J. Korean Math. Soc. **33** (1996), No 3, pp. 625–639

DEFORMATION SPACES OF CONVEX REAL–PROJECTIVE STRUCTURES AND HYPERBOLIC AFFINE STRUCTURES

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0. Introduction

A convex $\mathbb{RP}^n$-structure on a smooth manifold M is a representation of M as a quotient of a convex domain $\Omega \subset \mathbb{RP}^n$ by a discrete group Γ of collineations of $\mathbb{RP}^n$ acting properly on Ω. When M is a closed surface of genus $g > 1$, then the equivalence classes of such structures form a moduli space $\mathfrak{P}(M)$ homeomorphic to an open cell of dimension $16(g\text{-}1)$(Goldman [2]). This cell contains the Teichmüller space $\mathcal{T}(M)$ of M and it is of interest to know what of the rich geometric structure extends to $\mathfrak{P}(M)$. In [3], a symplectic structure on $\mathfrak{P}(M)$ is defined, which extends the symplectic structure on $\mathcal{T}(M)$ defined by the Weil-Petersson Kähler form.

The first step in this project is a construction of a Riemannian metric on $\mathfrak{P}(M)$. This metric exists for structures in all dimensions. The basic technique is that a canonically defined Riemannian metric on Ω/Γ defines a Riemannian structure on the moduli space via the Hodge theory of harmonic forms. Using Hodge theory and the canonical Riemannian metric constructed by Koszul [9] and Vinberg [16], we define a Riemannian metric g on $\mathfrak{P}(M)$. In dimension two, this deformation space enjoys a symplectic structure and combining these two structures we define a bundle map J on the tangent bundle $T\mathfrak{P}(M)$ and show that J is an almost complex structure. Thus $(\mathfrak{P}(M), g, J)$ is an almost Kähler

Received October 29, 1995.

1991 AMS Subject Classification: 53A15, 53A20, 53C15.

key words and phrases: affine connection, convex projective structure, sharp convex cone, Hessian metric, Weil-Petersson metric, Teichmuller space.

The authors acknowledge support from National Science Foundation grants DMS-9205139, and DMS-9504764.

structure and we conjecture that this is a Kähler structure, that is J is an integrable almost complex structure.

Another result is the following description of $\mathfrak{P}(M)$. Fix $\lambda > 1$ and let A be the multiplicative group of intergral powers of λ. For any manifold M, let M' denote the Cartesian product $M \times S^1$. Let $\frac{\partial}{\partial\theta}$ be the vector field on M' generating the flow

$$\Phi_t : (x, \theta) \longmapsto (x, \theta + t).$$

Let $d\theta$ the closed 1-form on M' defined by the projection $M' \to S^1$. The group $\mathcal{D}$ of diffeomorphisms of M isotopic to the identity acts on M' by

$$h : (x, \theta) \to (h(x), \theta)$$

and hence acts on the space of all affine connections on M'.

THEOREM 1. *The deformation space* $\mathfrak{P}(M)$ *of convex* $\mathbb{RP}^n$*-structures on* M *identifies with the space of* $\mathcal{D}$*-orbits of flat torsionfree affine connections* ∇ *on* M' *such that:*

- $\frac{\partial}{\partial\theta}$ *is radiant with respect to* ∇*: for any vector field* $X \in$ *Vect* (M'),
$$\nabla_X \frac{\partial}{\partial\theta} = X;$$
- $\nabla(d\theta) > 0$ *;*
- *Each trajectory of* Φ *is an closed geodesic affinely isomorphic to a Hopf circle* $\mathbb{R}_+/\wedge$*;*

1. Deformation space of convex $\mathbb{RP}^n$-structures

Let M be a smooth 2-manifold. An $\mathbb{RP}^n$-structure on M is maximal collection $\{(U_\alpha, \psi_\alpha)\}$, such that

- $\{U_\alpha\}$, is an open cover of M,
- For each $\alpha, \psi_\alpha : U_\alpha \to \mathbb{RP}^n$ is a surjective diffeomorphism,
- The change of coordinate are locally projective : If $\{(U_\alpha, \psi_\alpha)\}$ and $\{(U_\beta, \psi_\beta)\}$ are two such coordinate charts, then the restriction of $\psi_\beta \circ \psi_\alpha^{-1}$ to any connected component of $\psi_\beta^{-1}(\psi_\alpha(U_\alpha \cap U_\beta))$ is a projective transformation.

A manifold with an $\mathbb{RP}^n$-structure is called an $\mathbb{RP}^n$-manifold. A fundamental fact about $\mathbb{RP}^n$-structures is the following Development Theorem, due to Ch. Ehresmann in 1936 ([1]).

THEOREM 2. *Let M be an $\mathbb{RP}^n$-manifold and denote its a universal covering space by $p : \tilde{M} \to M$ Let π be the corresponding group of covering transformations.*

(1) *There exist a projective map $\text{dev} : \tilde{M} \to \mathbb{RP}^n$ and a homomorphism $h : \pi \to PGL(n+1, \mathbb{R})$ such that for each $\gamma \in \pi$, the following diagram commutes:*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\text{dev}} & \mathbb{RP}^n \\ \gamma \big\downarrow & & \big\downarrow h(\gamma) \\ \tilde{M} & \xrightarrow[\text{dev}]{} & \mathbb{RP}^n \end{array}$$

(2) *If (dev', h') is another such pair, then there exists a projective transformation $g \in PGL(n+1, \mathbb{R})$ such that $\text{dev}' = g \circ \text{dev}$ and $h' = i_g \circ h$ where*

$$i_g : PGL(n+1, \mathbb{R}) \to PGL(n+1, \mathbb{R})$$

denotes the inner automorphism defined by g [4].

An $\mathbb{RP}^n$-structure on M is called convex if dev is a diffeomorphism of $\tilde{M}$ onto a convex domain in $\mathbb{RP}^n$.

Let S denote a compact smooth n-manifold. Define

$$\varepsilon = \{(f, M) \mid f : S \to M \text{ is a diffeomorphism and } M \text{ is an } \mathbb{RP}^n - \text{manifold}\}.$$

Two elements $(f, M), (f', M') \in \varepsilon$ are equivalent if and only if there exists a projective isomorphism $h : M \to M'$ such that $h \circ f$ is isotopic to f'. The set of equivalence classes (denoted By $\mathbb{RP}^n(S)$) has a natural topology making it locally equivalent to Hom $(\pi, PGL(n+1, \mathbb{R}))/PGL(n+1, \mathbb{R})$. In other words there exists a map

$$\text{hol} : \mathbb{RP}^n(S) \to \text{Hom}(\pi, \text{PGL}(n+1, \mathbb{R}))/\text{PGL}(n+1, \mathbb{R})$$

which is a local diffeomorphism. Let $\mathfrak{P}(M)$ denote the subset of $\mathbb{RP}^n(S)$ corresponding to convex $\mathbb{RP}^n$-structures. $\mathfrak{P}(M)$ is an open set and the restriction of hol to $\mathfrak{P}(M)$ is an embedding of $\mathfrak{P}(M)$ onto an open subset of $\mathrm{Hom}(\pi, PGL(n+1,\mathbb{R}))/PGL(n+1,\mathbb{R})$ ([2]). This open set identifies with an open subset of a real algebraic variety $X(M)$. The Zariski tangent space to $X(M)$ at $\{\phi\}$ is isomorphic to $H^1(\pi, sl(n+1,\mathbb{R})_{Adh})$ which by de Rham's theorem is isomorphic to $H^1(S,\xi)$ where ξ is the flat $sl(n+1,\mathbb{R})$-bundle over S with holonomy representation $\mathrm{Ad}(h)$. (See [5]).

2. Hessian manifolds

Let M be a (flat) affine manifold. A Riemannian metric g on M is said to be Hessian if for each point p there exists a function f defined on a neighborhood of p such that $\nabla df > 0$. A flat affine manifold provided with a Hessian metric is called a Hessian manifold. (Compare Shima [12], [13]).

An open subset $\Omega \subset \mathbb{R}^{n+1}$ is said to a cone if it is invariant under the group $\mathbb{R}^+$ of positive homotheties. A convex cone is sharp if it does not contain any full straight line.

The following theorem is due to Koszul [9] and Vey [15]:

THEOREM 3. *Let M be a connected Hessian manifold with Hessian metric g. Suppose that admits a closed 1-form α such that $\nabla\alpha = g$ and there exists a group G of affine automorphisms of M preserving α.*

- *If M/G is quasi-compact, then the universal covering manifold of M is affinely isomorphic to a convex domain Ω real affine space not containing any full straight line;*
- *If M/G is compact, then Ω is a sharp convex cone.*

In the latter case, M is said to be hyperbolic affine manifold. The construction of an affinely invariant Hessian metric on a sharp convex cone Ω is due to Koszul [9] and Vinberg [16].

Suppose that $(f, M) \in \varepsilon$ corresponds to a convex $\mathbb{RP}^n$-structure on S as above. In fact M is a quotient Ω/Γ where $\Omega \subset \{\mathbb{RP}\}^n$ is a convex domain and $\Gamma \subset \mathrm{PGL}(n+1,\mathbb{R})$ is a discrete group acting properly on Ω. Let $\Omega' \subset \mathbb{R}^3$ be the corresponding cone in affine space $E = \mathbb{R}^3$.

The dual cone Ω^* is the subset of the dual vector space E^* consisting of linear functionals $\psi : E \to \mathbb{R}$ which are positive on Ω'. Recall the Koszul-Vinberg characteristic function: For x $\in \Omega'$, define

$$f(x) = \int_{\Omega^*} e^{-\psi(x)} d\psi$$

Then f satisfies

$$f(\gamma x) = \det(\gamma)^{-1} f(x) \tag{1}$$

for any $\gamma \in \mathrm{Aut}(\Omega')$ and the Hessian $d^2 \log f$ is a positive definite symmetric bilinear form on E invariant under $\mathrm{Aut}(\Omega')$.

Now consider the section

$$k : \Omega \to \Omega'$$
$$k([p]) \longmapsto f(p)^{1/(n+1)} p$$

By (1), k is well-defined and $k(\Omega) = f^{-1}(1)$.

$$g_\Omega = k^*(d^2 \log f) = Dk^*(d \log f)$$

is a Riemannian metric on Ω invariant under Γ. The closed 1-form

$$\alpha_\Omega = k^*(d \log f)$$

is also invariant under Γ, and satisfies

$$D\alpha_\Omega = g_\Omega$$

Hence $k^*(d^2 \log f)$ defines a Hessian metric on Ω/Γ.

3. A Weil-Petersson metric on $\mathfrak{P}(M)$

A convex $\mathbb{RP}^n$-structure on S determines a canonical metric on S, so there exists a Weil-Petersson metric on $\mathfrak{P}(M)$, defined as follows. Let $[M] \in \mathfrak{P}(S)$. From § 1,

$$T_{[M]}\mathfrak{P}(S) = H^1(M; \xi)$$

The space $\mathcal{A}^1(M,\xi)$ of all ξ-valued 1-forms on M consists of sections of the vector bundle $\mathrm{Hom}(TM,\xi)$. For every $x \in f^{-1}(1)$, the symmetric bilinear form $(d^2 \log f)_x$ defines an inner product

$$\mathrm{Hom}(\mathbb{R}^3,\mathbb{R}^3) \cong \mathbb{R}^3 \otimes (\mathbb{R}^3)^* \tag{2}$$

and therefore induces a Riemannian metric on the bundle ξ. If ϕ, ψ are sections of ξ, then the induced inner product is

$$d^2 \log f(\phi,\psi) = \mathrm{trace}(\phi \circ \tilde{\psi})$$

where $\tilde{\psi}$ is the adjoint of ψ with respect to $d^2 \log f$ ([7]).

On the other hand the Hodge star operator associated to the metric on M defines a metric on $\mathcal{A}^1(M)$. The metric on M induces a positive definite inner product g on $\mathcal{A}^1(M, sl(n+1,\mathbb{R})_{Adh})$. Let

$$\sigma \otimes \phi, \sigma' \otimes \phi' \in \mathcal{A}^1(M, sl(n+1,\mathbb{R})_{Adh})$$

where $\sigma, \sigma' \in \mathcal{A}^1(M)$ and ϕ', ϕ are sections of $sl(n+1,\mathbb{R})_{Adh})$. Then (see [17])

$$g(\sigma \otimes \phi, \sigma' \otimes \phi') \int_M (\sigma \wedge *\sigma')\ \mathrm{trace}(\phi \circ \tilde{\phi}')$$

where $\tilde{\phi}'$ denotes the adjoint of ϕ' as in (2).

This metric induces a metric g on the cohomology $H^1(M;\xi)$ as follows: Consider the operator δ adjoint to exterior differential d with respect to this inner product, and the corresponding Laplacian Δ on 1-forms:

$$\Delta = d\delta + \delta d$$

The kernel $\mathcal{H}^1(M;\xi)$ of Δ and the images of $d : \mathcal{A}^0(M;\xi) \to \mathcal{A}^1(M;\xi)$ and $\delta : \mathcal{A}^2(M;\xi) \to \mathcal{A}^1(M;\xi)$ decompose the vector space of 1-forms as an orthogonal direct sum

$$\mathcal{A}^1(M;\xi) = \mathcal{H}^1(M;\xi) \oplus d\mathcal{A}^0(M;\xi) \oplus \delta\mathcal{A}^2(M;\xi)$$

Consequently each de Rham cohomology class contains a unique harmonic representative. Define the pairing

$$g : \mathcal{H}^1(M;\xi) \times \mathcal{H}^1(M;\xi) \to \mathbb{R}$$

as the tensor product of the inner product on exterior differential forms and the metric on ξ induced from $x^* d^2 \log f$.

4. A symplectic form on $\mathfrak{P}(M)$

Now suppose $n = 2$. As in §1, the restriction of hol : $\mathbb{RP}^n(S) \to$ Hom $(\pi, \text{PGL}\ (n+1,\mathbb{R}))/\ \text{PGL}\ (n+1,\mathbb{R})$ to $\mathfrak{P}(S)$ embeds $\mathfrak{P}(S)$ as an open subset of Hom$(\pi, \text{PGL}\ (n+1,\mathbb{R}))/\ \text{PGL}\ (n+1,\mathbb{R})$. On the other hand the trace form

$$B : sl(n+1,\mathbb{R}) \times sl(n+1,\mathbb{R}) \to \mathbb{R}$$
$$B(X,Y) = \text{trace}(XY)$$

is an Ad-invariant bilinear form, so defines a bundle pairing $\xi \times \xi \to \mathbb{R}$ The natural dual pairing

$$w : H^1(M;\xi) \times H^1(M;\xi) \to H^2(M;\mathbb{R}) \cong \mathbb{R}$$

defined by the cup-product on M and with B as a coeficient pairing defines a symplectic form on $H^1(M:\xi)$. The induced symplectic structure on

$$\text{Hom}(\pi, \text{PGL}(n+1,\mathbb{R}))/\text{PGL}(n+1,\mathbb{R})$$

gives a symplectic structure on $\mathbb{RP}^n(S)$ and in particular one on $\mathfrak{P}(S)$. (See [5] and, for a more analytic treatment, [3].)

As in §1, identify $H^1(\pi; sl(n+1,\mathbb{R}))$ with $H^1(M;\xi)$. Let $\alpha, \alpha' \in H^1(M:\xi)$ and let $\sum_{i=1}^k \sigma_i \otimes \phi_i$ and $\sum_{i=1}^l \sigma_i' \otimes \phi_i'$ be harmonic forms representing α, α' respectively. Then

$$\sum_{1\le i\le k, 1\le j\le l} (\sigma_i \wedge \sigma_j') \otimes B(\phi_i, \phi_j')$$

is an exterior 2-form and its integral defines the symplectic structure on $\mathfrak{P}(S)$:

$$w(\alpha,\alpha') = \int_M \sum_{1\le i\le k, 1\le j\le l} (\sigma_i \wedge \sigma_j') \otimes B(\phi_i, \phi_j')$$

5. An almost complex structure on $\mathfrak{P}(M)$

Comparing the previous Riemannian and symplectic structures yields an almost complex structure on $\mathfrak{P}(M)$ as follows. Define an operator J on $\mathcal{A}^1(M;\xi)$ by

$$J(\sigma \otimes \phi) = - * \sigma \otimes \tilde{\phi} \tag{3}$$

where

$$* : \mathcal{A}^1(M) \to \mathcal{A}^1(M)$$

is the Hodge *-operator and $\tilde{\phi}$ denotes the adjoint of ϕ as in (2). For $n = 2$, the Hodge *-operator satisfies

$$* \circ * = -I$$

and since the adjoint operation

$$\phi \longmapsto \tilde{\phi}$$

has order two, it follows that

$$J \circ J = -I$$

that is, J, defines an almost complex structure.

LEMMA 4. *The Riemannian metric g, the symplectic structure w, and the almost complex structure J are related by*

(1) $w(\alpha, \alpha') = g(\alpha, J\alpha')$
(2) $g(\alpha, \alpha') = g(J\alpha, J\alpha')$

for $\alpha, \alpha' \in \mathcal{A}^1(M;\xi)$.

Proof. For both parts, it suffices to consider the case when $\alpha = \sigma \otimes \phi$ and $\alpha' = \sigma' \otimes \phi'$ where $\sigma, \sigma' \in \mathcal{A}^1(M)$ and σ, σ' are sections of ξ.

$$\begin{aligned} g(\alpha, J\alpha') &= g(\sigma \otimes \phi, *\sigma' \otimes \tilde{\phi}') \\ &= \int_S \sigma \wedge *(- * \sigma') \ \mathrm{trace}(\phi \circ \phi) \\ &= \int_S \sigma \wedge \sigma' \ \mathrm{trace}(\phi \circ \phi) \\ &= w(\alpha, \alpha') \end{aligned}$$

proving (1). For (2), use the facts that g is symmetric, w is alternating and (1):

$$\begin{aligned} g(J\alpha, J\alpha') &= w(J\alpha, \alpha') \\ &= -w(\alpha', J\alpha) \\ &= -g(\alpha', JJ\alpha) \\ &= g(\alpha', \alpha) \\ &= g(\alpha, \alpha') \end{aligned}$$

as desired $\square$

Thus we have proved that $(\mathfrak{P}(M), g, w, J)$ is an almost Kähler structure. We conjecture that J is an integrable almost complex structure, that is, this almost Kähler structure is an actual Kähler structure.

6. Affine connections

Now we shall describe an explicit construction associating to a convex $\mathbb{RP}^n$-manifold M a hyperbolic affine $n+1$-manifold, in fact a whole family of compact hyperbolic affine $n+1$-manifolds diffeomorphic to $M \times S^1$.

For each $\lambda \in \mathbb{R}^+$, let h_λ denote the homothety

$$S \times \mathbb{R}^+ \to S \times \mathbb{R}^+$$

$$(s, t) \longmapsto (s, \lambda t)$$

Let dt denote the 1-form on $S \times \mathbb{R}^+$ pulled back from dt on $\mathbb{R}^+$ by projection

$$t : S \times \mathbb{R}^+ \to \mathbb{R}^+.$$

The $t^{-1}dt$ is a 1-form on $S \times \mathbb{R}^+$ invariant under the homotheties above.

Lemma 5. *Let S be a closed manifold with convex $\mathbb{RP}^n$-structure. Then there exit a radiant affine manifold M, a diffeomorphism*

$$f : S \times \mathbb{R}^+ \to M$$

and an exact 1-form α_M on M. Let

$$\alpha_S = (f^{-1})^* \alpha_M$$

be the corresponding 1-form on $S \times \mathbb{R}^+$. Then

$$\alpha_S = t^{-1} dt, \qquad (h_\lambda)^* \alpha_S = \lambda \alpha_S$$

Proof. The convex $\mathbb{RP}^n$-structure on S induces a convex $\mathbb{RP}^n$-structure on $\tilde{S}$. Let x be a base-point in S. Let $\Pi : \tilde{S} \to S$ be the corresponding universal covering space and π the corresponding group of deck transformations. By the Development Theorem there exist a projective map dev and a homomorphism ρ such that dev is equivariant with respect to ρ. Let Ω' be the corresponding affine cone. Projectivization defines the structure of a principal $\mathbb{R}^+$-bundle $\Omega' \to \Omega$. By definition, dev is a diffeomorphism onto a convex domain Ω. By pulling back this bundle via dev, we obtain a principal $\mathbb{R}^+$-bundle. The open cone Ω' whose projectivization is Ω is the total space of a principal $\mathbb{R}^+$-bundle. Pulling back this bundle via dev produces a principal $\mathbb{R}^+$-principal bundle over $\tilde{M}$. The affine structure on Ω'. induced from $\mathbb{R}^{n+1} - \{0\}$, induces an affine structure on S'. There exists a lift

$$\tilde{h} : \pi_1(S) \to \mathrm{SL}(n+1, \mathbb{R})$$

of the homomorphism $h : \pi_1(S) \to \mathrm{PGL}\ (n+1, \mathbb{R})$ so that $\pi_1(S)$ acts affinely on S'. Clearly this action is proper and free. Hence the total space

$$\hat{S} = S'/\pi_1(S) \approx S \times \mathbb{R}^+$$

of a principal $\mathbb{R}^+$-bundle over S with holonomy representation h admits a radiant affine structure. The radiant vector field $\rho_{\hat{S}}$ generates the (fiberwise) affine action of $\mathbb{R}^+$ on $\hat{S}$, which is given locally in coordinates by homotheties. On the other hand every principal $\mathbb{R}^+$-bundle is trivial. Choose any $\lambda > 1$. The cyclic group $\langle \lambda \rangle \subset \mathbb{R}^+$ acts properly and freely by affine transformations on $\hat{S}$. The resulting affine manifold $\hat{S}/\langle \lambda \rangle$ is homeomorphic to $S \times S^1$. Corresponding to a convex real projective structure on S is a whole family of radiant affine structures on $S \times S^1$, one for each λ. The radiant vector field is, in fact, the vector field on $S \times S^1$ in the direction of S^1, which we denote by $\frac{\partial}{\partial \theta}$. Consider the characteristic function $f : \Omega' \to \mathbb{R}$ of Ω'. The logarithmic differential $d \log f$ is a closed 1-form on Ω' which is:

- positive definite ;
- invariant under $\mathrm{Aff}(\Omega')$

Then by the above construction, there exists a closed 1-form $\tilde{\alpha}$ on S' such that $\nabla \tilde{\alpha} > 0$ and is invariant under $\pi_1(S)$ and $\langle \lambda \rangle$. Consequently, there exists a closed 1-form α on $S \times S^1$ such that $\nabla \alpha > 0$. □

With the notation of Lemma 5,

LEMMA 6. *α represents the cohomology class of S^1, i.e. if $\pi_2 : S \times S^1 \to S^1$ denotes projection then $[\alpha] = \pi_2^*[S^1]$ where $[S^1] \in H^1(S^1)$ denotes a generator.*

The characteristic function on Ω' induces a function on S' which we again denote by $\log f$. The fundamental group of S naturally acts on S' by $\tilde{h}$. For all $\gamma \in \Gamma = \tilde{h}(\pi_1(S'))$,

$$\log f \circ \gamma = \log f - \log \det(\gamma)$$

But for $\gamma \in \pi_1(S'), \det(\gamma) = 1$, so $\log f = \log f \circ \gamma$. It follows that $\log f$ defines a function $l : \hat{S} \to \mathbb{R}$ such that $\alpha = dl$ is exact. Let $\Pi_\lambda : \hat{S} \to S_\lambda = \hat{S}/\langle \lambda \rangle$ denote projection and let $\hat{\alpha} = \Pi_\lambda^* \alpha$. There is a function l_λ related to l on S_λ such that $\hat{\alpha} = dl_\lambda$. The cyclic group $\langle \lambda \rangle$ is generated by

$$D_\lambda : z \longmapsto \lambda_z$$

and

$$\log(f \circ D_\lambda) = \log f - (n+1) \log \lambda$$

The α is a 1-form on $S \times S^1$. For every $\gamma \in \pi_1(S \times S^1) \cong \pi_1(S) \times \mathbb{Z}$,

$$\int_\gamma \alpha = \int_{\tilde{\gamma}} d \log f = \log f(\tilde{\gamma}\tilde{p}) - \log f(\tilde{p}) = -\log \det(\gamma)$$

where $\tilde{\gamma}$ is the lifting of γ and $\tilde{p}$ is an arbitrary point in S' whose projection by $S' \approx S \times S^1 \to S^1$ is the base point of S^1. Now

$$S'/\Gamma = \hat{S} \approx S \times \mathbb{R}^+$$

and $\pi_1(S) \cong \pi_1(\hat{S}) \cong \Gamma$ and for all $\gamma \in \Gamma$, the period of α around γ is zero. Also $\langle\lambda\rangle \subset \pi_1(S \times S^1)$, and $\forall\gamma \in \Gamma$ the period of α is zero. So by using the Hurewiez isomorphism

$$\begin{aligned} H^1(W, \mathbb{R}) &\cong \mathrm{Hom}(H_1(W, \mathbb{Z}), \mathbb{R}) \\ &\cong \mathrm{Hom}(\pi_1(W), \mathbb{R}) \end{aligned}$$

for $S \times S^1$, we have $[\alpha] = \pi_2^*[S^1]$.

Let $\frac{\partial}{\partial\theta}$ be the vector field on $S \times S^1$ in the direction of S^1, i.e. the infinitesimal generator of the flow:

$$\Theta_t : (s, u) \longmapsto (s, u + t)$$

for $u \in \mathbb{R}/\mathbb{Z}$. Let $d\theta$ be the 1-form dual to $\frac{\partial}{\partial\theta}$, i.e. $d\theta\frac{\partial}{\partial\theta} = 1$. Let C denote the set of all affine connections ∇ on $S \times S^1$ such that :

- ∇ is flat and torsionfree ;
- $\frac{\partial}{\partial\theta}$ is radiant with respect to ∇, i.e.

$$\nabla_X(\frac{\partial}{\partial\theta}) = X$$

  for all vector fields X on $S \times S^1$.
- $\nabla d\theta > 0$.

Before the main theorem, we prove the following lemma, which asserts that a radiant vector field is affine.

LEMMA 7. *Let ∇ be a flat torsionfree connection on N. Suppose ρ is a vector field on N and $\{\Theta_t\}_{t\in\mathbb{R}}$ be its flow. If ρ is radiant with respect to ∇, then ∇ is invariant under Θ_t.*

Proof. Define a derivation $A_\rho = L_\rho - \nabla_\rho$, where L_ρ is Lie derivative with respect to ρ. So

$$\begin{aligned} A_\rho(X) = L_\rho(X) - \nabla_\rho(X) &= [\rho, X] - (\nabla_X\rho + [\rho, X] + T(\rho, X)) \\ &= -\nabla_X\rho - T(\rho, X) \end{aligned}$$

since ρ is radiant and the torsion $T = 0$. Then $A_\rho(X) = -X$, that is, $A_\rho = -I$. By Prop. 2.6 of page 235 of [8], the vector field ρ is an infinitesimal affine transformation if and only if for all vector field Y on N

$$\nabla_Y(A_\rho) = R(\rho, Y)$$

where R is the curvature tensor. But $\nabla_Y(A_\rho) = \nabla_Y(-I) = 0$ (again since ∇ is torsionfree) and $R = 0$ (since ∇ is flat). Thus ρ is an infinitesimal affine transformation. Now by prop. 1.4, of [8] (p.228), ∇ is invariant with respect to Θ_t as desired. □

Let S be a closed surface with $\chi(S) < 0$ and fix a basepoint $p \in S$. Let ε denote the set of all pairs (f, M) where $f : S \to M$ is diffeomorphism and M is a convex $\mathbb{RP}^n$-manifold. Define a homomorphism

$$T : \mathrm{Diff}^0(S) \to \mathrm{Diff}(S \times S^1)$$

by $T(h)(s, u) = (h(s), u)$. If h fixes the basepoint $p \in S$, then $T(h)$ fixes the basepoint $(p, 0) \in S \times S^1$.

THEOREM 8. *The natural map*

$$\Phi : \varepsilon \to \mathcal{C}$$

is equivariant with respect to T and induces an isomorphism

$$\mathfrak{P}(S) \to \mathcal{C}/T(\mathit{Diff}^0(S))$$

By Lemma 5, corresponding to every (f, M) $\in \varepsilon$ is an element of $\mathcal{C}$. Conversely, let $\nabla \in \mathcal{C}$ and

$$\Omega_p \subset T_{(p,0)}(S \times S^1)$$

be the domain of the exponential map. By Koszul's theorem, Ω_p is a sharp convex cone and $\exp : \Omega_p \to S \times S^1$ is a covering map. Consider the projection map $\Pi : S \times S^1 \to S^1$. It is clear that $\frac{\partial}{\partial\theta}$ is transverse to level sets of Π. Lift the flow of $\frac{\partial}{\partial\theta}$ to Ω_p (denoting it by $\frac{\tilde\partial}{\partial\theta}$). The 1-form $\alpha = \exp^*(d\theta)$ is closed. $H^1(\Omega) = 0$ implies that there exists a function $\phi : \Omega \to \mathbb{R}$ such that $\alpha = d\phi$. Level sets of ϕ are transverse to the flow of $\frac{\tilde\partial}{\partial\theta}$ because the tangent space of a level set of an arbitrary point x is the kernel of $d\phi = \alpha$ at x and

$$1 = (d\theta)(\frac{\partial}{\partial\theta}) = (d\theta)(\exp \frac{\tilde\partial}{\partial\theta}) = (\exp^* d\theta)(\frac{\tilde\partial}{\partial\theta}) = d\phi(\frac{\tilde\partial}{\partial\theta})$$

Thus each level set $\phi^{-1}(c)$ is a cross-section of $\frac{\tilde{\partial}}{\partial\theta}$. On the other hand, Ω_p is a convex cone and the projectivization of $\phi^{-1}(c)$ is a convex domain in $\mathbb{RP}^n$. Now

$$\pi_1(S \times S^1) \cong \pi_1(S) \times \mathbb{Z}$$

and

$$\pi_1(S) \hookrightarrow SL(3, \mathbb{R})$$

so $\phi^{-1}(c)/\Gamma$ is a convex $\mathbb{RP}^n$-structure on S. The commutativity of the diagram:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\Phi} & \mathcal{C} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle T(h)} \\ \mathcal{E} & \xrightarrow[\Phi]{} & \mathcal{C} \end{array}$$

is obvious from the above construction and the proof of Lemma 5, that is, Φ is equivariant with respect to T, and induces the isomorphism

$$\mathfrak{P}(S) \to C/T(\mathrm{Diff}^0(S)).$$

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