{(f_k, g_k)} & B_k \\ {\scriptstyle i_k}\downarrow & & \downarrow{\scriptstyle j_k} \\ X & \xrightarrow{(f,g)} & Y \end{array}$$

where i_k, j_k are inclusions. X, Y and the components of $\hat{A}, \hat{B}$ have universal coverings

$$p : \widetilde{X} \to X, \qquad q : \widetilde{Y} \to Y$$
$$p_k : \widetilde{A}_k \to A_k, \qquad q_k : \widetilde{B}_k \to B_k, \qquad k = 1, \ldots, n$$

For each k, we pick a lifting $(\tilde{i}_k, \tilde{j}_k)$ of (i_k, j_k) such that the diagrams

$$\begin{array}{ccc} \widetilde{A}_k & \xrightarrow{\tilde{i}_k} & \widetilde{X} \\ {\scriptstyle p_k}\downarrow & & \downarrow{\scriptstyle p} \\ A_k & \xrightarrow{i_k} & X, \end{array} \qquad \begin{array}{ccc} \widetilde{B}_k & \xrightarrow{\tilde{j}_k} & \widetilde{Y} \\ {\scriptstyle q_k}\downarrow & & \downarrow{\scriptstyle q} \\ B_k & \xrightarrow{j_k} & Y \end{array}$$

commute. This $(\tilde{i}_k, \tilde{j}_k)$ determines a correspondence $(\tilde{i}_k, \tilde{j}_k)_{\mathrm{lift}}$ from liftings of (f_k, g_k) to liftings of (f, g). $(\tilde{i}_k, \tilde{j}_k)_{\mathrm{lift}}(\tilde{f}_k, \tilde{g}_k) = (\tilde{f}, \tilde{g})$ if $\tilde{j}_k \circ (\tilde{f}_k, \tilde{g}_k) = (\tilde{f}, \tilde{g}) \circ \tilde{i}_k$. And $(\tilde{i}_k, \tilde{j}_k)_{\mathrm{lift}}$ induces a correspondence from lifting classes of (f_k, g_k) to lifting classes of (f, g) which is independent of the choice of the liftings $(\tilde{i}_k, \tilde{j}_k)$ of $(i_k. j_k)$ and is determined by (i_k, j_k). It is denoted

$$(i_k, j_k)_C : C(f_k, g_k) \longrightarrow C(f, g)$$

where $C(f, g)$ is the set of all lifting classes of (f, g)[3].

PROPOSITION 2.1. *Every coincidence class $p_k\Gamma(\tilde{f}_k, \tilde{g}_k)$ of $(f_k, g_k) : A_k \to B_k$ belongs to some coincidence class $p\Gamma(\tilde{f}, \tilde{g})$ of $(f, g) : X \to Y$. When $p_k\Gamma(\tilde{f}_k, \tilde{g}_k)$ is non-empty, $p_k\Gamma(\tilde{f}_k, \tilde{g}_k)$ belongs to $p\Gamma(\tilde{f}, \tilde{g})$ if and only if $(i_k, j_k)_C[\tilde{f}_k, \tilde{g}_k] = [f, g]$.*

DEFINITION 2.2. A coincidence class $p\Gamma(\tilde{f}, \tilde{g})$ of $(f, g) : X \to Y$ is a weakly common coincidence class of (f, g) and $(\bar{f}, \bar{g})$ if it contains a coincidence class of $(f_k, g_k) : A_k \to B_k$ for some k. It is an essential weakly common coincidence class of (f, g) and $(\bar{f}, \bar{g})$ if it is an essential coincidence class of (f, g) as well as a weakly common coincidence class of (f, g) and $(\bar{f}, \bar{g})$. We write $E(f, g; \bar{f}, \bar{g})$ for the number of essential weakly common coincidence class of (f, g) and $(\bar{f}, \bar{g})$.

THEOREM 2.3. *A coincidence x_0 of (f, g) belongs to a weakly common coincidence class of (f, g) and $(\bar{f}, \bar{g})$ if and only if there is a path α from x_0 to A such that $f \circ \alpha \simeq g \circ \alpha : I, 0, 1 \longrightarrow Y, f(x_0), B$.*

Proof. "Only if". Let x_0 belong to a weakly common coincidence class $p\Gamma(\tilde{f}, \tilde{g})$ of (f, g) and $(\bar{f}, \bar{g})$. Suppose $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0)$. By assumption there exists a lifting $(\tilde{f}_k, \tilde{g}_k)$ of $(f_k, g_k) : A_k \to B_k$ so that $(\tilde{i}_k, \tilde{j}_k)_{\text{lift}}(\tilde{f}_k, \tilde{g}_k) = (\tilde{f}, \tilde{g})$. Pick a point $\tilde{a} \in \tilde{i}_k(\widetilde{A}_k)$, then $\tilde{f}(\tilde{a}), \tilde{g}(\tilde{a}) \in \tilde{j}_k(\widetilde{B}_k)$. Take a path $\tilde{\alpha}$ in $\widetilde{X}$ from $\tilde{x}_0$ to $\tilde{a}$. Since $\widetilde{Y}$ is 1-connected, there is a homotopy of the form

$$\tilde{f} \circ \tilde{\alpha} \simeq \tilde{g} \circ \tilde{\alpha} : I, 0, 1 \longrightarrow \widetilde{Y}, \tilde{f}(\tilde{x}_0), \tilde{j}_k(\widetilde{B}_k).$$

Projecting down to Y, we have

$$f \circ \alpha \simeq g \circ \alpha : I, 0, 1 \longrightarrow Y, f(x_0), B$$

where $\alpha = p \circ \tilde{\alpha}$.

"If". Suppose $x_0 \in p\Gamma(\tilde{f}, \tilde{g}), \tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0)$. Lift a path α from $\tilde{x}_0$ to get a path $\tilde{\alpha}$ in $\widetilde{X}$. Let $a = \alpha(1) \in A_k, b = f_k(a) \in B_k$, and pick $\tilde{a} \in p_k^{-1}(a), \tilde{b} \in q_k^{-1}(b)$, then there are liftings $\tilde{i}_k, \tilde{j}_k$ of i_k, j_k respectively such that

$$\begin{array}{ccccccc}
(\widetilde{A}_k, \tilde{a}) & \xrightarrow{\tilde{i}_k} & (\widetilde{X}, \tilde{\alpha}(1)) & & (\widetilde{B}_k, \tilde{b}) & \xrightarrow{\tilde{j}_k} & (\widetilde{Y}, \widetilde{f \circ \alpha}(1)) \\
{\scriptstyle p_k}\downarrow & & \downarrow{\scriptstyle p} & & {\scriptstyle q_k}\downarrow & & \downarrow{\scriptstyle q} \\
(A_k, a) & \xrightarrow{i_k} & (X, a). & & (B_k, b) & \xrightarrow{j_k} & (Y, b)
\end{array}$$

commutes[6,p.42, Proposition 1.2(i)]. Let $H : I \times I \to Y$ be the homotopy from $f \circ \alpha$ to $g \circ \alpha$, i.e. $H(t,0) = f \circ \alpha, H(t,1) = g \circ \alpha$. Then $\widetilde{f \circ \alpha}$ determines a lifting $\tilde{H} : I \times I \to \widetilde{Y}$ of H. Denote β the path $\{H(1,s)\}_{0 \le s \le 1}$ in B_k. Lift the path $\beta : I \to B_k$ from $\tilde{b}$ to get a path $\tilde{\beta}$ in $\widetilde{B}_k$, then $\tilde{j}_k \circ \tilde{\beta} : I \to \widetilde{Y}$ is a lifting from $\widetilde{f \circ \alpha}(1)$ in $\widetilde{Y}$ of the path $j_k \circ \beta$. By the unique lifting property of covering spaces, we have $\tilde{H}(1,s) = \tilde{j}_k \circ \tilde{\beta}(s)$. Then $\tilde{j}_k \circ \tilde{\beta}(0) = \tilde{H}(1,0) = \widetilde{f \circ \alpha}(1)$ and $\tilde{j}_k \circ \tilde{\beta}(1) = \tilde{H}(1,1) = \widetilde{g \circ \alpha}(1)$, there exists a unique lifting $(\tilde{f}_k, \tilde{g}_k)$ of $(f_k, g_k) : A_k \to B_k$ such that $\tilde{f}_k(\tilde{a}) = \tilde{\beta}(0), \tilde{g}_k(\tilde{a}) = \tilde{\beta}(1)$. Thus $\tilde{j}_k \circ \tilde{f}_k(\tilde{a}) = \tilde{f} \circ \tilde{i}_k(\tilde{a}), \tilde{j}_k \circ \tilde{g}_k(\tilde{a}) = \tilde{g} \circ \tilde{i}_k(\tilde{a})$. By the unique lifting property of covering spaces, we have $\tilde{j}_k \circ (\tilde{f}_k, \tilde{g}_k) = (\tilde{f}, \tilde{g}) \circ \tilde{i}_k$, i.e. $(\tilde{f}, \tilde{g}) = (\tilde{i}_k, \tilde{j}_k)_{\text{lift}}(\tilde{f}_k, \tilde{g}_k)$. This implies $[\tilde{f}, \tilde{g}] = (i_k, j_k)_C[\tilde{f}_k, \tilde{g}_k]$, i.e. $p\Gamma(\tilde{f}, \tilde{g})$ is a weakly common coincidence class of (f,g) and $(\bar{f}, \bar{g})$.

COROLLARY 2.4. *A coincidence class of $(f,g) : X \to Y$ containing a coincidence point on A is a weakly common coincidence class of (f,g) and $(\bar{f}, \bar{g})$.*

In [3], the number $N(f,g;\bar{f},\bar{g})$ of essential common coincidence classes of (f,g) and $(\bar{f},\bar{g})$ is introduced, and we have

PROPOSITION 2.5. $N(f,g;\bar{f},\bar{g}) \le E(f,g;\bar{f},\bar{g}) \le N(f,g)$.

Proof. By Corollary 2.4 and [3,Definition 4.1], we know that a common coincidence class is always a weakly common coincidence class. This implies the left inequality. The right one is obvious.

In general, $E(f,g;\bar{f},\bar{g})$ is different from $N(f,g;\bar{f},\bar{g})$. A simple example is the identity map $(f,g) : (D^2, S^1) \to (D^2, S^1)$ of the pair of a 2-disc and its boundary, it is easy to see $N(f,g) = 1$ and $N(\bar{f},\bar{g}) = 0$, then $N(f,g;\bar{f},\bar{g}) = 0$, but $E(f,g;\bar{f},\bar{g}) = 1$.

THEOREM 2.6 (HOMOTOPY INVARIANCE). *If $(f_0, g_0) \simeq (f_1, g_1) : (X, A) \to (Y, B)$ are homotopic, then*

$$E(f_0, g_0; \bar{f}_0, \bar{g}_0) = E(f_1, g_1; \bar{f}_1, \bar{g}_1).$$

Proof. Let $(f_t, g_t) : (X, A) \to (Y, B)$ be a homotopy between (f_0, g_0) and (f_1, g_1). There exists an index-preserving bijection $(\{f_t\}, \{g_t\})$:

$C(f_0, g_0) \to C(f_1, g_1)$. It suffices to show $(\{f_t\}, \{g_t\})$ sends weakly common coincidence classes to weakly common coincidence classes. Let $p\Gamma(\tilde{f}_0, \tilde{g}_0)$ be a weakly common coincidence class of (f_0, g_0) and $(\bar{f}_0, g_0)$, then there exists a component (A_k, B_k) of (A, B) and a lifting class $[\tilde{f}_{0,k}, \tilde{g}_{0,k}]$ of $(f_{0,k}, g_{0,k}) : A_k \to B_k$ such that

$$(i_k, j_k)_C[\tilde{f}_{0,k}, \tilde{g}_{0,k}] = [\tilde{f}_0, \tilde{g}_0].$$

Let $(\{f_{t,k}\}, \{g_{t,k}\})$, which is the restriction of (f_t, g_t) to A_k, send $[\tilde{f}_{0,k}, \tilde{g}_{0,k}]$ to $[\tilde{f}_{1,k}, \tilde{g}_{1,k}]$, then we have a commutative diagram [3]

$$\begin{array}{ccc}
[\tilde{f}_{0,k}, \tilde{g}_{0,k}] & \xrightarrow{(\{f_{t,k}\}, \{g_{t,k}\})} & [\tilde{f}_{1,k}, \tilde{g}_{1,k}] \\
{\scriptstyle (i_k, j_k)_C}\downarrow & & \downarrow{\scriptstyle (i_k, j_k)_C} \\
[\tilde{f}_0, \tilde{g}_0] & \xrightarrow{(\{f_t\}, \{g_t\})} & [\tilde{f}_1, \tilde{g}_1]
\end{array}$$

Thus $(\{f_t\}, \{g_t\})$ sends $[\tilde{f}_0, \tilde{g}_0]$ to $[\tilde{f}_1, \tilde{g}_1] = (i_k, j_k)_C[\tilde{f}_{1,k}, \tilde{g}_{1,k}]$, we get the conclusion.

DEFINITION 2.7. The number of essential coincidence classes of $(f, g) : X \to Y$ which are not weakly common coincidence classes is called the Nielsen number of (f, g) on the complementary space $X - A$, denoted $N(f, g; X - A)$.

By definition, $N(f, g; X - A)$ is a non-negative integer, and

$$N(f, g; X - A) + E(f, g; \bar{f}, \bar{g}) = N(f, g).$$

Hence, we also have the homotopy invariance of $N(f, g; X - A)$.

THEOREM 2.8(LOWER BOUND). *Any map $(f, g) : (X, A) \to (Y, B)$ has at least $N(f, g; X - A)$ coincidence points on $X - A$.*

Proof. Recall that each essential coincidence class at least one coincidence point. By Corollary 2.4, we get the conclusion.

3. Computation of $N(f,g;X-A)$

THEOREM 3.1. *Let $(f,g):(X,A)\to(Y,B)$ be a map of pairs of compact polyhedra. If there is a component A_k of $\hat{A}$ such that $j_{k,\pi}:\pi_1(B_k)\to\pi_1(Y)$ is onto, then $N(f,g;X-A)=0$.*

Proof. By [3;Proposition 2.4], $(i_k,j_k)_C$ is surjective. Then every coincidence class of $(f,g):X\to Y$ is a weakly common coincidence class of (f,g) and (f,g).

The computation of $N(f,g;X-A)$ is similar to the corresponding results for the relative Nielsen number $N_R(f,g)$ [3].

Pick a base point $a_k\in A_k\subset X$ and $b_k\in B_k\subset Y$ such that $f(a_k)=g(a_k)=b_k$. Then recall that points of universal covering spaces are identified with path classes in base spaces starting from base points. Under this identification, let $\tilde{a}_k\in p_k^{-1}(a_k),\tilde{x}_0\in p^{-1}(a_k),\tilde{b}_k\in q_k^{-1}(b_k)$ and $\tilde{y}_0\in q^{-1}(b_k)$ be the constant paths. Then there are unique lifting pairs $(\tilde{f}_k,\tilde{g}_k)$ of (f_k,g_k) and $(\tilde{f},\tilde{g})$ of (f,g) such that $\tilde{f}_k(\tilde{a}_k)=\tilde{g}_k(a_k)=\tilde{b}_k$ and $\tilde{f}(\tilde{x}_0)=\tilde{g}(\tilde{x}_0)=\tilde{y}_0$. By [2; Lemma 1.13], $\tilde{f}_{k\,\pi}=f_{k,\pi},\tilde{g}_{k,\pi}=g_\pi,\tilde{f}_\pi=f_\pi$ and $\tilde{g}_\pi=g_\pi$.

Throughtout this section, the lifting pair $(\tilde{f}_k,\tilde{g}_k)$ of (f_k,g_k) and the lifting pair $(\tilde{f},\tilde{g})$ of (f,g) are chosen as references.

LEMMA 3.2. *There exist one-to-one correspondences*

$$\phi_k : C(f_k,g_k)\longrightarrow \pi_1'(B_k,b_k)$$
$$\phi : C(f,g)\longrightarrow \pi_1'(Y,b_k)$$

defined by

$$\phi_k[\alpha_k\circ\tilde{f}_k,\beta_k\circ\tilde{g}_k]=[\alpha_k^{-1}\beta_k]$$
$$\phi[\alpha\circ\tilde{f},\beta\circ\tilde{g}]=[\alpha^{-1}\beta]$$

where $\alpha_k,\beta_k\in\pi_1(B_k,b_k);\alpha,\beta\in\pi_1(Y,b_k)$ and $\pi_1'(Y,b_k)$ is the set of f_π,g_π-conjugate classes in $\pi_1(Y,b_k)$.

Proof. See [3; Lemma 5.2].

If α_k and β_k are $f_{k,\pi},g_{k,\pi}$ conjugate classes in $\pi_1(B_k,b_k)$, then $j_{k,\pi}(\alpha_k)$ and $j_{k,\pi}(\beta_k)$ are f_π,g_π-conjugate classes in $\pi_1(Y,b_k)$. Then the homomorphism $j_{k,\pi}:\pi_1(B_k,b_k)\to\pi_1(Y,b_k)$ induces a transformation $\nu_k:\pi_1'(B_k,b_k)\to\pi_1'(Y,b_k)$.

THEOREM 3.3. *Let $(f,g):(X,A)\to(Y,B)$. A coincidence class of (f,g) is a weakly common coincidence class of (f,g) and $(\bar f,\bar g)$ if and only if it corresponds to an element in the image of ν_k.*

Proof. See [3; proposition 5.3].

Consider the commutative diagram

$$\begin{array}{ccccc}
\pi_1(B_k,b_k) & \xrightarrow{\theta_k} & H_1(B_k) & \xrightarrow{\eta_k} & \mathrm{coker}\,(g_{k,*}-f_{k,*}:H_1(A_k)\to H_1(B_k)) \\
\big\downarrow j_{k,\pi} & & \big\downarrow j_{k,*} & & \big\downarrow j_{k,*} \\
\pi_1(Y,b_k) & \xrightarrow{\theta} & H_1(Y) & \xrightarrow{\eta} & \mathrm{coker}\,(g_*-f_*:H_1(X)\to H_1(Y))
\end{array}$$

where θ_k,θ are abelianization and η_k,η are the natural projection. By some modification of [3; Lemma 5.4], we have

LEMMA 3.4. *The composition $\eta_k\circ\theta_k$ and $\eta\circ\theta$ induce correspondences*

$$\tau_k:\pi_1'(B_k,b_k)\longrightarrow \mathit{coker}(g_{k,*}-f_{k,*})$$
$$\tau:\pi_1'(Y,b_k)\longrightarrow \mathit{coker}(g_*-f_*)$$

and the diagram

$$\begin{array}{ccc}
\pi_1'(B_k,b_k) & \xrightarrow{\tau_k} & \mathit{coker}(g_{k,*}-f_{k,*}) \\
\nu_k\big\downarrow & & \big\downarrow j_{k,*} \\
\pi_1'(Y,b_k) & \xrightarrow{\tau} & \mathit{coker}(g_*-f_*)
\end{array}$$

commutes

THEOREM 3.5. *Let $(f,g):(X,A)\to(Y,B)$. Suppose Y is a Jiang space. If $\omega(f,g,X)=0$ then $N(f,g;X-A)=0$; if $\omega(f,g,X)\neq 0$ then*

$$N(f,g;X-A)=\sharp\{\mathit{coker}(g_*-f_*)\}-\sharp\{\cup_{k=1}^{n}j_{k,*}\mathit{coker}(g_{k,*}-f_{k,*})\}.$$

Proof. By [4;Corollary 4.16], the correspondence τ is bijective when Y is a Jiang space. Apply Theorem 3.3 and Lemma 3.4 to get the conclusion.

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