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BOUNDARIES OF THE CONE OF POSITIVE LINEAR MAPS AND ITS SUBCONES IN MATRIX ALGEBRAS

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1. Introduction

Let M_n be the C^*-algebra of all $n \times n$ matrices over the complex field, and $\mathbb{P}[M_m, M_n]$ the convex cone of all positive linear maps from M_m into M_n, that is, the maps which send the set of positive semi-definite matrices in M_m into the set of positive semi-definite matrices in M_n. The convex structures of $\mathbb{P}[M_m, M_n]$ are highly complicated even in low dimensions, and several authors [**CL**, **KK**, **LW**, **O**, **R**, **S**, **W**] have considered the possibility of decomposition of $\mathbb{P}[M_m, M_n]$ into subcones.

A linear map $\phi : A \to B$ between C^*-algebras is said to be *s-positive* if the linear map $\phi \otimes \mathrm{id}_s : A \otimes M_s \to B \otimes M_s$ is positive, and *completely positive* if it is s-positive for each natural number $s = 1, 2, \dots,$ where $\mathrm{id}_s : M_s \to M_s$ is the identity map. We denote by $\mathbb{P}_s[A, B]$ (respectively $\mathbb{P}_\infty[A, B]$) the convex cone of all s-positive (respectively completely positive) linear maps from A into B. Choi [**C1**] gave an example in $\mathbb{P}_{n-1}[M_n, M_n] \setminus \mathbb{P}_n[M_n, M_n]$, and showed that $\mathbb{P}_n[A, M_n]$ and $\mathbb{P}_m[M_m, B]$ coincide with the cone of all completely positive linear maps. In order to understand the boundary structures of $\mathbb{P}_1[M_n, M_n]$, the author has characterized the maximal faces of $\mathbb{P}_1[M_n, M_n]$ in [**K1**]. Using the methods in [**SW**], he [**K2**] has also found all maximal faces of $\mathbb{P}_\infty[M_m, M_n]$, and investigated how $\mathbb{P}_\infty[M_m, M_n]$ is sitting down in $\mathbb{P}_1[M_m, M_n]$.

In this note, we continue the same work for the cone $\mathbb{P}_s[M_m, M_n]$. Motivated by [**SW**] and [**K2**], we define for each $m \times n$ matrix V a

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linear functional

$$\phi \mapsto \theta_\phi(V) \tag{1.1}$$

on the space of all linear maps from M_m into M_n, in Section 2. It turns out that every maximal face of $\mathbb{P}_s[M_m, M_n]$ is exposed one obtained from this functional associated with a matrix V with rank $\le s$. With this information, we see in Section 3 how the cones $\mathbb{P}_s[M_m, M_n]$ and $\mathbb{P}_t[M_m, M_n]$ are related for different s and t.

Throughout this note, we fix natural numbers m and n, and denote by just $\mathbb{P}_s$ for the convex cone of all s-positive linear maps from M_m into M_n. We also denote by $m \wedge n$ the minimum of m and n, then $\mathbb{P}_\infty = \mathbb{P}_{m\wedge n}$ as was mentioned above. The vector space $M_{m,n}$ of all $m \times n$ complex matrices is endowed with the inner product arising from the usual trace of M_n. Then $M_{m,n}$ is isometrically isomorphic to the space $\mathbb{C}^m \otimes \mathbb{C}^n$. Every vector in $\mathbb{C}^n$ will be considered as an $n \times 1$ matrix. By the *interior* $\operatorname{int} C$ of a convex set C in $\mathbb{R}^\nu$, we mean the relative interior of C with respect to the affine manifold generated by C. The *boundary* ∂C of C is the set difference $C \setminus \operatorname{int} C$.

2. Maximal faces of $\mathbb{P}_s$

For an $m \times n$ matrix V with $\operatorname{rank} V = r$, choose an orthonormal basis $\{\eta_1, \dots, \eta_r\}$ of $(\operatorname{Ker} V)^\perp$. Then we have $V = \sum_{j=1}^r \xi_j \eta_j^*$ with $\xi_j = V\eta_j$ for $j = 1, \dots, r$. Denote by

$$P_V = \sum_{i,j=1}^r \xi_i \xi_j^* \otimes E_{ij} \in M_m \otimes M_r, \qquad \eta_V = \sum_{j=1}^r \eta_j \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^r,$$

where $\{E_{ij}\}$ and $\{e_j\}$ are the usual matrix units for M_r and orthonormal basis of $\mathbb{C}^r$. For a linear map $\phi : M_m \to M_n$, we define the scalar number

$$\theta_\phi(V) := \langle (\phi \otimes \mathrm{id}_r)(P_V)\eta_V, \eta_V \rangle. \tag{2.1}$$

Then we have

$$\begin{aligned}
\theta_\phi(V) &= \left\langle \sum_{i,j=1}^{r} \phi(\xi_i\xi_j^*)\eta_j \otimes e_i, \sum_{j=1}^{r} \eta_j \otimes e_j \right\rangle \\
&= \sum_{i,j=1}^{r} \langle \phi(\xi_i\xi_j^*)\eta_j, \eta_i\rangle \\
&= \sum_{i,j=1}^{r} \langle \phi \circ \sigma_V(\eta_i\eta_j^*)\eta_j, \eta_i\rangle,
\end{aligned}$$

where $\sigma_V : M_n \to M_m$ is given by $\sigma_V(X) = VXV^*$. If $\{\omega_1, \dots, \omega_r\}$ is an another orthonormal basis of $(\operatorname{Ker} V)^\perp$ and

$$\omega_i = \sum_{j=1}^{r} u_{ij}\eta_j, \qquad i = 1, \dots, r,$$

with an $r \times r$ unitary matrix $[u_{ij}]$, then we calculate

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{r} \langle \phi \circ \sigma_V(\omega_\alpha\omega_\beta^*)\omega_\beta, \omega_\alpha\rangle \\
&\qquad = \sum_{\alpha,\beta} \sum_{i,j,k,\ell} u_{\alpha i}\overline{u_{\beta j}}u_{\beta k}\overline{u_{\alpha \ell}}\langle \phi \circ \sigma_V(\eta_i\eta_j^*)\eta_k, \eta_\ell\rangle \\
&\qquad = \sum_{i,j,k,\ell} \left(\sum_{\alpha=1}^{r} u_{\alpha i}\overline{u_{\alpha\ell}}\right)\left(\sum_{\beta=1}^{r} \overline{u_{\beta j}}u_{\beta k}\right) \langle \phi \circ \sigma_V(\eta_i\eta_j^*)\eta_k, \eta_\ell\rangle \\
&\qquad = \sum_{i,j=1}^{r} \langle \phi \circ \sigma_V(\eta_i\eta_j^*)\eta_j, \eta_i\rangle.
\end{aligned}$$

Therefore, we see that the value $\theta_\phi(V)$ does not depend on the choice of $\{\eta_1, \dots, \eta_r\}$. For each $s = 1, \dots, m \wedge n$, we denote

$$\mathbb{M}_s := \{V \in M_{m,n} : \|V\| = 1, \ \operatorname{rank} V \le s\}.$$

Then $\mathbb{M}_s$ is a compact subset of $M_{m,n}$.

PROPOSITION 2.1. *For a linear map $\phi : M_m \to M_n$ and a na number $s = 1, \dots, m \wedge n$, the following are equivalent:*

(i) $\phi \in \mathbb{P}_s$.
(ii) $\theta_\phi(V) \geq 0$ *for each* $V \in \mathbb{M}_s$.

Proof. If $\phi \in \mathbb{P}_s$ and rank $V = r \leq s$ then $\phi \otimes \mathrm{id}_r$ is a positive li map, and so we have (i) $\Longrightarrow$ (ii). For the converse, put

$$(2.2)\ \ P = \sum_{i,j=1}^{s} \xi_i \xi_j^* \otimes E_{ij} \in M_m \otimes M_s, \qquad \eta = \sum_{j=1}^{s} \eta_j \otimes e_j \in \mathbb{C}^n \otimes$$

Then we have

$$\langle (\phi \otimes \mathrm{id}_s)(P)\eta, \eta \rangle = \theta_\phi \left(\sum_{j=1}^{s} \xi_j \eta_j^* \right),$$

which is nonnegative by the assumption (ii), because $\sum_{j=1}^{s} \xi_j \eta_j^*$ in $\mathbb{M}_s$ with normalization. Since every vector of $\mathbb{C}^n \otimes \mathbb{C}^s$ is of form η in (2.2), we see that $(\phi \otimes \mathrm{id}_s)(P)$ is positive semi-definite. conclude that $\phi \otimes \mathrm{id}_s$ is a positive linear map, because every pos semi-definite matrix in $M_m \otimes M_s$ is the nonnegative linear combina of matrices of the form P in (2.2). □

For each $s = 1, \dots, m \wedge n$ and $V \in \mathbb{M}_s$, we define

$$(2.3) \qquad F_s[V] := \{\phi \in \mathbb{P}_s : \theta_\phi(V) = 0\}.$$

If $s = 1$ and $V = \xi\eta^* \in \mathbb{M}_1$ then $F_1[V]$ is nothing but the maximal $F[\xi, \eta]$ of $\mathbb{P}_1$, as was introduced in [**K1**]. For a family $\mathcal{V} = \{V_k : 1, \dots, q\}$ of $m \times n$ matrices, we define the completely positive li map $\phi_\mathcal{V} : M_m \to M_n$ by

$$\phi_\mathcal{V} : X \mapsto \sum_{k=1}^{s} V_k^* X V_k, \qquad X \in M_m.$$

Recall [**C2**] that every element in the cone $\mathbb{P}_\infty = \mathbb{P}_{m\wedge n}$ arise this form. If $V = \sum_{j=1}^{r} (V\eta_j)\eta_j^* \in \mathbb{M}_r$ with orthonormal vec

$\{\eta_1, \dots, \eta_r\}$, then we have

$$
\begin{aligned}
\theta_{\phi_{\mathcal{V}}}(V) &= \sum_{k=1}^{q} \sum_{i,j=1}^{r} \langle V_k^* V \eta_i \eta_j^* V^* V_k \eta_j, \eta_i \rangle \\
&= \sum_{k=1}^{q} \sum_{i,j=1}^{r} \langle \eta_j^* V^* V_k \eta_j, \eta_i^* V^* V_k \eta_i \rangle \\
&= \sum_{k=1}^{q} \left(\sum_{j=1}^{r} \langle V^* V_k \eta_j, \eta_j \rangle \right) \left(\sum_{i=1}^{r} \overline{\langle V^* V_k \eta_i, \eta_i \rangle} \right) \\
&= \sum_{k=1}^{q} |\mathrm{Tr}\,(V^* V_k)|^2 = \sum_{k=1}^{q} |\langle V_k, V \rangle|^2.
\end{aligned}
\tag{2.4}
$$

Therefore, we see that

$$
F_{m\wedge n}[V] = \{\phi_{\mathcal{V}} \in \mathbb{CP} : \mathrm{span}\, \mathcal{V} \subset V^{\perp}\} \tag{2.5}
$$

is nothing but the maximal face $F[V]$ of $\mathbb{CP}$, as was shown in [**K2**].

Now, we fix an integer $s = 1, \dots, m \wedge n$. If $V \in \mathbb{M}_s$ then we see that $F_s[V]$ is an exposed face of $\mathbb{P}_s$ by Proposition 2.1. Because $F_{m\wedge n}[V]$ is nonempty by (2.5) and $F_{m\wedge n}[V] \subset F_s[V]$, we see that $F_s[V]$ is nonempty. Actually, we have seen in Theorem 3.2 of [**K2**] that every $F_{m\wedge n}[V]$ contains a unital linear map. We know that the trace map $\tau : M_m \to M_n$ given by $\tau(X) = \mathrm{Tr}\,(X)I$ is an interior point of $\mathbb{P}_s$. In order to characterize the interior of $\mathbb{P}_s$, we need to calculate $\theta_\tau(V)$ as follows:

$$
\begin{aligned}
\theta_\tau(V) &= \sum_{i,j=1}^{r} \langle \mathrm{Tr}\,[(V\eta_i)(V\eta_j)^*] \eta_j, \eta_i \rangle \\
&= \sum_{i=1}^{r} \mathrm{Tr}\,[(V\eta_i)(V\eta_i)^*] = \sum_{i=1}^{r} \langle V\eta_i, V\eta_i \rangle \\
&= \sum_{i=1}^{r} \langle V^* V \eta_i, \eta_i \rangle = \mathrm{Tr}\,(V^* V) = \|V\|^2.
\end{aligned}
$$

PROPOSITION 2.2. *Let $\phi \in \mathbb{P}_s$, where $s = 1, \dots, m \wedge n$ is a fixed integer. Then the following are equivalent:*

(i) *ϕ is an interior point of $\mathbb{P}_s$.*
(ii) *$\theta_\phi(V) > 0$ for each $V \in \mathbb{M}_s$.*

Proof. If $\phi \in \operatorname{int} \mathbb{P}_s$ then $\phi = (1-t)\tau + t\psi$ with $t < 1$ and $\psi \in \mathbb{P}_s$. Therefore, we have

$$\theta_\phi(V) = (1-t)\theta_\tau(V) + t\theta_\psi(V) = (1-t) + t\theta_\psi(V) > 0$$

whenever $V \in \mathbb{M}_s$. For the converse, we note that the map $V \mapsto \theta_\phi(V)$ is a continuous function on the compact set $\mathbb{M}_s$. We take ε with $0 < \varepsilon < 1$ such that

$$V \in \mathbb{M}_s \implies \theta_\phi(V) \geq \varepsilon,$$

and put $\psi = \left(1 - \frac{1}{1-\varepsilon}\right)\tau + \frac{1}{1-\varepsilon}\phi$. Then we have

$$\theta_\psi(V) = \left(1 - \frac{1}{1-\varepsilon}\right)\theta_\tau(V) + \frac{1}{1-\varepsilon}\theta_\phi(V) \geq 0,$$

for each $V \in \mathbb{M}_s$, and so $\psi \in \mathbb{P}_s$. Since $\frac{1}{1-\varepsilon} > 1$, we have $\phi \in \operatorname{int} \mathbb{P}_s$. □

Now, we are ready to characterize maximal faces of the convex cone $\mathbb{P}_s$ for $s = 1, \dots, m \wedge n$.

THEOREM 2.3. *Let $s = 1, \dots, m \wedge n$ be a fixed integer. If $V \in \mathbb{M}_s$ then the set $F_s[V]$ is a maximal face of the convex cone $\mathbb{P}_s$. Conversely, every maximal face of $\mathbb{P}_s$ is of the form $F_s[V]$ with an $V \in \mathbb{M}_s$.*

Proof. Take a family $\mathcal{V}$ of $m \times n$ matrices such that $\operatorname{span} \mathcal{V} = V^\perp$. Applying Theorem 2.5 (vii) of [**K1**], it suffices to show that

$$\phi \in \partial\mathbb{P}_s \setminus F_s[V] \implies \psi := \frac{1}{2}\phi_{\mathcal{V}} + \frac{1}{2}\phi \in \operatorname{int} \mathbb{P}_s.$$

Assume that there is $W \in \mathbb{M}_s$ such that $\theta_\phi(W) = 0$. Then we have

$$\theta_\phi(W) = \theta_{\phi_{\mathcal{V}}}(W) = 0.$$

The relation (2.4) and $\theta_{\phi_{\mathcal{V}}}(W) = 0$ shows that $W \in (\operatorname{span} \mathcal{V})^\perp$, and so W is a scalar multiple of V. Hence, we see that $\theta_\phi(W) > 0$ since $\phi \notin F_s[V]$. This contradiction says that $\theta_\phi(W) > 0$ for each $W \in \mathbb{M}_s$, and so $\psi \in \operatorname{int} \mathbb{P}_s$ by Proposition 2.2.

For the converse, let F be a maximal face of $\mathbb{P}_s$ with an interior point ϕ. Then there is $V \in \mathbb{M}_s$ such that $\theta_\phi(V) = 0$ by Proposition 2.2. Then $\phi \in F_s[V]$ by definition, and so it follows that $F \subset F_s[V]$ since $F_s[V]$ contains an interior point of F. The conclusion follows from the maximality of F. □

3. Relations between $\mathbb{P}_s$ and $\mathbb{P}_t$

By the discussion in the previous section, we see that the boundary of $\mathbb{P}_s$ is determined by the linear functional (1.1) with $V \in \mathbb{M}_s$. If s increase then more linear functionals are needed to determine the boundary of $\mathbb{P}_s$.

THEOREM 3.1. *Let $s, t = 1, 2, \ldots, m \wedge n$ be given with $s < t$. Then we have the following:*

(i) *If $V \in \mathbb{M}_s$ then $F_t[V] = F_s[V] \cap \partial\mathbb{P}_t \subset \partial\mathbb{P}_s$.*
(ii) *If $V \in \mathbb{M}_t \setminus \mathbb{M}_s$ then $\operatorname{int} F_t[V] \subset \operatorname{int} \mathbb{P}_s$.*

Proof. Assume that $V \in \mathbb{M}_s$. Then by definition (2.3), we have $F_s[V] \cap \mathbb{P}_t = F_t[V]$, and so $F_s[V] \cap \partial\mathbb{P}_t \subset F_t[V]$. Because $F_t[V]$ lies on the boundary of $\mathbb{P}_t$ the required relation hold.

For the second part, assume that there is $\phi \in \operatorname{int} F_t[V] \setminus \operatorname{int} \mathbb{P}_s$. Then ϕ lies on the boundary of $\mathbb{P}_s$ and so $\phi \in F_s[W]$ for a $W \in \mathbb{M}_s$. Since $\phi \in F_t[V] \in \partial\mathbb{P}_t$, we have $\phi \in F_t[W]$ by the first part. But, $\phi \in \operatorname{int} F_t[V]$ implies that $F_t[V] \subset F_t[W]$, and we have $F_t[W] = F_t[V]$ by the maximality. Considering completely positive linear maps in these two faces, we see that $\{V, W\}$ is linearly dependent. Therefore, we have $V \in \mathbb{M}_s$. □

COROLLARY 3.2. *Let $s, t = 1, 2, \ldots, m \wedge n$ be given with $s < t$, and $V \in \mathbb{M}_t$. Then we have*

$$F_t[V] \subset \partial\mathbb{P}_s \iff V \in \mathbb{M}_s.$$

In [**K2**], we have seen that every maximal face of $\mathbb{P}_{m\wedge n}$ contains a unital completely positive linear map. We denote by $\mathbb{P}_s[I]$ the convex compact set of all s-positive unital linear maps from M_m into M_n, and put

$$F_s[V,I] = F_s[V] \cap \mathbb{P}_s[I] \tag{3.1}$$

for $V \in \mathbb{M}_s$. Then every results in this note hold when $\mathbb{P}_s$ and $F_s[V]$ are replaced by $\mathbb{P}_s[I]$ and $F_s[V,I]$.

We conclude this note with comments on copositive linear maps. We denote by $\mathrm{tp}_s : M_s \to M_s$ the transpose maps. A linear map $\phi : A \to B$ between C^*-algebras is said to be *s-copositive* if the linear map $\phi \otimes \mathrm{tp}_s : A \otimes M_s \to B \otimes M_s$ is positive, and *completely copositive* if it is s-copositive for each natural number $s = 1, 2, \ldots$. We denote by $\mathbb{P}^s[A,B]$ (respectively $\mathbb{P}^\infty[A,B]$) the convex cone of all s-copositive (respectively completely copositive) linear maps from A into B.

In order to characterize the boundary of $\mathbb{P}^s$, we modify (2.1) to define

$$\begin{aligned} \theta^\phi(V) :&= \langle (\phi \otimes \mathrm{tp}_r)(P_V)\eta_V, \eta_V \rangle \\ &= \sum_{i,j=1}^{r} \langle \phi \circ \sigma_V(\eta_j \eta_i^*)\eta_j, \eta_i \rangle \end{aligned} \tag{3.2}$$

The proofs of the following theorem about the convex cone $\mathbb{P}^s$ are same as before, and will be omitted.

PROPOSITION 3.3. *For a fixed natural number $s = 1, \ldots, m \wedge n$, we have the following:*

(i) *A linear map $\phi : M_m \to M_n$ lies in $\mathbb{P}^s$ if and only if $\theta^\phi(V) \geq 0$ for each $V \in \mathbb{M}_s$.*

(ii) *$\phi \in \mathbb{P}^s$ is an interior point of $\mathbb{P}^s$ if and only if $\theta^\phi(V) > 0$ for each $V \in \mathbb{M}_s$.*

(iii) *For each $V \in \mathbb{M}_s$, the set*

$$F^s[V] := \{\phi \in \mathbb{P}^s : \theta^\phi(V) = 0\} \tag{3.3}$$

is a maximal face of the convex cone $\mathbb{P}^s$. Conversely, every maximal face of $\mathbb{P}^s$ is of the form $F^s[V]$ with an $V \in \mathbb{M}_s$.

(iv) *Let $s, t = 1, 2, \ldots, m \wedge n$ be given with $s < t$. If $V \in \mathbb{M}_s$ then $F^t[V] = F^s[V] \cap \partial\mathbb{P}^t$. If $V \in \mathbb{M}_t \setminus \mathbb{M}_s$ then $\mathrm{int}\, F^t[V] \subset \mathrm{int}\, \mathbb{P}^s$.*

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