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COUPLED FIXED POINT THEOREMS WITH APPLICATIONS

S. S. CHANG, Y. J. CHO AND N. J. HUANG

1. Introduction

Recently, existence theorems of coupled fixed points for mixed monotone operators have been considered by several authors (see [1]-[3], [6]).

In this paper, we are continuously going to study the existence problems of coupled fixed points for two more general classes of mixed monotone operators. As an application, we utilize our main results to show the existence of coupled fixed points for a class of non-linear integral equations.

2. Coupled Fixed Points for Set-Contractive Mixed Monotone Operators

Throughout this paper, E is a real Banach space, which is partially ordered by a cone P, i.e., $x \leq y$ if and only if $y - x \in P$. Let D be a subset of E. An operator $A : D \times D \to E$ is said to be *mixed monotone* if $A(x, y)$ is nondecreasing in x and nonincreasing in y. A point $(x^*, y^*) \in D \times D$ is called a *coupled fixed point* of an operator A if $x^* = A(x^*, y^*)$ and $y^* = A(y^*, x^*)$. When $x^* = y^*$, i.e., $x^* = A(x^*, x^*)$, we say that the point x^* is a *fixed point* of A.

Define the norm $\|\cdot\|$ on $E \times E$ by $\|(x, y)\| = \|x\| + \|y\|$ for $x, y \in E$ and let

$$\widetilde{P} = \{(x, y) \in E \times E : x \geq 0, y \leq 0\}. \tag{2.1}$$

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It is obvious that $\widetilde{P}$ is a cone in $E \times E$. Let us define a partial ordering $\leq$ in $E \times E$ by

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2, y_1 \geq y_2.$$

Let $D \subset E$ and $A : D \times D \to E$ be a mixed monotone operator. We define a mapping $\widetilde{A} : D \times D \to E \times E$ by

$$\widetilde{A}(u, v) = (A(u, v), A(v, u)), (u, v) \in D \times D. \tag{2.2}$$

If $(x_1, y_1) \leq (x_2, y_2)$, then we have

$$A(x_2, y_2) \geq A(x_1, y_2) \geq A(x_1, y_1)$$

and

$$A(y_2, x_2) \leq A(y_1, x_2) \leq A(y_1, x_1),$$

which implies that $\widetilde{A}(x_2, y_2) \geq \widetilde{A}(x_1, y_1)$, i.e., $\widetilde{A}$ is an increasing operator. Now (2.2) shows that $(x^*, y^*) \in D \times D$ is a fixed point of $\widetilde{A}$ if and only if (x^*, y^*) is a coupled fixed point of A.

DEFINITION 2.1. ([2]) Let D be a subset of E and $A : D \times D \to E$ be a mapping. Let $\widetilde{A} : D \times D \to E \times E$ be the mapping defined by (2.2). $\widetilde{A}$ is said to be *demi-compact* if for any bounded sequence $\{(x_n, y_n)\} \in D \times D$ such that $\{(x_n - A(x_n, y_n), y_n - A(y_n, x_n))\}$ is convergent, $\{(x_n, y_n)\}$ has a convergent subsequence.

DEFINITION 2.2. ([3]) Let D be a subset of E and $A : D \times D \to E$ be a continuous mapping. A is said to be *k-α-set contractive* if

$$\alpha(A(K)) \leq k \cdot \alpha(K),$$

where $\alpha(K)$ denotes the Kuratowski's measure of noncompactness of a bounded subset K of $D \times D$ and $k \geq 0$ is a constant.

LEMMA 2.1. ([2]) *Let A and B be two bounded subsets in E. Then we have*

$$\alpha(A \times B) \leq \alpha(A) + \alpha(B).$$

THEOREM 2.1. *Let $u_0, v_0 \in E$ and $u_0 < v_0$. Suppose that $A : [u_0, v_0] \times [u_0, v_0] \to E$ is a $\frac{1}{2}$-α-set-contractive mixed monotone mapping such that the mapping $\widetilde{A}$ defined by*

(2.2) is demi-compact and satisfies the following conditions:

(i) $u_0 \le A(u_0, v_0)$, $A(v_0, u_0) \le v_0$,

(ii) $A([u_0, v_0] \times [u_0, v_0])$ is bounded in E,

where $[u_0, v_0]$ is an order interval in E, i.e. $[u_0, v_0] = \{x \in E : u_0 \le x \le v_0\}$.
Then A has two coupled fixed points $(x^*, y^*), (x_*, y_*) \in [u_0, v_0] \times [u_0, v_0]$. Moreover, (x^*, y^*) and (x_*, y_*) are maximal and minimal, respectively, in the sense that $x_* \le \bar{x} \le x^*$ and $y^* \le \bar{y} \le y_*$ for any coupled fixed point $(\bar{x}, \bar{y}) \in [u_0, v_0] \times [u_0, v_0]$ of A.

Proof. From the cone $\widetilde{P}$ defined as in (2.1) and the mapping $\widetilde{A}$ defined by (2.2), it follows that $\widetilde{A}$ is increasing and demi-compact. It is also clear that $(u_0, v_0) < (v_0, u_0)$ and

$$(u_0, v_0) \le \widetilde{A}(u_0, v_0), \ \widetilde{A}(v_0, u_0) \le (v_0, u_0).$$

Let $K \subset [(u_0, v_0), (v_0, u_0)]$ be any bounded subset and denote $K' = \{(y, x) : (x, y) \in K\}$. Then $\alpha(K') = \alpha(K)$ and $\widetilde{A}(K) \subset A(K) \times A(K')$. It follows from Lemma 2.1 that

$$\begin{aligned} \alpha(\widetilde{A}(K)) &\le \alpha(A(K) \times A(K')) \\ &\le \alpha(A(K)) + \alpha(A(K')) \\ &\le \frac{1}{2}\alpha(K) + \frac{1}{2}\alpha(K') \\ &= \alpha(K), \end{aligned}$$

which implies that $\widetilde{A}$ is a 1-α-set-contractive operator.

Next, for any given $(x_0, y_0) \in [(u_0, v_0), (v_0, u_0)]$, let

$$\begin{aligned} \mathcal{Z} = \{S &: (x_0, y_0) \in S \subset [(u_0, v_0), (v_0, u_0)], \\ &S \text{ is a closed convex set and } \widetilde{A}(S) \subset S\}. \end{aligned}$$

Then $\mathcal{Z} \ne \emptyset$ since $[(u_0, v_0), (v_0, u_0)] \in \mathcal{Z}$. Letting $S(x_0, y_0) = \bigcap_{S \in \mathcal{Z}} S$, we can easily prove that

(1) $(x_0, y_0) \in S(x_0, y_0)$,
(2) $S(x_0, y_0)$ is a closed convex subset,
(3) $\widetilde{A}(S(x_0, y_0)) \subset S(x_0, y_0)$.

Since $\overline{co}\{\widetilde{A}(S(x_0, y_0)), (x_0, y_0)\} \subset S(x_0, y_0)$, we have

$$\begin{aligned}\widetilde{A}(\overline{co}\{\widetilde{A}(S(x_0, y_0)), (x_0, y_0)\}) &\subset \widetilde{A}(S(x_0, y_0)) \\ &\subset \overline{co}\{\widetilde{A}(S(x_0, y_0)), (x_0, y_0)\}\end{aligned}$$

and so we have

(4) $\overline{co}\{\widetilde{A}(S(x_0, y_0)), (x_0, y_0)\} \in \mathcal{Z}$,
(5) $\overline{co}\{\widetilde{A}(S(x_0, y_0)), (x_0, y_0)\} = S(x_0, y_0)$.

On the other hand, in view of the condition (ii), it is easy to see that $S(x_0, y_0)$ is a bounded closed convex subset. Especially, if we take $(x_0, y_0) = (u_0, v_0)$ and let

$$\widetilde{A}_n = \frac{1}{n}(u_0, v_0) + (1 - \frac{1}{n})\widetilde{A},$$

then $\widetilde{A}_n : S(u_0, v_0) \to S(u_0, v_0)$. Since $\widetilde{A}$ is a 1-α-set-contractive operator, $\widetilde{A}_n$ is a $(1 - \frac{1}{n})$-α-set-contractive operator. Hence by Sadovskii's fixed point theorem ([5]), there exists a point $(x_n, y_n) \in S(u_0, v_0)$ such that

$$(x_n, y_n) = \frac{1}{n}(u_0, v_0) + (1 - \frac{1}{n})\widetilde{A}(x_n, y_n).$$

By the condition (ii), we have

$$(x_n, y_n) - \widetilde{A}(x_n, y_n) = \frac{1}{n}((u_0, v_0) - \widetilde{A}(x_n, y_n)) \to 0 \text{ as } n \to \infty. \quad (2.3)$$

Since $\widetilde{A}$ is semi-compact, by (2.3), there exist a subsequence $\{(x_{n_i}, y_{n_i})\}$ of $\{(x_n, y_n)\}$ and a point $(x_*, y_*) \in S(u_0, v_0)$ such that $(x_{n_i}, y_{n_i}) \to (x_*, y_*)$ as $i \to \infty$. By the continuity of A and (2.3), we have

$$(x_*, y_*) = \widetilde{A}(x_*, y_*). \quad (2.4)$$

Similarly, taking $(x_0, y_0) = (u_0, v_0)$, then there exists a point $(x^*, y^*) \in S(v_0, u_0)$ such that

$$(x^*, y^*) = \widetilde{A}(x^*, y^*). \quad (2.5)$$

Finally, we prove that (x^*, y^*) and (x_*, y_*) are the maximal and minimal fixed points of $\widetilde{A}$, respectively. In fact, if $(\bar{x}, \bar{y}) \in [(u_0, v_0), (v_0, u_0)]$ is an arbitrary fixed point of $\widetilde{A}$, then we have

$$\widetilde{A}([(u_0, v_0), (\bar{x}, \bar{y})]) \subset [(u_0, v_0), (\bar{x}, \bar{y})],$$

$$\widetilde{A}([(\bar{x}, \bar{y}), (v_0, u_0)] \subset [(\bar{x}, \bar{y}), (v_0, u_0)]$$

and so

$$S(u_0, v_0) \subset [(u_0, v_0), (\bar{x}, \bar{y})], S(v_0, u_0) \subset [(\bar{x}, \bar{y}), (v_0, u_0)].$$

Thus, we have $(x_*, y_*) \in [(u_0, v_0), (\bar{x}, \bar{y})]$ and $(x^*, y^*) \in [(\bar{x}, \bar{y}), (v_0, u_0)]$, i.e.,

$$(x_*, y_*) \leq (\bar{x}, \bar{y}) \leq (x^*, y^*)$$

By virtue of the definition of $\widetilde{A}$, we know that (x^*, y^*) and $(x_*, y_*) \in [u_0, v_0] \times [u_0, v_0]$ are two coupled fixed points of A, and (x^*, y^*) and (x_*, y_*) are the maximal and minimal coupled fixed points of A, respectively, in the sense that $x_* \leq \bar{x} \leq x^*$ and $y^* \leq \bar{y} \leq y_*$ for any coupled fixed point $(\bar{x}, \bar{y}) \in [u_0, v_0] \times [u_0, v_0]$ of A. This completes the proof.

The following can be obtained from Theorem 2.1 immediately:

THEOREM 2.2. *Let P be a normal cone of E and $A : P \times P \to P$ be a mixed monotone operator. Suppose that*

(i) *A is $\frac{1}{2}$-α-set-contractive and $\widetilde{A}$ defined by (2.2) is demi-compact,*

(ii) *there exist a point $h \in P$ and a positive bounded linear operator $L : P \to E$ with spectral radius $r_\sigma(L) < 1$ such that*

$$A(u, 0) \leq L(u) + h, u \in P. \tag{2.6}$$

Then there is a point $v \in P$ such that $A : [0, v] \times [0, v] \to [0, v]$ and A has two coupled fixed points $(x^*, y^*), (x_*, y_*) \in [0, v] \times [0, v]$. Moreover, (x^*, y^*) and (x_*, y_*) are the maximal and minimal coupled fixed points of A in the sence that $x_* \leq \bar{x} \leq x^*$ and $y^* \leq \bar{y} \leq y_*$ for any coupled fixed point $(\bar{x}, \bar{y}) \in [0, v] \times [0, v]$ of A.

Proof. Since $r_\sigma(L) < 1$, the equation $(1 - L)x = h$ has a unique solution

$$v = (I - L)^{-1} h = \sum_{n=1}^{\infty} L^n h \in P.$$

By (2.6), we have $A(v,0) \leq L(v) + h = v$. Besides, since $A(0,v) \in P$, we have $A(0,v) \geq 0$. Further, since P is normal, the order interval $[0,v]$ is bounded ([3]). Hence all the conditions in Theorem 2.1 are satisfied. Therefore, the assertion follows from Theorem 2.1. This completes the proof.

3. Coupled Fixed Points for Generalized Condensing Mixed Monotone Operators

DEFINITION 3.1. ([8]) Let D be a subset of E. D is said to be *upper semi-closed* if for any sequence $\{x_n\}$ in D such that $x_n \to \bar{x}$ and $x_n \leq \bar{x}$, we have $\bar{x} \in D$.

Similarly we can define a lower semi-closed set in E. It is easy to see that D is an upper semi-closed set in E if and only if $-D = \{-x : x \in D\}$ is a lower semi-closed set in E.

LEMMA 3.1. ([8]) *Let D be a relatively compact upper (resp., lower) semi-closed set in E, then for any $a \in D$, there exists a maximal (resp., minimal) element $x_a \in D$ such that $a \leq x_a$ (resp., $x_a \leq a$).*

DEFINITION 3.2. Let D be a subset of E. A mapping $T : D \to E$ is said to be *generalized condensing* if for any $S \subset D$ such that $T(S) \subset S$ and $S \setminus \overline{co}(T(S))$ is relatively compact, S is relatively compact.

THEOREM 3.1. *Let $u_0, v_0 \in E$ be two given points with $u_0 < v_0$. Let $A : [u_0, v_0] \times [u_0, v_0] \to E$ be a mixed monotone operator such that*

$$u_0 \leq A(u_0, v_0), \; A(v_0, u_0) \leq v_0$$

and the mapping $\widetilde{A} : [u_0, v_0] \times [u_0, v_0] \to E \times E$ defined by (2.2) is a generalized condensing operator.
Then A has two coupled fixed points (x^*, y^*) and (x_*, y_*) in $[u_0, v_0] \times [u_0, v_0]$. Moreover, (x^*, y^*) and (x_*, y_*) are the maximal and minimal coupled fixed points of A, respectively, in the sense that $x_* \leq \bar{x} \leq x^*, y^* \leq \bar{y} \leq y_*$ for any coupled fixed point $(\bar{x}, \bar{y}) \in [u_0, v_0] \times [u_0, v_0]$ of A.

Proof. Let P be the cone defined by (2.1). Then $\widetilde{A}$ is increasing and

$$(u_0, v_0) \leq \widetilde{A}(u_0, v_0), \; \widetilde{A}(v_0, u_0) \leq (v_0, u_0).$$

For any given $(x_0, y_0) \in [(u_0, v_0), (v_0, u_0)]$, define

$$\mathcal{Z}' = \{S : (x_0, y_0) \in S \subset [(u_0, v_0), (v_0, u_0)],\ S \text{ is a closed subset and } \widetilde{A}(S) \subset S\}.$$

Then $\mathcal{Z}' \neq \emptyset$. Let $S(x_0, y_0) = \bigcap_{S \in \mathcal{Z}'} S$. Then it is clear that

(1) $(x_0, y_0) \in S(x_0, y_0)$,
(2) $S(x_0, y_0)$ is a closed subset,
(3) $\widetilde{A}(S(x_0, y_0)) \subset S(x_0, y_0)$.

Next, we prove that $S(x_0, y_0)$ is a compact set. In fact, from

$$\widetilde{A}(\overline{\{\widetilde{A}(S(x_0, y_0)), (x_0, y_0)\}}) \subset \widetilde{A}(S(x_0, y_0))$$

$$\subset \{\widetilde{A}(S(x_0, y_0)), (x_0, y_0)\} \subset S(x_0, y_0),$$

it follows that

$$\overline{\{\widetilde{A}(S(x_0, y_0)), (x_0, y_0)\}} \in \mathcal{Z}'$$

and so

$$S(x_0, y_0) = \overline{\{\widetilde{A}(S(x_0, y_0)), (x_0, y_0)\}}.$$

This implies that

$$\begin{aligned} S(x_0, y_0) \setminus \overline{co}(\widetilde{A}(S(x_0, y_0))) &\subset S(x_0, y_0) \setminus \overline{\widetilde{A}(S(x_0, y_0))} \\ &= \{(x_0, y_0)\}. \end{aligned} \tag{3.1}$$

Since the singleton $\{(x_0, y_0)\}$ is a compact set in $E \times E$, from (3.1), we know that $S(x_0, y_0) \setminus \overline{co}(\widetilde{A}(S(x_0, y_0)))$ is a relatively compact set. Therefore, $S(x_0, y_0)$ is a compact set. Taking $(x_0, y_0) = (u_0, v_0)$, we know that $S(u_0, v_0)$ is compact and $\widetilde{A}$ is a mapping from $S(u_0, v_0)$ into $S(u_0, v_0)$. Let

$$K = \{(x, y) \in S(u_0, v_0) : (x, y) \leq \widetilde{A}(x, y)\}.$$

Then $K \neq \emptyset$ since $(u_0, v_0) \in K$ and it is relatively compact.

Now we prove that K is an upper semi-closed set in $E \times E$. In fact, for any given sequence $\{(x_n, y_n)\} \in K$, if $(x_n, y_n) \to (\bar{x}, \bar{y})$ and $(x_n, y_n) \leq (\bar{x}, \bar{y})$, then we have

$$(x_n, y_n) \leq \widetilde{A}(x_n, y_n) \leq \widetilde{A}(x, y),$$

which means that $(\bar{x}, \bar{y}) \leq \widetilde{A}(\bar{x}, \bar{y})$ and so $(\bar{x}, \bar{y}) \in K$, i.e., K is an upper semi-closed set in $E \times E$. From Lemma 3.1, there exists a maximal element $(x_*, y_*) \in K$ such that $(x_*, y_*) = \widetilde{A}(x_*, y_*)$. In fact, if $\widetilde{A}(x_*, y_*) \neq (x_*, y_*)$, then $(x_*, y_*) < A(x_*, y_*)$ and so we have

$$(x_*, y_*) < \widetilde{A}(x_*, y_*) \leq \widetilde{A}(\widetilde{A}(x_*, y_*)).$$

This implies that $\widetilde{A}(x_*, y_*) \in K$, which is a contradiction. Taking $(x_0, y_0) = (v_0, u_0)$, we know that $S(v_0, u_0)$ is compact and $\widetilde{A}$ is a mapping from $S(v_0, u_0)$ into $S(v_0, u_0)$. Letting

$$K' = \{(x, y) \in S(v_0, u_0) : \widetilde{A}(x, y) \leq (x, y)\},$$

it is easy to see that $K' \neq \emptyset$ and K' is relatively compact. In addition, we can prove that K' is a lower semi-closed set in $E \times E$. Therefore, by Lemma 3.1, there exists a minimal element (x^*, y^*) such that $\widetilde{A}(x^*, y^*) = (x^*, y^*)$. Besides, by the same way as in the proof of Theorem 2.1, we can prove that (x^*, y^*), $(x_*, y_*) \in [u_0, v_0] \times [u_0, v_0]$ are two coupled fixed points of A and $(x^*, y^*), (x_*, y_*)$ are the maximal and minimal coupled fixed point of A, respectively, in the sence that $x_* \leq \bar{x} \leq x^*$ and $y^* \leq \bar{y} \leq y_*$ for any coupled fixed point $(\bar{x}, \bar{y}) \in [u_0, v_0] \times [u_0, v_0]$ of A. This completes the proof.

4. Applications

Let $C(I)$ be a Banach space of all real-valued continuous functions on $I = [0, a]$ and P be the cone of all nonnegative continuous functions in $C(I)$. In this section we use Theorem 2.2 to show the existence of coupled fixed points for the following nonlinear integral operator:

$$A(u(t), v(t)) = \varphi(t, u(t), v(t)) + \int_0^t K(t, s)\psi(s, u(s), v(s))ds \quad (4.1)$$

for $t \in I$, where $u, v \in C(I), K \in C(I \times I), \varphi, \psi \in C(I \times R^+ \times R^+), R^+ = [0, \infty)$, satisfy the following conditions :

(i) K is non-negative and $\varphi, \psi : I \times R^+ \times R^+ \to R^+$,

(ii)

$$|\varphi(t, u_1(t), v_1(t)) - \varphi(t, u_2(t), v_2(t))| \leq \frac{1}{2}\{|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|\}, \ t \in I,$$

(iii) $\varphi(t, u, v)$ and $\psi(t, u, v)$ are nondecreasing in u and nonincreasing in v for $t \in I$,

(iv) there exist nonnegative constants c_1, c_2, d_1, d_2 such that for all $t \in I$ and for each $u \in P$,

$$\varphi(t, u(t), 0) \leq c_1 \cdot u(t) + d_1, \psi(t, u(t), 0) \leq c_2 \cdot u(t) + d_2.$$

Set

$$L(u(t)) = c_1 \cdot u(t) + c_2 \cdot \int_0^t K(t, s)u(s)ds, \tag{4.2}$$

$$F(u(t), v(t)) = \varphi(t, u(t), v(t)),$$

$$h(t) = d_1 + d_2 \cdot \int_0^t K(t, s)ds. \tag{4.3}$$

Let us define a mapping $\widetilde{F} : P \times P \to P \times P$ as follows :

$$\widetilde{F}(u(t), v(t) = (F(u(t), v(t)), F(v(t), u(t))).$$

We assume further that

(v) $\widetilde{F}$ is a semi-compact operator,

(vi) the spectral radius of $L, r_\sigma(L) < 1$.

By the conditions (i) and (iii), it is clear that $A : P \times P \to P$ is mixed monotone. The integral part on the right-hand side of (4.1) defines a compact operator $G : P \times P \to P$. By the assumptions (ii) and (v), it is easy to see that the operator A is $\frac{1}{2}$-α-set-contractive and the mapping A defined by

$$\widetilde{A}(u(t), v(t)) = (A(u(t), v(t)), A(v(t), u(t))), u, v \in P,$$

is semi-compact.

Now note that

$$A(u(t),0) \leq L(u(t)) + h(t)$$

can be obtained from (iv), (4.2) and (4.3). Therefore, all the conditions in Theorem 2.2 are satisfied. Hence there exists a positive function $v \in C(I)$ such that A has two coupled fixed points $(x^*, y^*), (x_*, y_*) \in [0, v] \times [0, v]$ which are maximal and minimal, respectively, in the sence that $x_* \leq \bar{x} \leq x^*$ and $y^* \leq \bar{y} \leq y_*$ for any coupled fixed point $(\bar{x}, \bar{y}) \in [0, v] \times [0, v]$ of A.

REMARK. It should be pointed out that the existence problems of coupled fixed points for the nonlinear integral operator defined by (4.1) was first considered by Chen ([2]).

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Shih-sen Chang and Nan-jing Huang
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
People's Republic of China

Yeol Je Cho
Department of Mathematics
Gyeongsang National University
Jinju 660-701, Korea