J. Korean Math. Soc. **33** (1996), No. 3, pp. 587–599

A UNIFIED TREATMENT OF SOME SPECIAL PROPERTIES OF CONVEX CONFORMAL FUNCTIONS

JONG SU AN AND TAI SUNG SONG

1. Introdution

We assume throughout the paper that $f(z)$ is a conformal function in the open unit disk $D = \{z : |z| < 1\}$. In the rest of paper, conformal means analytic and locally univalent. The Schwarzian derivative of a function f is defined by

$$S_f(z) = \varphi'_f(z) - (1/2)\varphi_f(z)^2, \ \varphi_f(z) = f''(z)/f'(z).$$

A set is convex if it contains the line segment between any two of its points. It can be shown that $f(z)$ is convex in D if and only if $f(z)$ satisfies either one of the following two inequalities (see[1] p.5):

(1) $1 + \mathrm{Re} z\varphi_f(z) > 0, \ z \in D$

(2) $|(1 - |z|^2)\varphi_f(z) - 2\bar{z}| \leq 2, \ z \in D.$

We define the following class $C(r)$ in D. Let $C(r)$ be the class of all conformal functions in D which are convex on every hyperbolic disk in D of hyperbolic radius $\rho, r = \tanh\rho$, for some $r \leq 1$ ([1, p.2]). We have already noted the following four result:

(1) Nehari's result ([7]); Let $f(z)$ be a convex function in D. Then

$$(1 - |z|^2)^2 |S_f(z)| \leq 2.$$

(2) Pommerenke's result (Theorem 2.4 in [10]); For $\alpha > 1$, if f satisfies the inequality $|(1-|z|^2)\varphi_f(z) - 2\bar{z}| \leq 2\alpha$. for all $z \in D$, then

$$(1 - |z|^2)^2 |S_f(z)| \leq 2(\alpha^2 + 3\sqrt{3}\alpha + 3).$$

Received September 22, 1995.
1991 AMS Subject Classification: 30C99.
Key words: Schwarzian derivative, hyperbolic radius, convex function .

(3) Pommerenke's result (Theorem 2.5 in [10]): For $\alpha \geq 1$, if f satisfies the inequality $|(1-|z|^2)\varphi_f(z)-2\overline{z}| \leq 2\alpha$, for all $z \in D$, then

$$f \in C(\alpha - \sqrt{\alpha^2 - 1}).$$

(4) Pommerenke's result (Corollary 2.3 in [10]); For $\beta \geq 0$, if f satisfies the inequality $(1-|z|^2)^2|S_f(z)| \leq 2\beta$, for all $z \in D$, then

$$|(1-|z|^2)\varphi_f(z) - 2\overline{z}| \leq 2\sqrt{1+\beta}.$$

In this paper we show Theorem 2.2 and Theorem 3.1. Theorem 3.1 is an improvement of Pommerenke's result (2). Both Theorem 2.2 and Theorem 3.1 follow consequences of Theorem 2.1. We find values for α and β such that f satisfies

$$(*) \qquad |(1-|z|^2)\varphi_f(z) - 2\overline{z}| \leq 2\alpha, \text{ for some } \alpha \geq 1, z \in D$$

and

$$(**) \qquad (1-|z|^2)^2|S_f(z)| \leq 2\beta, \text{ for some } \beta \geq 0, z \in D.$$

Using results of P.Beesack-B.Schwarz[3] and D.Minda[5] we derive an estimation for the uniform radius of univalence of functions in $C(r)$ satisfying the above property (*).

2. The lower bounds of functions having special property

Let Ω be a simply connected region in the complex plane $\mathbf{C}$ ($\mathbf{C} \neq \Omega$) and $r \in (0,1)$. Denote

$$\varphi_{fog}(z) = \frac{(f \circ g)''(z)}{(f \circ g)'(z)}, \text{ where } g(z) = \frac{z+a}{1+\overline{a}z}, \; a \in D.$$

If the condition $w = 1 + z\varphi_{fog} \in \Omega$, for every $z \in D_r = \{z : |z| < r\}$ and for all $a \in D$, holds, then we say that $f(z)$ is Ω-locally convex with radius r, and denote by $C(\Omega, r)$, the class of all such functions. We note $1 \in \Omega$. Hence by the Riemann Mapping Theorem there exists a unique analytic function $h(w)$ which maps Ω onto D, such that $h(1) = 0$ and $h'(1) > 0$.

LEMMA 2.1. *For* $(z,a) \in D \times D$,

$$(1+\overline{a}z)\varphi_{f\circ g}(z) = (1-|a|^2)\varphi_f(g(z)) - 2\overline{a}(1+\overline{a}z).$$

Proof.

$$\varphi_{f\circ g}(z) = \frac{(f\circ g)''(z)}{(f\circ g)'(z)} = \frac{f''(g(z))(1-|a|^2)}{f'(g(z))(1+\overline{a}z)^2} - \frac{2\overline{a}(1+\overline{a}z)}{(1+\overline{a}z)^2}.$$

This completes the proof. □

LEMMA 2.2. *For* $z \in D$,

$$\left|\left(\frac{h(1+z\varphi_{f\circ g}(z))}{z}\right)'\right|_{z=0} + \left|\frac{h(1+z\varphi_{f\circ g}(z))}{z}\right|^2_{z=0} \leq \frac{1}{r^2}.$$

Proof. By the Schwarz-Pick Lemma we have

$$\frac{r\left|\left(\frac{h(1+z\varphi_{f\circ g}(z))}{z}\right)'\right|}{1-r^2\left|\frac{h(1+z\varphi_{f\circ g}(z))}{z}\right|^2} \leq \frac{r}{r^2-|z|^2}, \text{ for } z \in D_r,$$

$$(r^2-|z|^2)\left|\left(\frac{h(1+z\varphi_{f\circ g}(z))}{z}\right)'\right| \leq 1-r^2\left|\frac{h(1+z\varphi_{f\circ g}(z))}{z}\right|^2.$$

For near z=0, we derive the result. □

LEMMA 2.3.

$$\left.\frac{d}{dz}\varphi_{f\circ g}(z)\right|_{z=0} = (1-|a|^2)^2\varphi_f'(a) - 2\overline{a}[(1-|a|^2)\varphi_f(a) - \overline{a}].$$

Proof.

$$\varphi_{f\circ g}(z) = \frac{f''(g(z))(g'(z))}{f'(g(z))} + \frac{g''(z)}{g'(z)}.$$

$$\frac{d}{dz}(\varphi_{f\circ g}(z)) = \frac{f'''(g(z))(g'(z))^2}{f'(g(z))} - \left(\frac{f''(g(z))g'(z)}{f'(g(z))}\right)^2 + \frac{f''(g(z))g''(z)}{f'(g(z))} + \frac{g'''(z)}{g'(z)} - \left(\frac{g''(z)}{g'(z)}\right)^2.$$

Since $g(z) = (z+a)/(1+\overline{a}z)$, we have

$$\left.\frac{d}{dz}(\varphi_{f\circ g}(z))\right|_{z=0} = (1-|a|^2)^2\left\{\frac{f'''(a)}{f'(a)} - \left(\frac{f''(a)}{f'(a)}\right)^2\right\} - 2\overline{a}(1-|a|^2)\varphi_f(a) + 2(\overline{a})^2$$

$$= (1-|a|^2)^2\varphi_f'(a) - 2\overline{a}[(1-|a|^2)\varphi_f(a) - \overline{a}].$$

This completes the proof. □

THEOREM 2.1. *If $f \in C(\Omega, r)$ for a given Ω and $r \in (0,1]$, then*

$$|(1-|z|^2)\varphi_f(z) - 2\overline{z}| \le [h'(1)]^{-1}r^{-1},$$

$$(1-|z|^2)^2|S_f(z)| + c|(1-|z|^2)\varphi_f(z) - 2\overline{z}|^2 \le [h'(1)]^{-1}r^{-2}$$

where $z \in D$ and $c = h'(1) - \frac{1}{2}|1+\rho_h(1)|$.

Proof. By definition of $C(\Omega, r)$ and $h(w)$, it readily follows that the composition $h(1+z\varphi_{f\circ g}(z))$ is an analytic function of z which maps D_r into D and it vanishes at $z=0$. Hence by the Schwarz Lemma we conclude that the function $h(1+z\varphi_{f\circ g}(z))/z$ is also analytic in z and maps D_r into $D_{1/r}$. Therefore we have in particular

$$|h(1+z\varphi_{f\circ g}(z))/z| \le 1/r.$$

For near $z=0$, using Lemma 2.1, we have

$$\varphi_{f\circ g}(0) = (1-|a|^2)\varphi_f(a) - 2\overline{a},\ a \in D.$$

We get

$$h'(1)\varphi_{f\circ g}(0) = h'(1)[(1-|a|^2)\varphi_f(a) - 2\overline{a}] \le r^{-1}.$$

Hence

$$|(1-|z|^2)\varphi_f(z)-2\overline{z}| \le [h'(1)]^{-1}r^{-1},\ z \in D.$$

On the other hand, using Lemma 2.1, we get

$$\left.\frac{h(1+z\varphi_{f\circ g}(z))}{z}\right|_{z=0} = h'(1)\varphi_{f\circ g}(0) = h'(1)[(1-|a|^2)\varphi_f(a)-2\overline{a}].$$

Expands $h(1+z\varphi_{f\circ g}(z))/z$ into a power series in z :

$$h(1+z\varphi_{f\circ g}(z))/z$$

$$= h'(1)\varphi_{f\circ g}(0) + \left\{h'(1)\left.\frac{d}{dz}\varphi_{f\circ g}(z)\right|_{z=0} + \frac{1}{2}h''(1)\varphi^2_{f\circ g}(0)\right\}z + \dots .$$

For near $z=0$,

$$\left.\left(\frac{h(1+z\varphi_{f\circ g}(z))}{z}\right)'\right|_{z=0} = h'(1)\left.\frac{d}{dz}\varphi_{f\circ g}(z)\right|_{z=0} + \frac{1}{2}h''(1)\varphi^2_{f\circ g}(0).$$

By Lemma 2.3,

$$[h'(1)]^{-1}\left.\left(\frac{h(1+z\varphi_{f\circ g}(z))}{z}\right)'\right|_{z=0} = \left.\frac{d}{dz}\varphi_{f\circ g}(z)\right|_{z=0} + \frac{1}{2}\varphi_h(1)\varphi^2_{f\circ g}(0)$$

$$= (1-|a|^2)^2\varphi'_f(a) - 2\overline{a}[(1-|a|^2)\varphi_f(a)-\overline{a}] + \frac{1}{2}\varphi_h(1)\varphi^2_{f\circ g}(0)$$

$$= (1-|a|^2)^2[\varphi'_f(a) - \frac{1}{2}\varphi^2_f(a)] + \frac{1}{2}(1+\varphi_h(1))\ (1-|a|^2)\varphi_f(a)-2\overline{a}]^2.$$

By Lemma 2.2, we have the result. □

COROLLARY 2.1.1. *Let Ω be a simply connected region in $\mathbf{C}$ ($\mathbf{C} \ne \Omega$) and $r \in (0,1)$. If $f \in C(\Omega, r)$ then f satisfies (*) and (**), where*
$2\alpha = 2\alpha^\Omega_r = [h'(1)]^{-1}r^{-1}$ and
$2\beta = 2\beta^\Omega_r = \max\{h'(1), (1/2)|1+\varphi_h(1)|\}[h'(1)]^{-2}r^{-2}$.

Proof. By Theorem 2.1, f satisfies the inequality

$$|(1-|z|^2)\varphi_f(z)-2\overline{z}| \le [h'(1)]^{-1}r^{-1},\ z \in D.$$

Also if $h'(1) \geq \frac{1}{2}|1+\varphi_h(1)|$, then $c \geq 0$ and by Theorem 2.1, we have

$$(1-|z|^2)^2|S_f(z)| \leq [h'(1)]^{-1}r^{-2} = 2\beta_r^{\Omega}$$

for every $f \in C(\Omega, r)$ and all $z \in D$. Finally, if $h'(1) < (1/2)|1+\varphi_h(1)|$, so that $c < 0$, then Theorem 2.1 implies

$$\begin{aligned}(1-|z|^2)^2|S_f(z)| &\leq -c|(1-|z|^2)\varphi_f(z) - 2\bar{z}|^2 + [h'(-1)]^{-1}r^{-2} \\ &\leq \{(1/2)|1+\varphi_h(1)| - h'(1)\}|(1-|z|^2)\varphi_f(z) - 2\bar{z}|^2 + [h'(-1)]^{-1}r^{-2} \\ &\leq \{(1/2)|1+\varphi_h(1)| - h'(1)\}[h'(1)]^{-2}r^{-2} + [h'(-1)]^{-1}r^{-2} \\ &\leq (1/2)|1+\varphi_h(1)|[h'(1)]^{-2}r^{-2} = 2\beta_r^{\Omega}\end{aligned}$$

for all $f \in C(\Omega, r)$ and $z \in D$. □

Now we can derive coefficient inequalites for normalized functions in $C(\Omega, r)$.

COROLLARY 2.1.2. *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in C(\Omega, r)$ for given Ω and $r \in (0,1]$ then*

$$|a_2| \leq \frac{1}{2}[h'(1)]^{-1}r^{-1}, \ |a_3| \leq \frac{1}{6}\max\{h'(1), |1-\frac{1}{2}\varphi_h(1)|\}h(1)^{-2}r^{-2}.$$

By applying Theorem 2.1 to the class $C(\Omega^\delta, r)$ where

$$\Omega^\delta = \{w : |\arg w| < \pi\delta/2\} \qquad \text{for } 0 \leq r \leq \min(1,\delta),\ \delta > 0,$$

we obtain generalized form.

THEOREM 2.2. *If $f \in C(\Omega^\delta, r)$, then*

$$(1-|z|^2)^2|S_f(z)| + \frac{1}{2\delta}\left|(1-|z|^2)\varphi_f(z) - 2\bar{z}\right|^2 \leq 2\delta r^{-2}, z \in D.$$

Hence we have in particular

$$|(1-|z|^2)\varphi_f(z) - 2\bar{z}| \leq (2\delta/r) \text{ and } (1-|z|^2)^2|S_f(z)| \leq (2\delta/r^2).$$

Proof. Let $h(w) = (w^{1/\delta}-1)(w^{1/\delta}+1)$. Then h maps Ω^δ onto D with

$$h'(1) = 1/2\delta,\ \varphi_h(1) = -1 \text{ and } c = h'(1) = 1/2\delta.$$

Thus Theorem 2.1 readily yields Theorem 2.2 as well. □

Notice that in the case $r = \delta = 1$, Theorem 2.2 implies an improvement of Nehari's result.

COROLLARY 2.2.1. *Let $f(z)$ be a convex function in D. For $z \in D$, then*

$$(1-|z|^2)^2|S_f(z)| + (1/2)|(1-|z|^2)\varphi_f(z) - 2\bar{z}|^2 \leq 2.$$

Minda[5] establishes the following geometric interpretation of Theorem 2.2 in the case $\delta = 1$. Let $r(z,f)$ be the hyperbolic radius of the largest disk in D centered at z in which f is univalent. Define $r(f) = \inf\{r(z,f) : z \in D\}$. The function f is called uniformly locally univalent in D provided that $r(f) > 0$. If

$$(1-|z|^2)^2|S_f(z)| \leq 2(1+k^2),$$

then $r(f) \geq \pi/2k$ for all $k \geq 0$.

COROLLARY 2.2.2. *If $f \in C(\Omega, r)$, then $f(z)$ is uniformly locally univalent, that is, $f(z)$ is univalent in every hyperbolic disk in D, of hyperbolic radius*

$$r(f) \geq \pi/2k = \pi r/2\sqrt{1-r^2} = (\pi/2)sinh\rho(f)$$

where $\rho(f) = (1/2)log[(1+r)/(1-r)]$ is the uniform hyperbolic radius of convexity of $f(z)$.

Proof. By Theorem 3 in [5], $f(z)$ is univalent in every hyperbolic disk in D of the hyperbolic radius $r(f) \geq \pi/2k$, provided that $(1-|z|^2)^2|S_f(z)| \leq 2(1+k^2)$. On the other hand by Theorem 2.2, $f \in C(\Omega, r)$ then $(1-|z|^2)|S_f(z)|^2 \leq 2/r^2$. Hence $k = \sqrt{(1/r^2)-1}$, and the result follows. □

REMARK. From Theorem 4 in [5], f satisfies the inequality

$$(1-|z|^2)^2|S_f(z)| \leq 6/\tanh^2(r(f)).$$

By Pommerenke's result(4), we have

$$|(1-|z|^2)\varphi_f(z) - 2\bar{z}| \leq 2\sqrt{1+3/\tanh^2(r(f))}$$

and

$$f \in C\left(\sqrt{1+3/\tanh^2(r(f))} - \sqrt{3/\tanh^2(r(f))}\right).$$

3. An estimate of the class of special functions

In this section we use Theorem 2.1 and find an estimate for β, such that if f satisfies (*) then f satisfies (**). First we show that every function f which satisfies (*) is Ω_r^α-locally convex with radius r, for all $r \in (0,1)$, where

$$\Omega_r^\alpha = \left\{ w : \left| w - \frac{1+r^2}{1-r^2} \right| < \frac{2\alpha r}{1-r^2} \right\}.$$

LEMMA 3.1. *If f satisfies (*), then $f \in C(\Omega_r^\alpha, r)$ for every $r \in (0,1)$.*

Proof. Note that if f satisfies (*) then $f \circ g$ satisfies the inequality $|(1-|z|^2)\varphi_{f\circ g}(z) - 2\bar{z}| \le 2\alpha$ for every Möbius automorphism $g(a) = (\alpha a + z)/(1+\alpha \bar{z} a)$ of D. Since f satisfies (*) it can be written in the form

$$\left| 1 + z\varphi_f(z) - \frac{1+|z|^2}{1-|z|^2} \right| \le \frac{2\alpha|z|}{1-|z|^2}.$$

Hence, $f \circ g$ satisfies (*) which is equivalent to

$$\left| 1 + z\varphi_{f\circ g}(z) - \frac{1+|z|^2}{1-|z|^2} \right| \le \frac{2\alpha|z|}{1-|z|^2}.$$

This shows that $w = 1 + z\varphi_{f\circ g}(z) \in \overline{\Omega_{|z|}^\alpha}$. Note $\overline{\Omega_{|z|}^\alpha} \subset \Omega_r^\alpha$ whenever $|z| < r$ for every $r \le 1 \le \alpha$. So that it completes the proof. □

THEOREM 3.1. *If f satisfies (*), then*

$$(1-|z|^2)^2|S_f(z)| + (1/2)|(1-|z|^2)\varphi_f(z) - 2\bar{z}|^2$$
$$\le 2\{\alpha^2 + (\alpha + (1/2)\alpha)\sqrt{\alpha^2-1}\},$$
$$(1-|z|^2)^2|S_f(z)| \le 2\beta(\alpha)$$

where $1 \le \beta(\alpha) \le 1+\alpha^2$ is given by

$$\beta(\alpha) = 1+\alpha^2, \text{ for } \alpha \ge \sqrt{1+\sqrt{2}}$$

and

$$\beta(\alpha) = \sqrt{(1/8\alpha^2)[27\alpha^4 - 18\alpha^2 - 1 + \sqrt{(\alpha^2-1)(9\alpha^2-1)^3}]}$$

for $1 \leq \alpha \leq \sqrt{1+\sqrt{2}}$.

Proof. By Lemma 3.1 we can apply Theorem 2.1 to the case that $\Omega = \Omega_r^\alpha$ for a fixed $\alpha \geq 1$ and a variable $r \in (0,1)$. Thus

$$h(w) = \frac{\alpha(1-r^2)(w-1)}{r[(1-r^2)w + (2\alpha^2 - r^2 - 1)]} : \Omega_r^\alpha \longrightarrow D.$$

So we have

$$h'(1) = \frac{\alpha(1-r^2)}{2r(\alpha^2 - r^2)}, \ \varphi_h(1) = -\frac{1-r^2}{\alpha^2 - r^2}$$

and

$$c = h'(1) - \frac{1}{2}|1+\varphi_h(1)| = \frac{1-\alpha r}{2r(\alpha - r)} = c(r).$$

By Theorem 2.1 the following inequality holds:

$$(1-|z|^2)^2|S_f(z)| + c(r)|(1-|z|^2)\varphi_f(z) - 2\overline{z}|^2 \leq 2\left[\frac{\alpha^2 - r^2}{\alpha r(1-r^2)}\right]$$

where

$$(1/2)[h'(1)]^{-1}r^{-2} = (\alpha^2 - r^2)/\alpha r(1-r^2).$$

Now put $r = \alpha - \sqrt{\alpha^2 - 1}$. Then

$$\begin{aligned}&(1-|z|^2)^2|S_f(z)| + (1/2)|(1-|z|^2\varphi_f(z) - 2\overline{z}|^2\\ &\leq 2\alpha^2 + 2(\alpha + (1/2)\alpha)\sqrt{\alpha^2 - 1}.\end{aligned}$$

Next, note that $c(r) > 0$ only for $r \in (0, 1/\alpha)$ and therefore

$$(1-|z|^2)^2|S_f(z)| \leq \begin{cases} 2\left[\frac{\alpha^2 - r^2}{\alpha r(1-r^2)}\right], & \text{for } 0 < r \leq 1/\alpha, \\ 2\left[\frac{\alpha^2 - r^2}{\alpha r(1-r^2)}\right] - 4\alpha^2 c(r), & \text{for } r \geq 1/\alpha. \end{cases}$$

Observe now that the minimum value of $2\left[\frac{\alpha^2 - r^2}{\alpha r(1-r^2)}\right]$ in $(0,1)$ is obtained at

$$r_0 = \sqrt{(1/2)[(3\alpha^2 - 1) - \sqrt{(\alpha^2 - 1)(9\alpha^2 - 1)}]},$$

and let $\beta(\alpha) = 2\left[\frac{\alpha^2 - r_0^2}{\alpha r_0(1-r_0^2)}\right]$. If $1 \le \alpha \le \sqrt{1+\sqrt{2}}$ then $r_0 \le \frac{1}{\alpha}$ and if $\alpha \ge \sqrt{1+\sqrt{2}}$ then $\left[\frac{\alpha^2-r^2}{\alpha r(1-r^2)}\right] \ge 1+\alpha^2$. On the other hand

$$\frac{\alpha^2 - r^2}{\alpha r(1-r^2)} - 2\alpha^2 c(r) = \frac{\alpha^2 - 1}{\alpha}\left(\frac{\alpha^2}{\alpha - 1} + \frac{r}{1-r^2}\right)$$

which obviously is an increasing function of r in $[1/\alpha, 1)$, and therefore we get in the interval

$$\begin{aligned}\frac{\alpha^2 - r^2}{\alpha r(1-r^2)} - 2\alpha^2 c(r) &\ge \frac{\alpha^2 - (1/\alpha)^2}{[1-(1/\alpha)^2]} - 2\alpha^2 c(1/\alpha)\\ &= \frac{\alpha^2 - (1/\alpha)^2}{[1-(1/\alpha)^2]} \ge 1+\alpha^2.\end{aligned}$$

This completes the proof □

Using the argument of the proof of Corollary 2.2.2 we obtain the following result.

COROLLARY 3.1.1. *Let f satisfies (*). Then $f(z)$ is univalent in every hyperbolic disk in D with the hyperbolic radius*

$$r(f) \ge \pi/2\sqrt{\beta(\alpha) - 1} \ge \pi/2\alpha.$$

Next we derive some coefficient inequalites for normalized functions.

COROLLARY 3.1.2. *Let $f(z) = z + a_2 z^2 + \dots$ and f satisfies (*). Then*

$$\begin{aligned}&|a_3 - a_2^2| + 1/3|a_2|^2 \le (1/3)[\alpha^2 + (\alpha + 1/2\alpha)\sqrt{\alpha^2 - 1}],\\ &|a_3 - a_2^2| \le \beta(\alpha)/3 \le (1+\alpha^2)/3,\\ &|a_3| \le \frac{1}{3}\max[(1-r^2)\alpha, (1+2\alpha^2 - 3r^2)r]\frac{\alpha^2 - r^2}{\alpha^2 r(1-r^2)^2},\end{aligned}$$

for every $r \in (0,1)$.

Proof. The inequalities are localized versions of Theorem 3.1 at near $z = 0$. If we substitute the values of $h'(1)$ and $\varphi_h(1)$, in the proof of Theorem 3.1 into Corollary 2.1.2 we obtain the result. □

4. The equivalent properties of convex functions

If $f(z)$ satisfies the improved convexity condition

$$1 + \mathrm{Re} z\varphi(z) > \sigma, \ \ z \in D,$$

then we say that $f(z)$ is a convex function of order σ, for some $\sigma \in (0,1)$(see[4]). For $\alpha > 0$, the technique of the proof of Theorem 2.1 may be applied here to derive sharp bounds for $|(1-|z|^2)\varphi_f(z) - 2\bar{z}|$ and for $(1-|z|^2)^2|S_f(z)|$.

THEOREM 4.1. *The following statements are equivalent:*

(1) $f(z)$ *is a convex function of order* σ *in* D.
(2) $\left|(1-|z|^2)\varphi_f(z) - 2(1-\sigma)\bar{z}\right| \leq 2(1-\sigma)$.
(3) $\left|(1-|z|^2)\varphi_f(z) - 2\bar{z}\right| \leq 2[1-\sigma(1-|z|)]$.
(4) $\left|(1-|z|^2)\varphi_f(z) - 2\bar{z}\right| \leq 2[1-\sigma(1-|z|^2)]^{1/2}$.
(5) $(1-|z|^2)^2|S_f(z)| + \frac{1}{2}\left|(1-|z|^2)\varphi_f(z) - 2\bar{z}\right|^2 \leq 2[1-\sigma(1-|z|^2)]$.

Proof. We show that (1) $\Longrightarrow$ (2). Since $f(z)$ is a convex function of order σ in D, $f(z)$ satisfies the improved convexity condition $1 + \mathrm{Re} z\varphi_f(z) > \sigma, z \in D$. The above inequality tells us that the analytic function $w = 1 + z\varphi_f(z)$ maps the unit disk D into the half plane $\{w : \mathrm{Re} w > \sigma\}$, and $g(w) = (w-1)/(w+1-2\sigma)$ maps the half plane onto D. The composition of two functions

$$g(1 + z\varphi_f(z)) = \frac{z\varphi_f(z)}{2(1-\sigma) + z\varphi_f(z)}$$

satisfies the requirement of the Schwarz Lemma and therefore

$$\frac{g(1 + z\varphi_f(z))}{z} = \frac{\varphi_f(z)}{2(1-\sigma) + z\varphi_f(z)}$$

map D into itself. Hence we have $|g(1 + z\varphi_f(z))/z| \leq 1$ for all $z \in D$. By simple computation we see:

$$|\varphi_f(z)|^2 \leq 4(1-\sigma)^2 + 4(1-\sigma)\mathrm{Re} z\varphi_f(z) + |z\varphi_f(z)|^2,$$

$$(1-|z|^2)|\varphi_f(z)|^2 \leq 4(1-\sigma)^2 + 4(1-\sigma)\mathrm{Re} z\varphi_f(z),$$

$$(1-|z|^2)\{|\varphi_f(z)|^2 - \frac{4(1-\sigma)}{1-|z|^2}\mathrm{Re} z\varphi_f(z) + \frac{4(1-\sigma)}{(1-|z|^2)^2}|z|^2\}$$

$$\leq 4(1-\sigma)^2(1 + |z|^2/1 - |z|^2)$$

which is inequality (2). (2)$\Longrightarrow$(3) and (3)$\Longrightarrow$(4) are trivial. We show that (4)$\Longrightarrow$(5). If we square (4) we have $|(1-|z|^2)\varphi_f(z)-2\bar{z}|^2 \leq 4[1-\sigma(1-|z|^2)]$ then simply we obtain

$$0 \leq (1-|z|^2)|\varphi_f(z)|^2 \leq 4(1-\sigma+\mathrm{Re}z\varphi_f(z)).$$

Hence $1+\mathrm{Re}z\varphi_f(z)>\sigma$. The above inequality tells us that the analytic function $w = 1+z\varphi_f(z)$ maps the unit disk D into the half plane $\{w : \mathrm{Re}w > \sigma\}$, and $g(w)=(w-1)/(w+1-2\sigma)$ maps the half plane onto D. The composition of two functions

$$g(1+z\varphi_f(z)) = \frac{z\varphi_f(z)}{2(1-\sigma)+z\varphi_f(z)}$$

satisfies the requirements of the Schwarz-Pick Lemma. Hence we have

$$(1-|z|^2)\left|\left(\frac{\varphi_f(z)}{2(1-\sigma)+z\varphi_f(z)}\right)'\right| \leq 1-\left|\frac{\varphi_f(z)}{2(1-\sigma)+z\varphi_f(z)}\right|^2$$

So we have

$$(1-|z|^2)|2(1-\sigma)\varphi_f'(z)-\varphi_f^2(z)| \leq |2(1-\sigma)+z\varphi_f(z)|^2-|\varphi_f(z)|^2.$$

This inequality yields

$$\begin{aligned}&(1-|z|^2)\{2(1-\sigma)|\varphi_f'(z)-(1/2)\varphi_f^2(z)|-\sigma|\varphi_f(z)|^2\}\\ &\leq 4(1-\sigma)[1-\sigma+\mathrm{Re}z\varphi_f(z)]-(1-|z|^2)|\varphi_f(z)|^2\end{aligned}$$

and simplifying above inequality we get

$$\begin{aligned}&(1-|z|^2)\left\{|\varphi_f'(z)-\frac{1}{2}\varphi_f^2(z)|+\frac{1}{2}|\varphi_f(z)|^2-\frac{2}{1-|z|^2}\mathrm{Re}z\varphi_f(z)\right\}\\ &\leq 2(1-\sigma).\end{aligned}$$

Finally, adding $(2|z|^2)/(1-|z|^2)$ to both side, then the inequality (5) follows at once. We show that (5)$\Longrightarrow$(1). Suppose the below inequality is holds:

$$(1-|z|^2)^2|S_f(z)|+\frac{1}{2}|(1-|z|^2)\varphi_f(z)-2\bar{z}|^2 \leq 2[1-\sigma(1-|z|^2)].$$

Then we have $|(1-|z|^2)\varphi_f(z)-2\overline{z}| \le 2[1-\sigma(1-|z|^2)]^{1/2}$. Square above inequality and simplify. Then we obtain

$$0 \le (1-|z|^2)|\varphi_f(z)|^2 \le 4(1-\alpha+\operatorname{Re} z\varphi_f(z)).$$

Hence $f(z)$ is a convex function of order σ in D. □

From the inequalities (2) and (5), we obtain the following consequence.

COROLLARY 4.1.1. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ *be a convex function of order* σ *in* D, *for some* $\sigma \in (0,1)$. *Then*

$$|a_2| \le 1-\sigma, \qquad |a_3| \le (1/3)(1-\sigma)(3-2\sigma).$$

References

1. L. V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York, 1973.
2. A. F. Beardon and F. W. Gehring, *Schwarzian derivative, the Poincaré metric and the kernel function*, Comment. Math. Helv. **55** (1980), 50-64.
3. P. Beesack and B. Schwarz, *On the zeros of solutions of second-order linear differential equations*, Can. J. Math. **8** (1956), 504-515.
4. A. W. Goodman, *Univalent Functions*, Mariner Publishing Company, Inc, 1983.
5. D. Minda, *The Schwarzian derivative and univalence criteria*, Contemp. Math. **38** (1985), 43-52.
6. Z. Nehari, *The Schwarzian derivative and schlicht function*, Bull. Amer. Math. Soc. **55** (1949), 545-551.
7. ———, *A property of convex conformal maps*, J. Analyse Math. **30** (1976), 390-393.
8. B. Osgood, *Univalence criteria in multiply connected domains*, Trans. Amer. Math. Soc. **260** (1980), 459-473.
9. ———, *Some properties of f''/f' and the Poincaré metric*, Indiana Univ. Math. J. **31** (1982), 449-461.
10. Ch. Pommerenke, *Linear-invariante Familien analytischer Funktionen I*, Math. Ann. **155** (1964), 108-154.

Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea