J. Korean Math. Soc. **33** (1996), No. 3, pp. 557–573

ON THE GENERALIZED KORTEWEG–DE VRIES EQUATION WITH DISSIPATION

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1. Introduction

When we attempt to describe the propagation of nonlinear dispersive waves, it is frequently necessary to take account of dissipative effects. In this paper we consider the initial value problem (IVP) for the generalized nonlinear-dispersive-dissipative equations

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^3 u}{\partial x^3} + D_x^1 u + f(u)\dfrac{\partial u}{\partial x} = 0 \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), \end{cases}$$

where $f(\cdot)$ is a nonlinear function, the regularity of which will be specified later. Without loss of generality, we suppose $f(0) = 0$, since we can change the variable $x \to x - f(0)t$. D_x^α is the Fourier multiplier operator defined by $(D_x^\alpha v)^\wedge(\xi) = |\xi|^\alpha \hat{v}(\xi)$.

In particular, the IVP (1.1) with $f(u) = u$ has been derived by Ott and Sudan [7] in order to describe the development of nonlinear ion acoustic waves in a plasma. For the large time asymptotic behavior of solutions with the dissipation term $D_x^\alpha u$, $\alpha > 1$, we refer Biler [1], Dix [2], and references therein. Also the recent work of Naumkin does this as $t \to \infty$ and $\xi = |x|t^{-1/\alpha} \to \infty$ simultaneously in the case $f(u) = u$ and $0 < \alpha < 3$ (See [6], and references therein).

Here we are interested in the local and global well-posedness result for the IVP (1.1) in the classical Sobolev spaces $H^s(\mathbb{R})$. The well-posedness means that there is a unique solution which depends continuously upon the data and which has the persistence property (i.e.

Received August 24, 1995. Revised February 13, 1996
1991 AMS Subject Classification: Primary 35Q53, 35J05, 35J10, 35K05, 58F07.
Key words: KdV equations, dispersion and dissipation.
This work was partially supported by KOSEF–GARC

the solution describes a continuous flow in a Hilbert space X whenever $u_0 \in X$).

Since $\{e^{-tD_x^1}\} = \{P_t(\cdot)\}$, $P_t(\cdot)$ is the Poisson kernel, is a contraction semigroup in $H^s(\mathbb{R})$, $s \in \mathbb{R}$, reproducing Kato's proof in [3], the IVP (1.1) is locally (*resp.* globally) well-posed in $H^s(\mathbb{R})$ with $s > 3/2$ (*resp.* $s \geq 2$) (See also Saut [8]). Our main results show that the IVP (1.1) is locally well-posed in $H^s(\mathbb{R})$ with $s > 1/2$ and globally well-posedness for small enough data $H^s(\mathbb{R})$ with $s > 1/2$.

To prove these results, we begin by considering the associated linear problem

$$\tag{1.2} \begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial^3 v}{\partial x^3} + D_x^1 v = 0 & x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \\ v(x,0) = u_0(x) \in H^s(\mathbb{R}), \end{cases}$$

which has a solution $v(\cdot)$ given by

$$v(x,t) = W^1(t)u_0(x) = c\int_{-\infty}^{\infty} e^{i(x\xi + t\xi^3) - t|\xi|}\hat{u}_0(\xi)\, d\xi,$$

where $\hat{u}_0$ denotes the Fourier transform of u_0. Then the solution of IVP (1.1) is furnished according to Duhamel's principle by the formula

$$u(t) = W^1(t)u_0 - \int_0^t W^1(t-t')\big(f(u)\frac{\partial u}{\partial x}\big)\, dt'.$$

Roughly speaking, our method of proof depends heavily on the several mixed-norm estimates for the contraction semigroup $\{W^1(t)\}_0^\infty$ in $H^s(\mathbb{R})$ and the contraction mapping principle. This approach follows the work of Kenig, Ponce, and Vega ([4], [5]).

We use energy method (L^2-theory) for the linear equation (1.2) (See [1]–[2] for the nonlinear equation (1.1)) to get the regularity gain on the $L_T^2 L_x^2$-space ;

$$\tag{1.3} \int_0^\infty \int_{-\infty}^\infty \left| D_x^{1/2} W^1(t)u_0(x)\right|^2 \, dx\, dt \leq \|u_0\|_2^2.$$

Also using duality and the group properties of $\{W^1(t)\}_0^\infty$ in $H^s(\mathbb{R})$ we can obtain $L_T^\infty L_x^2$ and $L_T^2 L_x^2$ estimates for the inhomogeneous version of (1.3)

$$\sup_{t\in[0,T]}\left(\int_{-\infty}^{\infty}\left|D_x^{1/2}\int_0^t W^1(t-t')g(\cdot,t')\,dt'\right|^2 dx\right)^{1/2} \le \left(\int_0^T\int_{-\infty}^{\infty}|g(x,t)|^2\,dx\,dt\right)^{1/2}, \tag{1.4}$$

$$\int_0^T\int_{-\infty}^{\infty}\left|D_x^1\int_0^t W^1(t-t')g(\cdot,t')\,dt'\right|^2 dx\,dt \le \int_0^T\int_{-\infty}^{\infty}|g(x,t)|^2\,dx\,dt, \tag{1.5}$$

which has twice the regularity gain of (1.3).

Once that (1.3)–(1.5) has been established, we use these results to prove well-posedness of (1.1). The difficulty in this process is that the non-linear terms involve fractional derivatives in the space variable. This can be solved by employing the Leibniz rule and chain rule for the fractional derivatives in the form of Riesz potentials (See [5]).

We will now give the precise statement of our results.

THEOREM 1. *Let $f(\cdot)$ be a C^1-function defined in $\mathbb{R}$ with the growth condition $|f(v)| \le c|v|(1+|v|^p)$ for some $p\in(0,\infty)$. Then for any $u_0\in H^s(\mathbb{R})$, $s>1/2$, there exist $T=T(\|u_0\|_{H^s})>0$ (with $T(\rho)\to\infty$ as $\rho\to 0$) and a unique strong solution $u(t)$ of the IVP (1.1) satisfying*

$$u\in C([0,T]:H^s(\mathbb{R})), \tag{1.6}$$

$$\|D_x^1 u\|_{L_T^2L_x^2}+\|D_x^{(2s+1)/2}u\|_{L_T^2L_x^2}<\infty, \tag{1.7}$$

and

$$\|u\|_{L_T^\infty L_x^\infty}+\|D_t^{(2s-1)/8}u\|_{L_T^\infty L_x^\infty}<\infty. \tag{1.8}$$

For any $T'\in(0,T)$, there exists a neighborhood V of u_0 in $H^s(\mathbb{R})$ such that the map $\tilde u_0\to\tilde u(t)$ from V into the class defined by (1.6) – (1.8) with T' instead of T is Lipschitz.

The outline of this paper is as follows. In Section 2 we discuss all the preliminary estimates to be used in this paper. Section 3 is devoted to show the well-posedness for the IVP (1.1).

Finally, we introduce basic notations and definitions that will be used throughout this paper.

DEFINITION. We let $L^p_x L^q_T$, $L^q_T L^p_x$, $1 \le p, q \le \infty$ denote the space of all Lebesque measurable functions $f : \mathbb{R} \times [-T, T] \to \mathbb{R}$ with the norms

$$\|f\|_{L^p_x L^q_T} \equiv \left(\int_{-\infty}^{\infty} \left(\int_0^T |f(x,t)|^q dt \right)^{p/q} dx \right)^{1/p}$$

and

$$\|f\|_{L^q_T L^p_x} \equiv \left(\int_0^T \left(\int_{-\infty}^{\infty} |f(x,t)|^p dx \right)^{q/p} dt \right)^{1/q}$$

When $T = t$ it indicates the case of $[0, T] = \mathbb{R}^+$.

For $\alpha \in \mathbb{R}$, we define the Riesz potentials in the x variable by

$$D^\alpha_x f(x,t) = c \int_{-\infty}^{\infty} e^{ix\xi} |\xi|^\alpha \hat{f}^{(x)}(\xi, t)\, d\xi,$$

where $\hat{f}^{(x)}$ denotes the partial Fourier transform of f in the x variable. $D^\alpha_t f$ can be defined in a similar way. The norm in the classical Sobolev space $H^s(\mathbb{R}) = (1 - \Delta)^{-s/2} L^2(\mathbb{R})$ will be denoted by $\|\cdot\|_{s,2}$, i.e. $\|u_0\|_{s,2} = \|(1-\Delta)^{s/2} u_0\|_2$.

2. Linear Estimates

In this section we obtain several estimates concerning the Poisson kernel, Airy kernel, and their fractional derivatives. Also we show the Leibniz rule and chain rule for the fractional derivatives in the form of Riesz potentials to handle the nonlinear terms.

The solution of the linear IVP

$$(2.1)\qquad \begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial^3 v}{\partial x^3} + D_x^1 v = 0 & x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \\ v(x,0) = v_0(x) \end{cases}$$

is given by the contraction semigroup $\{W^1(t)\}_0^\infty$, i.e., $v(\cdot,t) = W^1(t)v_0 = P_t * W(t)v_0 = P_t * (S_t * v_0)$, where

$$P_t(x) = c\int_{-\infty}^{\infty} e^{ix\xi - t|\xi|}\, d\xi \qquad \text{and}$$

$$S_t(x) = c\int_{-\infty}^{\infty} e^{i(x\xi + t\xi^3)}\, d\xi.$$

Note that $P_t(x)$ is the Poisson kernel, $S_t(x)$ is the Airy kernel, and $W(t)u_0(x)$ is the solution for the associated linear problem of the KdV equation.

Our first result in this section is concerned with estimates for fractional derivatives of order $\alpha \in [0,\infty)$ of the Poisson kernel.

LEMMA 2.1. *If $\alpha \in [0,\infty)$ and*

$$(2.2)\qquad D_x^\alpha P_t(x) = c\int_{-\infty}^{\infty} e^{ix\xi - t|\xi|}|\xi|^\alpha\, d\xi,$$

then

$$\int_{-\infty}^{\infty} |D_x^\alpha P_t(x)|\, dx \le ct^{-\alpha}$$

for any $t > 0$.

Proof. It is best to begin with the simple case $\alpha \in [0,1)$. Once this case has been set down, the proof for the case $\alpha \in [1,\infty)$ can be easily accomplished through the integration by parts. Using the homogeneity in the t-variable, i.e.,

$$\left|D_x^\alpha P_t(x)\right| = |t|^{-(1+\alpha)}\left|D_x^\alpha P_1(x/t)\right|$$

and a simple computation, it suffices to show that

$$\left|D_x^\alpha P_1(x)\right| \le c\frac{1+\alpha x^{1-\alpha}}{1+x^2}$$

For $0 \leq \alpha < 1$, applying the integration by parts, we have

$$\begin{aligned}
&D_x^\alpha P_1(x)\\
&= \int_{-\infty}^{\infty} e^{ix\xi - |\xi|}|\xi|^\alpha \, d\xi\\
&= -\alpha \int_0^\infty \xi^{\alpha-1} e^{-\xi} \left(\frac{e^{ix\xi}}{-1+ix} + \frac{e^{-ix\xi}}{-1-ix}\right) d\xi\\
&= \frac{2\alpha}{1+x^2} \int_0^\infty \xi^{\alpha-1} e^{-\xi} \cos(x\xi)\, d\xi - \frac{\alpha x}{1+x^2} \int_0^\infty \xi^{\alpha-1} e^{-\xi} \sin(x\xi)\, d\xi\\
&= \frac{2\alpha}{1+x^2} (I_1(x) - I_2(x)).
\end{aligned}$$

First, since $|\cos\theta| \leq 1$ and $\alpha\xi^{\alpha-1}$ is integrable on $[0,1]$, $|\alpha I_1(x)| \leq c$ independently of α.

For the term $I_2(x)$ we can suppose $x > 0$, since $x \sin(x\xi) = -x \sin(-x\xi)$. To estimate $I_2(x)$, let us decompose $I_2(x)$ as in

$$\begin{aligned}
I_2(x) &= \left(\int_{0 \leq x\xi < \pi/2} + \int_{\pi/2 \leq x\xi} \right) \xi^{\alpha-1} e^{-\xi} x \sin(x\xi)\, d\xi\\
&= I_2'(x) + I_2''(x).
\end{aligned}$$

Since $|\sin\theta| \leq \theta$ on $[0, \pi/2]$, $I_2'(x)$ is bounded by

$$\begin{aligned}
|I_2'(x)| &\leq \int_0^{\pi/2x} x^2 \xi^\alpha \, d\xi\\
&= x^2 \left. \frac{\xi^{1+\alpha}}{1+\alpha} \right|_0^{\pi/2x} = c x^{1-\alpha}.
\end{aligned}$$

To bound the term $I_2''(x)$, we use the integration by parts to get that

$$
\begin{aligned}
|I_2''(x)| &= \left| \int_{\pi/2x}^{\infty} \xi^{\alpha-1} e^{-\xi} \frac{d(-\cos(x\xi))}{d\xi}\, d\xi \right. \\
&= \left| \int_{\pi/2x}^{\infty} \cos(x\xi)\left((\alpha-1)\xi^{\alpha-2}e^{-\xi} - \xi^{\alpha-1}e^{-\xi}\right) d\xi \right| \\
&\leq \int_{\pi/2x}^{\infty} \left((1-\alpha)\xi^{\alpha-2}e^{-\xi} + \xi^{\alpha-1}e^{-\xi}\right) d\xi \\
&= -\xi^{\alpha-1}e^{-\xi}\Big|_{\pi/2x}^{\infty} \\
&\leq c x^{1-\alpha},
\end{aligned}
$$

which completes the proof. □

Next result is concerned with estimates for the time behavior of the fractional derivatives of order $\alpha \in [0, 1/2]$ of the oscillatory kernel $S_t(\cdot)$ proven in Lemma 1 in [4].

LEMMA 2.2. *If* $\alpha \in [0, 1/2]$ *and*

$$D_x^{\alpha} S_t(x) = c \int_{-\infty}^{\infty} e^{i(x\xi + t\xi^3)} |\xi|^{\alpha}\, d\xi,$$

then

$$|D_x^{\alpha} S_t(x)| \leq c|t|^{-(1+\alpha)/3}. \tag{2.3}$$

Combining Lemma 2.1–2.2 with complex interpolation argument, we have

PROPOSITION 2.3. *For any* $(\theta, \alpha) \in [0, 1] \times [0, \infty)$

$$\|D_x^{\theta\alpha} W^1(t) u_0\|_{2/(1-\theta)} \leq c\beta(t) \|u_0\|_{2/(1+\theta)}, \tag{2.4}$$

where $\beta(t) = t^{-\theta(\alpha+1)/3}$ *for* $\alpha \in [0, 1/2]$ *and* $\beta(t) = t^{-\theta\alpha}$ *for* $\alpha \in [1/2, \infty)$.

Proof. In case of $\alpha \in [0, 1/2]$, for the proof we refer to Proposition 2.3 in [4].

For $\alpha \in [1/2, \infty)$, we introduce the analytic family of operators

$$D_x^{\alpha+i\delta} W^1(t) u_0 = D_x^{\alpha+i\delta}(P_t * S_t * u_0), \quad (\alpha, \delta) \in [1/2, \infty) \times \mathbb{R}.$$

Using Young's inequality and decay (in time) estimates (2.2)–(2.3), we obtain that

$$\begin{aligned} \|D_x^{\alpha+i\delta} W^1(t) u_0\|_\infty &\le \|D_x^{\alpha-1/2} P_t(\cdot)\|_1 \|D_x^{1/2+i\delta}(S_t * u_0)\|_\infty \\ &\le c t^{-\alpha} \|u_0\|_1. \end{aligned}$$

Since $\{W^1(t)\}_0^\infty$ is a contraction semigroup on $L^2(\mathbb{R})$,

$$\|D_x^{i\delta} W^1(t) u_0\|_2 \le \|u_0\|_2.$$

By Stein's complex interpolation, we can obtain the desired result. □

Note that in case of the KdV equation, which does not have the dissipation effect in the linear terms, we can obtain estimate (2.4) only for $(\theta, \alpha) \in [0, 1] \times [0, 1/2]$.

Using duality, the P. Tomas [9] argument, and Proposition 2.3, we have the global smoothing effect in the space variable.

Lemma 2.4. *For any $(\theta, \alpha) \in [0, 1] \times [0, \infty)$ with $\theta\alpha \in [0, 1)$ and $T > 0$*

$$\|D_x^{\theta\alpha/2} W^1(t) u_0\|_{L_t^q L_x^p} \le c\|u_0\|_2, \tag{2.5}$$

$$\sup_{t \in [0,T]} \left\| D_x^{\theta\alpha/2} \int_0^t W^1(t-t') g(\cdot, t')\, dt' \right\|_2 \le c\|g\|_{L_T^{q'} L_x^{p'}}, \tag{2.6}$$

and

$$\left\| D_x^{\theta\alpha} \int_0^t W^1(t-t') g(\cdot, t')\, dt' \right\|_{L_T^q L_x^p} \le c\|g\|_{L_T^{q'} L_x^{p'}}, \tag{2.7}$$

where $p = \frac{2}{1-\theta}$, $q = \frac{6}{\theta(1+\alpha)}$ for $\alpha \in [0, 1/2]$ and $q = \frac{2}{\theta\alpha}$ for $\alpha \in (1/2, \infty)$, and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

Proof. We shall follow the method used in [4], Lemma 2.4.

First, by duality

$$\iint D_x^{\theta\alpha/2} W^1(t) u_0(\cdot) g(x,t)\, dx\, dt = \int u_0(x) \Big(\int D_x^{\theta\alpha/2} W^1(t) g(\cdot,t)\, dt \Big)\, dx$$

(2.5) is equivalent to the easy fact

$$\Big\| D_x^{\theta\alpha/2} \int_0^\infty W^1(t) g(t)\, dt \Big\|_2 \le c \|g\|_{L_t^{q'} L_x^{p'}}. \tag{2.8}$$

Combining (2.8) with the contraction semigroup $\{W^1(t)\}$ in the homogeneous Sobolev space, yields (2.6).

Once that the equivalence relationship between (2.5) and (2.8) has been established, the proof of (2.7)–(2.8) follows the familiar P. Tomas [9] argument and fractional integration. □

By analyzing the dissipative character in the linear IVP (2.1), we can obtain another global smoothing effect in the space variable. In particular, we get linear estimates involving fractional derivatives in the space as well as in the time variable.

DEFINITION. We shall say that $f(x,t) \in \mathcal{D}_{\otimes}(\mathbb{R} \times [0,T])$ if $f(x,t) = \sum_{i=1}^{N} f_i(x) g_i(t)$ with $f_i \in C_0^\infty(\mathbb{R})$ and $g_i \in C_0^\infty([0,T])$ for $i = 1, \ldots, N$.

Notice that $\mathcal{D}_{\otimes}(\mathbb{R} \times [0,T])$ is dense in $L_T^q L_x^p(\mathbb{R} \times [0,T])$ for $p, q \in [1, \infty)$.

LEMMA 2.5. (i) *If $u_0 \in L^2(\mathbb{R})$ then for any $\beta \in (1/2, 1)$ and $T > 0$*

$$\|D_x^{1/2} W^1(t) u_0\|_{L_T^2 L_x^2} \le \|u_0\|_2 \tag{2.9}$$

and

$$\|D_x^{(7-6\beta)/8} D_t^{(2\beta-1)/8} W^1(t) u_0\|_{L_T^2 L_x^2} \le c\beta(T) \|u_0\|_2. \tag{2.10}$$

(ii) *If $g \in L_t^2 L_x^2$ then for any $T > 0$*

$$\sup_{t\in[0,T]} \Big\| D_x^{1/2} \int_0^t W^1(t-t') g(\cdot,t')\, dt' \Big\|_2 \le \|g\|_{L_T^2 L_x^2} \tag{2.11}$$

and

$$\left\| D_x^1 \int_0^t W^1(t-t')g(\cdot,t')\,dt' \right\|_{L_T^2 L_x^2} \le \|g\|_{L_T^2 L_x^2}. \tag{2.12}$$

(iii) *If* $g \in L_t^2 L_x^2$ *then for any* $\beta \in (1/2, 1)$ *and* $T > 0$

(2.13)
$$\sup_{t\in[0,T]} \left\| D_x^{(7-6\beta)/8} D_t^{(2\beta-1)/8} \int_0^t W^1(t-t')g(\cdot,t')\,dt' \right\|_{2} \le c\beta(T)\|g\|_{L_T^2 L_x^2}$$

and

(2.14)
$$\left\| D_x^{(7-6\beta)/4} D_t^{(2\beta-1)/4} \int_0^t W^1(t-t')g(\cdot,t')\,dt' \right\|_{L_T^2 L_x^2} \le c\beta(T)^2 \|g\|_{L_T^2 L_x^2},$$

where $\beta(T) = 1 + T^{(2\beta-1)/2}$.

Proof. (i) It suffices to establish the result for $f \in C_0^\infty(\mathbb{R})$, since $C_0^\infty(\mathbb{R})$ is dense in $L_x^2(\mathbb{R})$. By a simple computation, $v(t) = W^1(t)u_0$ satisfies the linear equation (2.1) $\partial_t v + \partial_x^3 v + D_x^1 v = 0$ with $v(0) = u_0$. Multiplying the above equation by v and integrating in x-variable, we have that

$$\frac{d}{dt}\int |v(t)|^2\,dx + 2\int |D_x^{1/2} v(t)|^2\,dx = 0$$

and after integrating in t-variable, for any $T > 0$

$$\int |v(T)|^2\,dx + 2\int_0^T \int |D_x^{1/2} v(t)|^2\,dx\,dt = \|u_0\|_2^2,$$

which concludes (2.9).

Since $v(t) = W^1(t)u_0$ satisfies the linear equation (2.1), we have the following identity

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^6 v}{\partial x^6} + 2\frac{\partial^3 (D_x^1 v)}{\partial x^3} + D_x^2 v, \tag{2.15}$$

which combined with the interpolation inequality and (2.9) yields

$$\begin{aligned}\|D_x^{(7-6\beta)/8}D_t^{(2\beta-1)/8}W^1(t)u_0\|_{L^2_TL^2_x}\\ &\le c\|(D_x^{1/2}+D_x^{(3-2\beta)/4})W^1(t)u_0\|_{L^2_TL^2_x}\\ &\le c(1+T^{(2\beta-1)/2})\|u_0\|_2,\end{aligned}$$

which completes (i).

(ii) Once that the estimate (2.9) has been established, the proof of (2.11)-(2.12) follows the familiar argument as in (2.6)-(2.7).

(iii) By reproducing their argument in [5], Lemma 3.4, we obtain the following relationship :

For any $f\in\mathcal{D}_{\otimes}(\mathbb{R}\times[0,T])$

(2.16)

$$\begin{aligned}B(f)(x,t)&=\lim_{\epsilon\to0}\iint\limits_{\epsilon<|\xi^3+i|\xi|-\tau|}e^{i(x\xi+t\tau)}\frac{\hat{f}_1(\xi,\tau-2i|\xi|)}{\xi^3+i|\xi|-\tau}\,d\tau\,d\xi\\ &=2\int_0^tW^1(t-t')f(t')\,dt'-\int_0^TW^1(t-t')f(t')\,dt',\end{aligned}$$

where $f_1(x,t)=f(x,t)$ if $t\in[0,T]$ and $f_1(x,t)=0$ otherwise.

It is not hard to see that the relationship (2.16), the duality argument to (2.10), and the estimate (to be proven)

$$\|D_x^{(7-6\beta)/4}D_t^{(2\beta-1)/4}B(f)\|_{L^2_TL^2_x}\le c(1+T^{2\beta-1})\|f\|_{L^2_TL^2_x} \tag{2.17}$$

yield (2.13)-(2.14). Thus it remains to prove (2.17).

Since

$$D_x^{(7-6\beta)/4}D_t^{(2\beta-1)/4}B(f)=$$
$$\lim_{\epsilon\to0}\iint\limits_{\epsilon<|\xi^3-i|\xi|-\tau|}e^{i(x\xi+t\tau)}\frac{|\xi|^{(7-6\beta)/4}|\tau|^{(2\beta-1)/4}}{\xi^3-\tau+i|\xi|}\hat{f}_1(\xi,\tau-2i|\xi|)\,d\tau\,d\xi,$$

by Plancherel's Theorem, it suffices to verify that

$$g(\xi,\tau)=\frac{|\xi|^{(7-6\beta)/2}|\tau|^{(2\beta-1)/2}}{(\xi^3-\tau)^2+\xi^2}\le C.$$

But a simple computation shows that g is uniformly bounded in $\mathbb{R}^2$, which completes (iii). □

Next we have the Leibniz rule and chain rule for the fractional derivatives in the forms of Riesz potentials. The main idea for the proof is derived from the work of R. R. Coifman and Y. Meyer for multilinear pseudo-differential operators.

To obtain these results, suppose that $F \in C^1(\mathbb{C})$ satisfy

$$|F'(\theta x + (1-\theta)y)| \le c(\theta)(|F'(x)| + |F'(y)|) \tag{2.18}$$

for all $x, y \in \mathbb{C}$ and $\int_0^1 c(\theta)\, d\theta < \infty$. Observe that $F(x) = |x|^p x$ with $p > 0$ satisfies the condition (2.18).

THEOREM 2.6. *Suppose that $F \in C^1(\mathbb{C})$ satisfy the condition (2.18). Let $0 < \alpha < 1$ and $1 < p < \infty$. Then*

$$\|D^\alpha F(f)\|_p \le c\|F'(f)\|_\infty \|D^\alpha f\|_p \tag{2.19}$$

and

$$\|D^\alpha(fg) - fD^\alpha g\|_p \le c\|g\|_\infty \|D^\alpha f\|_p. \tag{2.20}$$

Proof. For (2.19) see Theorem A.7 in [5] and for (2.20) see Theorem A.12 in [5]. □

3. Proof of Theorem 1

In this section we combine the estimates (2.9), (2.11)–(2.13) for the group $\{W^1(t)\}_0^\infty$ proven in section 2 with Sobolev embedding Theorem and the interpolation argument to prove Theorem 1

Proof. To simplify the exposition we restrict ourselves to the case $s \in (1/2, 1)$. If $s = m + \sigma$, where m is a positive integer and $0 \le \sigma < 1$, the proof is similar and simpler, since the highest derivatives in our estimates below always appear linear.

For $w : \mathbb{R} \times [0,T] \to \mathbb{R}$, define

$$\gamma_1^T(w) = \max_{t\in[0,T]} \|w(t)\|_{s,2},$$
$$\gamma_2^T(w) = \|D_x^1 w\|_{L_T^2 L_x^2},$$
$$\gamma_3^T(w) = \|w\|_{L_T^\infty L_x^\infty} + (1-T)^{-(2s-1)/2}\|D_t^{(2s-1)/8} w\|_{L_T^\infty L_x^\infty},$$
$$\Gamma^T(w) = \max\{\gamma_1^T(w), \gamma_2^T(w) + \gamma_2^T(D_x^{(2s-1)/2}w), \gamma_3^T(w)\},$$

and

$$\mathcal{X}^T = \{w \in C([0,T] : H^s(\mathbb{R})) \setminus \Gamma^T(w) < \infty\}.$$

Applying (2.9) and the group properties of $\{W^1(\cdot)\}$ and combining (2.15) with the interpolation argument and Sobolev embedding Theorem, we see that if $u_0 \in H^s(\mathbb{R})$, then for any $T > 0$, $W^1(t)u_0 \in \mathcal{X}^T$ with $\Gamma^T(W^1(t)u_0)$ depending on $\|u_0\|_{s,2}$ but not on T.

For $u_0 \in H^s(\mathbb{R})$, we denote by $u = \Phi(v) = \Phi_{u_0}(v)$ the solution of the linear inhomogeneous IVP

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^3 u}{\partial x^3} + D_x^1 u + f(v)\dfrac{\partial v}{\partial x} = 0 \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), \end{cases}$$

where $v \in \mathcal{X}_a^T = \{w \in \mathcal{X}^T \setminus \Gamma^T(w) \le a\}$.

We shall prove that there exist T and a (depending only on $\|u_0\|_{s,2}$) such that if $v \in \mathcal{X}_a^T$ then $u = \Phi(v) \in \mathcal{X}_a^T$ and

$$\Phi : \mathcal{X}_a^T \to \mathcal{X}_a^T$$

is a contraction map. For this purpose we will use the integral equation version of this IVP (3.1), i.e.

$$\text{(3.2)} \qquad u(t) = \Phi(v)(t) = W^1(t)u_0 - \int_0^t W^1(t-t')\big(f(v)\frac{\partial v}{\partial x}\big)\, dt'.$$

First to estimate the bound of $\Gamma^T(\Phi(v))$ for the nonlinear term, we need the following proposition.

PROPOSITION. *If $v \in \mathcal{X}_a^T$ then*

$$\left\|D_x^{(2s-1)/2}\Big(f(v)\frac{\partial v}{\partial x}\Big)\right\|_{L_T^2 L_x^2} + \left\|f(v)\frac{\partial v}{\partial x}\right\|_{L_T^2 L_x^2} \le cT^{1/q}A(T,v), \tag{3.3}$$

where $A(T,v) = \big(\Gamma^T(v)\big)^2\big(1+(\Gamma^T(v))^p\big)$.

Proof of Proposition. First we use Sobolev embedding Theorem and Hölder's inequality to get

$$\begin{aligned} \|v\|_{L_T^\infty L_x^\infty} &\le c\|(1+D_t^{(2s-1)/8})v\|_{L_x^\infty L_T^q} \\ &\le cT^{1/q}\|(1+D_t^{(2s-1)/8})v\|_{L_T^\infty L_x^\infty} = cT^{1/q}\gamma_3^T(v), \end{aligned} \tag{3.4}$$

where $q = q(s) < \infty$ satisfies $q(2s-1) > 8$.

Next we use the growth condition on $f(v)$ and (3.4) to get

$$\begin{aligned} \|f(v)\|_{L_T^\infty L_x^\infty} &\le c\|v\|_{L_T^\infty L_x^\infty}(1+\|v\|_{L_T^\infty L_x^\infty}^p) \\ &\le cT^{1/q}\Gamma^T(v)(1+\Gamma^T(v)^p). \end{aligned} \tag{3.5}$$

Finally before we begin to show (3.3), we need the following inequality, which obtained by the interpolation argument of $\|v\|_{L_T^\infty L_x^\infty}$ and $\gamma_2^T(D_x^{(2s-1)/2}v)$,

$$\|D_x^{(2s+1)\theta/2}v\|_{L_T^{2/\theta} L_x^{2/\theta}} \le \|v\|_{L_T^\infty L_x^\infty}^{1-\theta}\|D_x^{(2s+1)/2}v\|_{L_T^2 L_x^2}^{\theta}, \tag{3.6}$$

where $\theta \in [0,1]$.

To estimate (3.3), we use Theorem 2.6 and (3.5)–(3.6) to obtain

$$\begin{aligned} &\left\|D_x^{(2s-1)/2}\Big(f(v)\frac{\partial v}{\partial x}\Big)\right\|_{L_T^2 L_x^2} + \left\|f(v)\frac{\partial v}{\partial x}\right\|_{L_T^2 L_x^2} \\ &\quad \le c\|D_x^{(2s+1)/2}v\|_{L_T^2 L_x^2}\|f(v)\|_{L_T^\infty L_x^\infty} + \|D_x^1 v\|_{L_T^2 L_x^2}\|f(v)\|_{L_T^\infty L_x^\infty} \\ &\qquad + c\|D_x^1 v\|_{L_T^{2/\theta_0} L_x^{2/\theta_0}}\|D_x^{(2s-1)/2}(f(v))\|_{L_T^{2/\theta_1} L_x^{2/\theta_1}} \\ &\quad \le cT^{1/q}\big(\Gamma^T(v)\big)^2\big(1+(\Gamma^T(v))^p\big) \\ &\qquad + cT^{(1-\theta_0)/q}\Gamma^T(v)\|D_x^{(2s-1)/2}v\|_{L_T^{2/\theta_1} L_x^{2/\theta_1}}(1+\|f'(v)\|_{L_T^\infty L_x^\infty}) \\ &\quad \le cT^{1/q}\big(\Gamma^T(v)\big)^2\big(1+(\Gamma^T(v))^p\big). \end{aligned}$$

where $\theta_0 + \theta_1 = 1$ and $\theta_0 = \frac{2}{2s+1}$, which concludes (3.3). □

Now we are ready to prove Theorem 1. Combining the group properties of $\{W^1(\cdot)\}$ and Sobolev embedding Theorem with the estimates (2.11) and (3.3), we easily see that

(3.7)
$$\begin{aligned}\gamma_1^T(\Phi(v)) + \|\Phi(v)\|_{L_T^\infty L_x^\infty} &\leq c \max_{t\in[0,T]} \|\Phi(v)\|_{s,2} \\ &\leq c\|u_0\|_{s,2} + c\left\|D_x^{(2s-1)/2}(f(v)\frac{\partial v}{\partial x})\right\|_{L_T^2 L_x^2} + cT^{1/2}\left\|f(v)\frac{\partial v}{\partial x}\right\|_{L_T^2 L_x^2} \\ &\leq c\|u_0\|_{s,2} + cT^{1/q}(1+T^{1/2})A(T,v).\end{aligned}$$

Similarly (2.9), (2.12), and (3.3) lead to

$$\begin{aligned}&\gamma_2^T(\Phi(v)) + \gamma_2^T(D_x^{(2s-1)/2}\Phi(v)) \\ &\leq c\|u_0\|_{s,2} + \left\|f(v)\frac{\partial v}{\partial x}\right\|_{L_T^2 L_x^2} + \left\|D_x^{(2s-1)/2}(f(v)\frac{\partial v}{\partial x})\right\|_{L_T^2 L_x^2} \\ &\leq c\|u_0\|_{s,2} + cT^{1/q}A(T,v).\end{aligned} \tag{3.8}$$

Finally applying Sobolev embedding Theorem and combining (2.15) with the interpolation argument and estimates (2.13), (3.3), we get

$$\begin{aligned}&(1+T)^{-(2s-1)/2}\|D_t^{(2s-1)/8}\Phi(v)\|_{L_T^\infty L_x^\infty} \\ &\leq c(1+T)^{-(2s-1)/2}\|(1+D_x^{(2s+3)/8})D_t^{(2s-1)/8}\Phi(v)\|_{L_T^\infty L_x^2} \\ &\leq c\|u_0\|_{s,2} + (1+T^{\delta(s)})\left\|D_x^{(2s-1)/2}(f(v)\frac{\partial v}{\partial x})\right\|_{L_T^2 L_x^2} \\ &\leq c\|u_0\|_{s,2} + cT^{1/q}(1+T^{\delta(s)})A(T,v)\end{aligned} \tag{3.9}$$

where $0 < \delta(s) < 1$.

Hence collecting the estimates (3.7)–(3.9) we can conclude that

$$\Gamma^T(\Phi(v)) \leq c\|u_0\|_{s,2} + cT^{1/q}(1+T)^{\delta(s)}\left(\Gamma^T(v)\right)^2\left(1+(\Gamma^T(v))^p\right) \tag{3.10}$$

Similar arguments show that

$$\Gamma^T(\Phi(v)-\Phi(\tilde{v})) \leq cT^{1/q}(1+T)^{\delta(s)}\Gamma^T(v-\tilde{v})B(T;v,\tilde{v}), \tag{3.11}$$

where $B(T;v,\tilde{v}) = \Gamma^T(v) + \Gamma^T(\tilde{v}) + \left(\Gamma^T(v)\right)^{p+1} + \left(\Gamma^T(\tilde{v})\right)^{p+1}$ and for $T_1 \in (0,T)$, we have

$$\begin{aligned} \Gamma^{T_1}(\Phi_{u_0}(v) - \Phi_{\tilde{u}_0}(\tilde{v})) &\le c\|u_0 - \tilde{u}_0\|_{s,2} \\ &+ cT_1^{1/q}(1+T_1)^{\delta(s)}\Gamma^{T_1}(v-\tilde{v})B(T;v,\tilde{v}). \end{aligned} \tag{3.12}$$

Thus we first choose

$$a = 2c\|u_0\|_{s,2} \tag{3.13}$$

and then T such that

$$4cT^{1/q}(1+T)^{\delta(s)}a(1+a^p) < 1. \tag{3.14}$$

Since it is easy to see that if $v \in \mathcal{X}_a^T$ then $u = \Phi(v) \in C([0,T] : H^s(\mathbb{R}))$, we can conclude that

$$\Phi : \mathcal{X}_a^T \to \mathcal{X}_a^T$$

is a contraction map.

By the contraction mapping principle, we know that there exists a unique $u \in \mathcal{X}_a^T$ with $\Phi_{u_0}(u) = u$, i.e.

$$u(t) = W^1(t)u_0 - \int_0^t W^1(t-t')(f(u)\frac{\partial u}{\partial x})\, dt'. \tag{3.15}$$

Moreover, (3.12)–(3.14) show that for $T_1 \in (0,T)$, the map $\tilde{u}_0 \to \tilde{u}$ from V (neighborhood of u_0 depending on T_1) to $\mathcal{X}_a^{T_1}$ is Lipschitz. Hence our solution $u(\cdot) \in \mathcal{X}_a^T$ of the integral equation (3.15) is a strong solution of the IVP (1.1). In particular, u satisfies the equation in (1.1) in the distribution sense.

Next we can extend our uniqueness result to the class $\mathcal{X}^T$. Suppose that $w \in \mathcal{X}^{T_1}$ for some $T_1 \in (0,T)$ is a strong solution of the IVP (1.1). The argument used in (3.10) shows that for some $T_2 \in (0,T_1)$, $w \in \mathcal{X}_a^{T_2}$. Hence (3.14) implies that $w \equiv u$ in $\mathbb{R} \times [0,T_2]$. Reapplying this argument, this result can be extended to the whole interval $[0,T]$. This yields the uniqueness result in $\mathcal{X}^T$. □

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