

AN EXISTENCE OF THE INERTIAL MANIFOLD FOR NEW DOMAINS

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1. Introduction

Consider a specific class of scalar-valued reaction diffusion equations of the form

$$(1.1) \quad u_t = \nu \Delta u + f(u), \quad u \in R$$

where $\nu > 0$ is viscosity parameter and $f : R \rightarrow R$ is sufficiently smooth. We are concerned with the existence of inertial manifold of the equation (1.1) on the following type of domains

$$(1.2) \quad \begin{cases} \Omega_2 = (\text{equilateral triangle of side } \pi) \\ \Omega_3 = (\text{equilateral triangle of side } \pi) \times [0, L\pi], \end{cases}$$

with Neumann boundary conditions. This problem was studied by Mallet-Paret and Sell (1988) with more general nonlinear function f for 2-dimensional rectangular domains and 3-dimensional cubic domain $\Omega_3 = (0, 2\pi)^3$. For their result, they introduce the new property, the Principle of Spatial Averaging (PSA), which plays important role in their theory.

The purpose of this paper is to extend their result into the domains Ω given in (1.2) by restricting a class of nonlinear functions f . For doing this, we introduce an abstract result established by Mallet-Paret and Sell and then we will show, under suitable conditions, that there exists an inertial manifold for the equation (1.1).

Received May 1, 1996.

1991 AMS Subject Classification: 35P20, 34C29, 35K57.

Key words: dissipative system, inertial manifold, reaction diffusion, Principle of Spatial Averaging.

2. An Abstract Invariant Manifold Theory

For convenience we present here some ideas developed in Mallet-Paret and Sell (1988) throughout this section.

Let H be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$, and let \mathcal{P} be a finite dimensional subspace of H with orthogonal projection P , and let \mathcal{Q} be the orthogonal complement with complementary projection $Q = I - P$.

Writing $u \in H$ as $u = (p, q)$ where

$$p = Pu \in PH \equiv \mathcal{P}, \quad q = Qu \in QH \equiv \mathcal{Q},$$

we consider an abstract differential equation

$$(2.1) \quad \begin{aligned} p' &= F(p, q), \\ q' &= -Aq + G(p, q). \end{aligned}$$

We assume A is a closed selfadjoint linear operator on \mathcal{Q} with dense domain $D(A) \subset \mathcal{Q}$. We assume further that $-A$ generates a C^0 -semigroup e^{-At} in \mathcal{Q} for $t > 0$, and that A has a compact resolvent on \mathcal{Q} . The (nonlinear) functions

$$F : H \rightarrow \mathcal{P}, \quad G : H \rightarrow \mathcal{Q}$$

are assumed to be locally Lipschitz continuous in H . These assumptions on A, F , and G are standing assumptions throughout this section. Under these assumptions, the system (2.1) with initial condition $u_0 = (p_0, q_0) \in H$ has a unique, maximally defined solution $u(t) = u(t, u_0) = (p(t, p_0, q_0), q(t, p_0, q_0))$ on some interval $[0, \omega)$ where $\omega = \omega(p_0, q_0) \in (0, \infty]$.

Furthermore, the existence of invariant manifold \mathcal{M} for the system (2.1) can be proved under the following five hypotheses.

(I) (Regularity Condition) There exist constants R_1 and R_2 such that both F and G are C^1 in the convex set $\mathcal{A} \times \mathcal{C}$, where

$$\begin{aligned} \mathcal{A} &= \{p \in \mathcal{P} : \|p\| \leq R_1\}, \\ \mathcal{C} &= \{q \in D(A) \subset \mathcal{Q} : \|Aq\| \leq R_2\}. \end{aligned}$$

(II) (Dissipative Condition) If $p \in Cl(\mathcal{P} \setminus \mathcal{A})$, then

$$\langle p, F(p, 0) \rangle < 0 \quad \text{and} \quad G(p, 0) = 0.$$

(III) (Sobolev Condition) If $p_0 \in \mathcal{A}$ and $t_0 > 0$ are such that $p(t, p_0, 0) \in \mathcal{A}$ holds in $[0, t_0]$, then one also has $q(t, p_0, 0) \in \mathcal{C}$ in $[0, t_0]$.

(IV) (Linear Stability Condition) One has $\langle q, Aq \rangle \geq \Lambda \|q\|^2$ for all $q \in \mathcal{D}$, for some $\Lambda > 2\gamma$, where $\gamma = \sup\{\|DG(p, q)\|_{\mathcal{L}} : (p, q) \in \mathcal{A} \times \mathcal{C}\}$, and $\mathcal{L} = \mathcal{L}(H, \mathcal{Q})$.

(V) (Uniform Cone Condition) With $V \equiv \frac{1}{2}\|\sigma\|^2 - \frac{1}{2}\|\rho\|^2$ and $V' \equiv \langle \sigma, \sigma' \rangle - \langle \rho, \rho' \rangle$, there exists a $\xi > 0$ such that $V' \leq -\xi$ whenever $(p, q) \in \mathcal{A} \times \mathcal{C}$ and $\|\rho\| = \|\sigma\| \neq 0$, where $\rho \in \mathcal{P}, \sigma \in \mathcal{D} \subset \mathcal{Q}$, and ρ' and σ' are given by the linear variational equation form of (2.1), i.e.,

$$(2.2) \quad \begin{aligned} \rho' &= DF(p, q)(\rho, \sigma), \\ \sigma' &= -A\sigma + DG(p, q)(\rho, \sigma). \end{aligned}$$

We refer to Mallet-Paret and Sell (1988) for a better understanding of the analytical and geometrical significances of these five assumptions.

Before we state the Theorem we introduce the following notation: Let $\Phi : \mathcal{P} \rightarrow \mathcal{Q}$ be a function. The graph and the support of Φ are

$$\begin{aligned} graph(\Phi) &= \{(p, \Phi(p)) : p \in \mathcal{P}\}, \\ supp(\Phi) &= Cl\{p \in \mathcal{P} : \Phi(p) \neq 0\}. \end{aligned}$$

Let \mathcal{E} denote the following subset of $\mathcal{A} \times \mathcal{C}$:

$$\mathcal{E} = \{(p, q) \in \mathcal{A} \times \mathcal{C} : \|q\| \leq dist(p, bdy\mathcal{A})\}.$$

Finally we let

$$(2.3) \quad \mathcal{G} = \mathcal{E} \cup (\mathcal{P} \times \{0\}).$$

THEOREM 2.1. *Assume that the differential equation (2.1) satisfies conditions (I)-(V) in addition to the standing assumptions on $A, F,$ and $G.$ Then there exists a Lipschitz function $\Phi : \mathcal{P} \rightarrow \mathcal{Q}$ with Lipschitz constant at most one, satisfying*

$$\Phi(p) \in \mathcal{C} \quad \text{for } p \in \mathcal{P},$$

$$\text{supp}\Phi \subset \mathcal{A}$$

and such that the graph, $\mathcal{M} = \text{graph}(\Phi),$ is an invariant manifold for (2.1) with $\mathcal{M} \subset \mathcal{G},$ where \mathcal{G} is given by (2.3). Furthermore \mathcal{M} is locally attracting in the following sense; there exists an $\alpha > 0$ such that if $u(t) = (p(t), q(t))$ is a solution of (2.1) satisfying $u(t) \in \mathcal{E}$ for all $t > 0,$ then

$$\text{dist}(u(t), \mathcal{M}) \leq 2e^{-\alpha t}(\text{diam}\mathcal{C}), \quad t > 0.$$

That is, $u(t)$ approaches \mathcal{M} at a uniform exponential rate. Moreover Φ is a C^1 -function and the derivative $\Psi(p) = D\Phi(p)$ satisfies $\|\Psi\|_\infty \leq 1.$

3. The Existence of Inertial Manifolds

We turn our attention now to a specific class of scalar partial differential equations of the form

$$(3.1) \quad \frac{\partial u}{\partial t} = \nu \Delta u + f(u), \quad u \in R$$

where the domain Ω_n is given in (1.2). The main goal is to show the existence of inertial manifold for (3.1) and each domain Ω_n provided L^2 is rational number.

3.1. Basic Assumptions and Preliminary Result

Here we follow Mallet-Paret and Sell's approach. The nonlinearity

$$f : R \rightarrow R$$

is assumed to satisfy the following conditions for some positive constants K_1 and K_2 :

$$(3.2) \quad \begin{cases} f \text{ is } C^1 \text{ in } R, \\ |f(u)| \leq K_1|u| + K_2 \text{ in } R. \text{ and} \\ |D_u f(u)| \leq K_1 \text{ in } R. \end{cases}$$

We consider Neumann boundary conditions for the equation (3.1); i.e.,

$$(3.3) \quad \text{Neumann : } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_n.$$

Then equation (3.1) can be written as an abstract differential equation

$$(3.4) \quad \frac{du}{dt} = \nu\Delta u + \tilde{f}(u)$$

in the phase space $H = L^2(\Omega_n)$ and \tilde{f} is a Lipschitz continuous mapping on H such that

$$(3.5) \quad \begin{cases} \|\tilde{f}(u_1) - \tilde{f}(u_2)\| \leq K_1\|u_1 - u_2\|, \text{ for all } u_1, u_2 \in H \\ \|\tilde{f}(u)\| \leq K_1\|u\| + K_3 \text{ for all } u \in H \end{cases}$$

where $K_3 = (\text{vol}\Omega_n)^{\frac{1}{2}}K_2$. Let

$$(3.6) \quad \mathcal{D} = \{u \in H^2 : \text{the boundary conditions (3.3) hold}\}.$$

For simplicity we shall assume $\nu = 1$. For any $\lambda > 0$ let P_λ denote the canonical orthogonal projection onto the finite dimensional subspace

$$\mathcal{P}_\lambda = \text{Span}\{e_m : \lambda_m \leq \lambda\}$$

of H where $\{e_j : j = 1, 2, \dots\}$ be a complete orthonormal set of eigenfunctions e_j corresponding to eigenvalues λ_j of $-\Delta$ and let $Q_\lambda = I - P_\lambda$.

Then the modified equations, to which the Theorem 2.1 will be applied, are

$$(3.7) \quad \begin{aligned} p' &= -\phi(\|Ap\|^2)Ap + \psi(\|p\|^2)[p + P_\lambda\tilde{f}(p, q)], \\ q' &= -Aq + q + \psi(\|p\|^2)Q_\lambda\tilde{f}(p, q) \end{aligned}$$

where $\lambda > 0$ is appropriately chosen and A is the positive self-adjoint operator

$$A = I - \Delta,$$

$\phi, \psi : [0, \infty) \rightarrow [0, 1]$ are C^1 functions such that with a sufficiently large fixed $R > 0$, ϕ, ψ satisfy

$$(3.8) \quad \begin{cases} \phi'(\tau) \leq 0 & \text{in } [0, \infty), \\ \tau\phi'(\tau) + \phi(\tau) \geq 0 & \text{in } [0, \infty), \\ \phi(\tau) = 1 & \text{in } [0, R^2], \\ \phi(\tau) = \frac{1}{2} & \text{in } [K_4 R^2, \infty) \text{ for some } K_4 > 1, \\ \psi(\tau) = 1 & \text{in } [0, K_4 R^2], \\ \psi(\tau) = 0 & \text{in } [K_5 R^2, \infty) \text{ for some } K_5 > K_4. \end{cases}$$

In order to assert the existence of invariant manifold for (3.1), we have to show that (3.7) satisfies main hypotheses (I)-(V).

For positive constants R_1 and R_2 and for $\lambda > 0$ we define

$$\begin{aligned} \mathcal{A} &= \{p \in \mathcal{P}_\lambda : \|p\| \leq R_1\} \\ \mathcal{C} &= \{q \in \mathcal{D} \subset \mathcal{Q}_\lambda : \|Aq\| \leq R_2\}. \end{aligned}$$

The following lemma is proved for any bounded Lipschitz domain and hence it is available for the domains Ω_n we study here.

LEMMA 3.1. *Let $\Omega_n \subset R^n$ be a bounded Lipschitz domain with $n = \dim \Omega \leq 3$, and let f satisfy (3.2). Fix ϕ and ψ satisfying (3.8), and fix the boundary conditions (3.3). Then there exist R_1 and R_2 such that the hypotheses (I)-(IV) of the Theorem 2.1, hold for the system (3.7) for all sufficiently large $\lambda > 0$. Here one has*

$$\begin{aligned} A &= I - \Delta, \\ F(p, q) &= -\phi(\|Ap\|^2)Ap + \psi(\|p\|^2)[p + P_\lambda \tilde{f}(p, q)], \\ G(p, q) &= q + \psi(\|p\|^2)Q_\lambda \hat{f}(p, q) \end{aligned}$$

for the abstract system (2.1).

Moreover we obtain a partial verification of the Uniform Cone Condition.

LEMMA 3.2. For any λ and any (p, q) and (ρ, σ) in $\mathcal{P}_\lambda \times \mathcal{Q}_\lambda$ one has

$$V' \leq -\langle \sigma, A\sigma \rangle + \phi(\|Ap\|^2)\|A\rho\| \|\rho\| + W$$

where

$$(3.9) \quad \begin{aligned} W = & \|\sigma\|^2 + 2\psi'(\|p\|^2)\langle p, \rho \rangle [\langle (-\rho, \sigma), \tilde{f}(p, q) \rangle - \langle p, \rho \rangle] \\ & + \psi(\|p\|^2)[\langle (-\rho, \sigma), D\tilde{f}(p, q)(\rho, \sigma) \rangle - \|\rho\|^2]. \end{aligned}$$

Furthermore, If $(p, q) \in \mathcal{A} \times \mathcal{C}$ and $\|\rho\|, \|\sigma\| \leq 1$, then the estimate

$$(3.10) \quad |W| \leq K_6$$

holds for some K_6 depending only on R_1 and R_2 , but not on λ .

This lemma implies the partial proof of the Uniform Cone Condition as done in Mallet-Paret and Sell (1988).

LEMMA 3.3. Given R_1 and R_2 , then for all sufficiently large λ the inequality $V' \leq -\xi$ of the Uniform Cone Condition holds for any $(p, q) \in \mathcal{A} \times \mathcal{C}$ and $(\rho, \sigma) \in \mathcal{P}_\lambda \times \mathcal{Q}_\lambda$ with $\|\rho\| = \|\sigma\| = 1$, provided

$$(3.11) \quad \|Ap\| \geq K_4^{\frac{1}{2}} R$$

also holds.

3.2. Verifying The Uniform Cone Condition

Now the only thing we need to show is that

$$V' \leq -\xi$$

for $(p, q) \in \mathcal{A} \times \mathcal{C}$ with $\|Ap\| \leq K_4^{\frac{1}{2}} R$ and for $(\rho, \sigma) \in \mathcal{P}_\lambda \times \mathcal{Q}_\lambda$. Since $\|p\| \leq \|Ap\|$, for such points one has $\psi(\|p\|^2) = 1$, $\psi'(\|p\|^2) = 0$, and $\phi(\|Ap\|^2) \leq 1$. Thus with $\|\rho\| = \|\sigma\| = 1$, Lemma 3.2 yields

$$(3.12) \quad \begin{cases} V' \leq -\langle \sigma, A\sigma \rangle + \|A\rho\| + W \\ W = \langle (-\rho, \sigma), D\tilde{f}(p, q)(\rho, \sigma) \rangle. \end{cases}$$

To show this, we need good information about the eigenvalues and eigenfunctions of $-\Delta$ for Neumann boundary conditions. By using M. Pinsky's idea (1981) and separation of variables, one obtains the following lemma. Let Z_+ denote the positive integers and $Z_\oplus = Z_+ \cup \{0\}$:

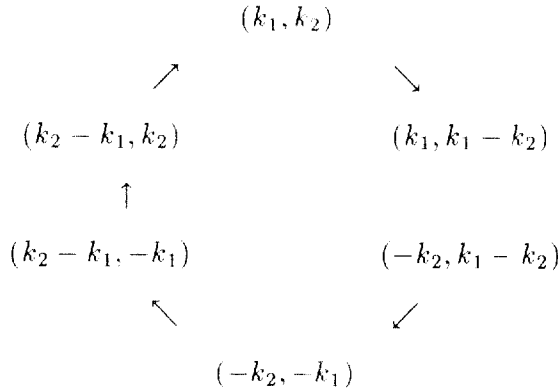
LEMMA 3.4. Let $\Omega_n \subset R^n$ be given in (1.2) for $n = 2, 3$. Then the eigenvalues and the eigenfunctions of $-\Delta$ for Neumann boundary conditions are of the form: for $\Omega_2 \subset R^2$,

$$(3.13) \quad \begin{cases} \lambda_k = \frac{16}{27}(k_1^2 + k_2^2 - k_1 k_2), \\ f_k(x_1, x_2) = \sum_{(k_1, k_2)} \exp\left(\frac{2i}{3}\right)(k_2 x_1 + \frac{2k_1 - k_2}{\sqrt{3}} x_2), \end{cases}$$

and for $\Omega_3 \subset R^3$,

$$(3.14) \quad \begin{cases} \lambda_k = \frac{16}{27}(k_1^2 + k_2^2 - k_1 k_2) + \frac{k_3^2}{L^2}, \\ f_k(x_1, x_2, x_3) = \sin\frac{k_3}{L} x_3 \sum_{(k_1, k_2)} \exp\left(\frac{2i}{3}\right)(k_2 x_1 + \frac{2k_1 - k_2}{\sqrt{3}} x_2), \end{cases}$$

where $k = (k_1, k_2) \in Z^2$ ($k = (k_1, k_2, k_3) \in Z^2 \times Z_{\oplus}$ for $n = 3$) satisfies (i) $k_1 + k_2$ is multiple of 3, and (ii) (k_1, k_2) in the summation ranges over $S \subset Z^2, |S| = 6$, which are given as follows:



The proof that $V' \leq -\xi$, for domains of interest, involves considering two possibilities: either (i) $-\langle \sigma, A\sigma \rangle + \|A\rho\|$ is sufficiently negative to overcome the bound $|W| \leq K_6$, where K_6 is given by (3.10): or (ii) $-\langle \sigma, A\sigma \rangle + \|A\rho\|$ is negative and bounded away from zero, while W is close to zero.

The case (i) is related to the occurrence of large gaps in the spectrum of $A = I - \Delta$. Such is the case for the 2-dimensional domains considered here. Therefore, one can prove Uniform Cone Condition completely in these cases.

THEOREM 3.1. *Assume that Ω_2 is 2-dimensional domain given in (1.2). Fix Neumann boundary conditions. Assume f satisfies the regularity and growth conditions (3.2); assume also the function f is C^3 in R . Fix functions ϕ and ψ satisfying (3.8). Then there exists arbitrarily large λ such that the system (3.7) satisfies all the hypotheses of the Invariant Manifold Theorem 2.1.*

Proof. Since the eigenvalues of $-\Delta$ for Neumann boundary conditions are of the form

$$\lambda_k = \frac{16}{27}(k_1^2 + k_2^2 - k_1k_2)$$

where $k = (k_1, k_2) \in Z^2$, there exists arbitrarily large m such that

$$\lambda_k = \frac{16}{27}(k_1^2 + k_2^2 - k_1k_2) \notin [m, m + h]$$

for any $k = (k_1, k_2) \in Z^2$ which implies the existence of arbitrarily large spectral gaps (Mallet-Paret and Sell (1988)). Therefore, one can choose arbitrarily large $\lambda > 0$ which insure that $-\langle \sigma, A\sigma \rangle + \|A\rho\|$ is sufficiently negative to overcome the bound $|W| \leq K_6$, where K_6 is given by (3.10).

If there are no large gaps in the spectrum of $A = I - \Delta$ to overcome $|W|$, we must use show that $-\langle \sigma, A\sigma \rangle + \|A\rho\|$ is negative and bounded away from zero, while W is close to zero from which we obtain

$$(3.15) \quad V' \leq -\xi.$$

In order to do this we define a multiplication operator on L^2 as follows: for any $v \in L^\infty$ we let B_v denote the operator on L^2 defined by

$$(B_v u)(x) = v(x)u(x), \quad u \in L^2$$

and let \tilde{v} denote the mean value

$$\tilde{v} = (\text{vol}\Omega)^{-1} \int_{\Omega} v(x)dx.$$

Then we obtain the following results.

LEMMA 3.5. Let Ω_3 be the domain given in (1.2) and let $\{e_m\}_{m=1}^\infty$ denote a complete orthonormal set of eigenfunctions of $-\Delta$ with Neumann boundary conditions. For any $v \in H^2(\Omega_3)$, let $v_m = \langle v, e_m \rangle$ be the m th Fourier coefficient. Then there exists a monotone function $\Theta(\lambda) > 0$ satisfying $\Theta(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$, such that for any $\lambda > 0$ and $v \in \mathcal{D} \subset H^2$, one has

$$(3.16) \quad \sum_{\lambda_m \geq \lambda} |v_m| \leq \Theta(\lambda) \|v\|_{H^2}.$$

Proof. For each m , one has

$$(3.17) \quad \begin{aligned} (1 + \lambda_m)v_m &= \langle v, Ae_m \rangle = \langle v, e_m \rangle - \langle v, \Delta e_m \rangle \\ &= \langle v, e_m \rangle - \int_{\Omega} (\Delta v) \bar{e}_m dx + \int_{\partial\Omega} \frac{\partial v}{\partial n} \bar{e}_m d\tilde{x} \\ &= \langle A_* v, e_m \rangle, \end{aligned}$$

where A_* denotes the operator $I - \Delta$ extended to the domain H^2 . Then

$$\begin{aligned} \sum_{\lambda_m \geq \lambda} |v_m| &\leq \left(\sum_{\lambda_m \geq \lambda} (1 + \lambda_m)^{-2} \right)^{\frac{1}{2}} \left(\sum_{\lambda_m \geq \lambda} |\langle A_* v, g_m \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{\lambda_m \geq \lambda} (1 + \lambda_m)^{-2} \right)^{\frac{1}{2}} \|A_* v\|_{L^2} \\ &\leq \Theta(\lambda) \|v\|_{H^2} \end{aligned}$$

where $\Theta(\lambda) = K_7 \leq (\sum_{\lambda_m \geq \lambda} (1 + \lambda_m)^{-2})^{\frac{1}{2}}$ and K_7 does not depend on λ .

LEMMA 3.6. Assume that L^2 is rational number. Let $k = (k_1, k_2, k_3) \in Z^3$ and consider

$$(3.18) \quad \|[k]\|_1^2 \equiv \frac{16}{27}(k_1^2 + k_2^2 - k_1 k_2) + \frac{k_3^2}{L^2}.$$

Then there exists $\xi > 0$ such that for any $\kappa > 1$ and $d > 0$, there exists an arbitrarily large λ satisfying two statements :

- (i) whenever $\|[k]\|_1^2, \|[l]\|_1^2 \in (\lambda - \kappa, \lambda + \kappa)$ with $k, l \in Z^3$, one has either $k = l$ or $\|[k - l]\|_1 \geq d$,
- (ii) $\|[k]\|_1^2 \notin (\lambda - \frac{\xi}{2}, \lambda + \frac{\xi}{2})$ for each $k \in Z^3$.

Since the proof of lemma is similar as one in Mallet-Paret and Sell (1988), we omit the proof. From above lemmas, one proves the following result.

LEMMA 3.7. Let Ω_3 be given in (1.2) and fix Neumann boundary conditions for the Laplacian. Then there exists a quantity $\xi > 0$ such that for every $\epsilon > 0, \kappa > 0$, there exist arbitrarily large $\lambda > \kappa$, such that

$$(3.19) \quad \| (P_{\lambda+\kappa} - P_{\lambda-\kappa})(B_v - \tilde{v}I)(P_{\lambda+\kappa} - P_{\lambda-\kappa}) \|_{op} \leq \epsilon \|v\|_{H^2}$$

holds for any $v \in \mathcal{D} \subset H^2$; and such that

$$(3.20) \quad \lambda_{m+1} - \lambda_m \geq \xi$$

where m satisfies $\lambda_m \leq \lambda < \lambda_{m+1}$.

Proof. For this proof, we follow Mallet-Paret and Sell's approach. Let $\epsilon > 0$ and $\kappa > 0$ be given and fix $d > 0$ sufficiently large so that

$$\theta(d^2) < \frac{\epsilon}{6}$$

where the function θ is as in Lemma 3.5. With κ and d now given, let ξ and λ be as in Lemma 3.6; note that λ can be selected to be arbitrarily large with the other quantities kept fixed. Then (3.20) is immediately satisfied. Let $\{e_k : k \in Z^2 \times Z_{\oplus}\}$ be a complete orthonormal set of eigenfunctions of $-\Delta$ for Neumann boundary condition on the domain Ω_3 . Fix $v \in \mathcal{D} \subset H^2$ and expand this function as a Fourier series in the eigenvalues of $-\Delta$:

$$v(x) = \sum_{k \in Z^2 \times Z_{\oplus}} v_{(k)} e_k.$$

Note the mean value

$$\hat{v} = v_{(0)}$$

of the function v , as well as the summability

$$(3.21) \quad \sum |v_{(k)}| < \infty$$

of the coefficients. Now consider a function

$$\rho \in \text{Range} (P_{\lambda+\kappa} - P_{\lambda-\kappa}) \subset L^2 : \quad \|\rho\| = 1.$$

Its Fourier expansion

$$\rho(x) = \sum \rho_{(k)} e_k$$

involves only terms for which

$$\lambda - \kappa < \|[k]\|_1^2 \leq \lambda + \kappa.$$

Since the operator B_v is selfadjoint, it is sufficient to show

$$\left| \langle \rho, (B_v - \tilde{v}I)\rho \rangle \right| \leq \epsilon \|v\|_{H^2}$$

to show the required inequality (3.19).

The summability condition (3.21) permits one to write term by term

$$\begin{aligned} \langle \rho, (B_v - \tilde{v}I)\rho \rangle &= \sum_{j \neq 0} \sum_k \sum_l \tilde{v}_{(j)} \bar{\rho}_{(k)} \rho_{(l)} \int_{\Omega_3} \bar{e}_j \bar{e}_k e_l dx \\ (3.22) \qquad \qquad \qquad &= \sum_{s=1}^6 \sum_{j \neq 0} \sum_k \sum_l \tilde{v}_{(j)} \bar{\rho}_{(k)} \rho_{(l)} \int_{\Omega_3} \bar{e}_j e_{k-\delta_s} dx \end{aligned}$$

where δ_s are

$$\begin{aligned} \delta_1 &= (l_1, l_2, l_3), & \delta_2 &= (l_1, l_1 - l_2, l_3), \\ \delta_3 &= (-l_2, l_1 - l_2, l_3), & \delta_4 &= (-l_2, -l_1, l_3), \\ (3.23) \quad \delta_5 &= (-l_1 + l_2, -l_1, l_3), & \delta_6 &= (l_2 - l_1, l_2, l_3) \end{aligned}$$

and k, l are restricted to the ranges

$$\|[k]\|^2, \quad \|[l]\|^2 \in (\lambda - \kappa, \lambda + \kappa].$$

However, since $\|[l]\|_1^2 \in (\lambda - \kappa, \lambda + \kappa]$ and $\|[\delta_s]\|_1^2 = \|[l]\|_1^2$ for all $s = 1, 2, \dots, 6$, by the property (i) of Lemma 3.6, one obtains, for each s , either

$$(3.24) \qquad \|[k - \delta_s]\|_1 \geq d, \qquad \text{or} \qquad k = \delta_s.$$

For each $s, 1 \leq s \leq 6$, let h_s be a function on Z^3 given by $h_s(l) = \delta_s$, where δ_s is given in (3.23). Then, by (3.22) and a Young inequality, one has

$$\begin{aligned} & \sum_{s=1}^6 \sum_{j \neq 0} \sum_k \sum_l \bar{v}_{(j)} \bar{\rho}_{(k)} \rho_{(l)} \int_{\Omega_3} e_j e_{k-h_s(l)} dx \\ &= (\text{vol } \Omega_3) \sum_{s=1}^6 \sum_{k \neq h_s(l)} v_{(k-h_s(l))} \bar{\rho}_{(k)} \rho_{(l)}, \\ &\leq (\text{vol } \Omega_3) \sum_{s=1}^6 \sum_{k \neq h_s(l)} |v_{(k-h_s(l))}| \sum |\rho_{(k)}|^2, \\ &\leq 6 \sum_{k \neq h_s(l)} |v_{(k-h_s(l))}|, \\ &\leq 6 \theta(d^2) \|v\|_{H^2}, \\ &\leq \epsilon \|v\|_{H^2} \end{aligned}$$

as required.

REMARK. The definition of PSA used in Mallet-Paret and Sell requires that the inequality (3.19) holds for any $v \in H^2$. However due to the reduction of a class of nonlinear functions f , one obtains the last part of the next lemma and hence the result of Lemma 3.7 can replace the role of PSA.

LEMMA 3.8. Assume that f is a C^3 -function with respect to $u \in R$. Then for any $u \in R, v(x) = D_u f(u(x))$ belongs to H^2 . Moreover, given C_1 there exists C_2 such that

$$\|u\|_{H^2} \leq C_1 \quad \rightarrow \quad \|v\|_{H^2} \leq C_2.$$

In particular, if $u \in \mathcal{D} \subset H^2$, then $v \in \mathcal{D} \subset H^2$.

THEOREM 3.2. Assume that the domain Ω_3 given in (1.2). Assume the same conditions as in Theorem 3.1 except the dimension of domains. Then there exist arbitrary large λ such that the system (3.7) satisfies all the hypotheses of the Invariant Manifold Theorem 2.1.

For this result, we mention Mallet-Paret and Sell (1988). From all these results, one obtains the following result.

THEOREM 3.3. *Assume that (3.1) is dissipative and that $f : R \rightarrow R$ is of class C^3 . Let the domain $\Omega_n \subset R^n$ be given in (1.2). Then for every $\nu > 0$ and for Neumann boundary conditions, there exists an inertial manifold \mathcal{M} for (3.1).*

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