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HEAT EQUATION IN WHITE NOISE ANALYSIS

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1. Introduction

The Fourier transform plays a central role in the theory of distribution on Euclidean spaces. Although Lebesgue measure does not exist in infinite dimensional spaces, the Fourier transform can be introduced in the space $(\mathcal{S})^*$ of generalized white noise functionals. This has been done in the series of paper by H.-H. Kuo [1, 2, 3], [4] and [5]. The Fourier transform $\mathcal{F}$ has many properties similar to the finite dimensional case; e.g., the Fourier transform carries coordinate differentiation into multiplication and vice versa. It plays an essential role in the theory of differential equations in infinite dimensional spaces. An important example of a partial differential equation in the infinite dimensional space is the heat equation with Gross Laplacian operator. This equation has been studied by Gross [6].

We shall construct and investigate a solution of the heat equation in the white noise set-up. In the following we briefly describe the white noise noise framework. For more details and proofs we refer to [7] and references we quoted here.

Let $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ be the white noise probability space, where $\mathcal{B}$ is the Borel algebra of the weak topology of $\mathcal{S}'(\mathbb{R})$ and the measure μ is determined by

$$\int_{\mathcal{S}'(\mathbb{R})} \exp(i\langle x, \xi\rangle) d\mu(x) = \exp(-\frac{1}{2}|\xi|_2^2), \quad \xi \in \mathcal{S}(\mathbb{R}),$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, and $|\cdot|_2$ denotes the norm of $L^2(\mathbb{R})$.

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Let (L^2) denote the space $L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$. We will consider the triple

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*,$$

where $(\mathcal{S})$ is the space of white noise test functionals and its dual $(\mathcal{S})^*$ is the space of Hida distributions. These spaces are defined in [7]. The canonical dual pairing between $(S)^*$ and (S) will be denoted by $\langle\langle \Phi, \varphi \rangle\rangle$ for $\Phi \in (S)^*, \varphi \in (S)$.

The $\mathcal{S}$– and $\mathcal{T}$–transformations play important roles in white noise analysis. They are defined as follows:
For $\varphi \in (L^2)$ and $\xi \in \mathcal{S}(\mathbb{R})$, the $\mathcal{S}$–and $\mathcal{T}$–transforms of φ are given by [8]:

$$\mathcal{S}\varphi(\xi) = \int_{\mathcal{S}'(\mathbb{R})} \varphi(x + \xi) d\mu(x),$$

$$\mathcal{T}\varphi(\xi) = \int_{\mathcal{S}'(\mathbb{R})} \exp(i\langle x, \xi\rangle)\varphi(x) d\mu(x),$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$. From the translation formula for Gaussian measures [9], we have

$$\mathcal{S}\varphi(\xi) = \exp\left(\frac{-|\xi|_2^2}{2}\right) \int_{\mathcal{S}'(\mathbb{R})} \exp(\langle x, \xi\rangle)\varphi(x) d\mu(x).$$

Since for all $\xi \in \mathcal{S}(\mathbb{R})$ and $\lambda \in \mathbb{C}$, the functional $x \longmapsto \exp(\lambda\langle x, \xi\rangle)$ is an element in $(\mathcal{S})$, we can extend the $\mathcal{S}$– and $\mathcal{T}$– transforms to $(\mathcal{S})^*$ in the following way:
For $\Phi \in (\mathcal{S})^*$,

$$\mathcal{S}\Phi(\xi) = \langle\langle \Phi, e^{\langle\cdot,\xi\rangle} \rangle\rangle e^{-\frac{1}{2}|\xi|_2^2} = \langle\langle \Phi, :e^{\langle\cdot,\xi\rangle}: \rangle\rangle,$$
$$\mathcal{T}\Phi(\xi) = \langle\langle \Phi, e^{i\langle\cdot,\xi\rangle} \rangle\rangle,$$

where $:e^{\langle\cdot,\xi\rangle} := e^{\langle\cdot,\xi\rangle} e^{-\frac{1}{2}|\xi|_2^2}$.

The Fourier transform $\mathcal{F}$ was introduced by H.-H. Kuo on $(\mathcal{S})^*$ [2, 3] as follows:

$$(\mathcal{S}\mathcal{F}\Phi)(\xi) = \mathcal{S}\Phi(-i\xi) e^{-\frac{1}{2}|\xi|_2^2}.$$

or equivalently,

$$= \langle\langle \Phi, e^{-i<\cdot,\xi>} \rangle\rangle.$$

Using the analytic property of $\mathcal{S}\Phi$, both sides of this equation are well defined and $\mathcal{F}$ is a continuous linear transformation on $(\mathcal{S})^*$.

In this paper, we will investigate the fundamental solution of the heat equation on $\mathcal{S}'(\mathbb{R})$. Consider a function u on $\mathbb{R}_+ \times \mathcal{S}'(\mathbb{R})$,

$$\frac{d}{dt}u(t,x) = \frac{1}{2}\Delta_G u(t,x), \quad t > 0, \ x \in \mathcal{S}'(\mathbb{R})$$
$$u(0,x) = f(x), \quad f \in (\mathcal{S}),$$

where the Gross' Laplacian Δ_G is an infinite dimensional analogue of the finite dimensional Laplacian on $\mathbb{R}^n$.

It turns out that one can construct a solution $u(t,\cdot)$ of the above heat equation as

$$\begin{aligned} u(t,x) &= P_t f(x) \\ &= \int f(x+y)dP_t(y), \end{aligned}$$

where P_t denote the Gaussian measure on $\mathcal{S}'(\mathbb{R})$ with variance $t > 0$. We show that $u(t,\cdot)$ belongs to $(\mathcal{S})$ for all $t \geq 0$. This representation of u gives a fundamental solution of the heat equation, which is represented by $\frac{d\mu_{t,x}}{d\mu} \in (\mathcal{S})^*$ in the sense of a generalized Radon-Nikodym derivative.

We now give a white noise description of Gross Laplacian which was originally formulated in abstract Wiener spaces.

The Gross Laplacian Δ_G was introduced by Gross [6] for functions defined on an abstract Wiener space (H, B). Suppose that φ is a twice differentiable function on B such that $\varphi''(x)$ is a trace class operator on H for every $x \in B$. Then the Gross' Laplacian $\Delta_G\varphi$ of φ is defined by :

$$(\Delta_G\varphi)(x) = trace_H \varphi''(x).$$

White noise calculus is the study of functionals on the white noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ using white noise $\{\dot{B}(t); t \in \mathbb{R}\}$ as coordinates.

The partial differentiation $\partial_t = \frac{\partial}{\partial \dot{B}(t)}$ with respect to these coordinates are special cases of differentiation in the sense of Gâteaux. Namely, we set $\partial_t = D_{\delta_t}$, where

$$D_y \phi(x) := \frac{d}{d\lambda}\phi(x+\lambda y)|_{\lambda=0}, \ y \in \mathcal{S}'(I\!R).$$

Therefore ∂_t is the Gâteaux derivative in the direction of δ_t, $t \in I\!R$.

As mentioned in [7], for every $t \in I\!R$, ∂_t is continuous on $(\mathcal{S})$. By ∂_t^* , $t \in I\!R$, we denote the adjoint of the operator ∂_t. and it is a linear operator from $(\mathcal{S})^*$ into $(\mathcal{S})^*$, continuous with respect to the weak topology.

Note that in the white noise space $(\mathcal{S}'(I\!R), B, \mu)$, the Gaussian measure is actually supported in the space $\mathcal{S}_{-p}(I\!R)$ for any $p > \frac{1}{2}$. Thus $(L^2(I\!R), \mathcal{S}_{-p}(I\!R))$ is an abstract Wiener space. Therefore for a function φ defined on $\mathcal{S}'(I\!R)$, we can consider its restriction to some $\mathcal{S}_{-p}(I\!R)$ for $p > \frac{1}{2}$ and define $\Delta_G\varphi$ if it exists.

We can use the following :

THEOREM 1.1. *Suppose φ is twice $L^2(I\!R)$– differentiable on some $\mathcal{S}_{-p}(I\!R)$, $p > 1$, such that $\varphi'(x) \in \mathcal{S}_{-p}(I\!R)$ and $\varphi''(x)$ is a trace class operator of $L^2(I\!R)$ for all x in $\mathcal{S}_{-p}(I\!R)$. Then φ is twice $\dot{B}(t)$–differentiable and*

$$\Delta_G\varphi = \int_{I\!R} {\partial_t}^2 \varphi dt.$$

For the proof, see [10].

REMARK. Note that in white noise calculus, $\{\dot{B}(t); t \in I\!R\}$ is regarded as a coordinate system so that $\int_{I\!R} \partial_t^2 \varphi dt$ is the infinite dimensional analogue of $\sum_{j=1} \frac{\partial}{{\partial_j}^2}\varphi$.

We have the following useful result in [7].

THEOREM 1.2. *The Gross Laplacian Δ_G is a continuous linear operator from $(\mathcal{S})$ into itself. Moreover, for $\varphi \in (\mathcal{S})$ represented by*

$$\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle,$$

$\Delta_G\varphi$ is given by

$$\Delta_G\varphi(x) = \sum_{n=2}^{\infty} n(n-1)\langle : x^{\otimes(n-2)} :, trf_n\rangle.$$

For the proof, see [7].

2. Gross Theorem

In this section we will look at the solution of the heat equation in the finite dimensional case and in abstract Wiener spaces. Consider the finite dimensional heat equation on $\mathbb{R}^k$

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta_x u(t,x), \quad t > 0, \ x \in \mathbb{R}^k,$$

(1) $$u(0,x) = f(x).$$

where f is a suitable function for the initial condition. This equation can be solved conveniently by using the Fourier transform. We shall denote the Fourier transform of a Schwartz function $g \in \mathcal{S}(\mathbb{R}^k)$ by

(2) $$\hat{g}(y) = (\frac{1}{\sqrt{2\pi}})^k \int_{\mathbb{R}^k} e^{-i<x,y>} g(x)dx.$$

For a function $u(t,x)$ depending on $t \in \mathbb{R}_+$ and on $x \in \mathbb{R}^k$, we denote its Fourier transform (2) with respect to the x variable also by $\hat{u}(t,y)$. Then equation (1) reads as follows:

$$\frac{\partial}{\partial t}\hat{u}(t,y) = -\frac{1}{2}|y|^2\hat{u}(t,y)$$

$$\hat{u}(0,y) = \hat{f}(y).$$

Hence, the solution is given by

$$\hat{u}(t,y) = \hat{f}(y)e^{-\frac{1}{2}t|y|^2}. \tag{3}$$

By taking the inverse Fourier transform, we obtain the solution of (1)

$$u(t,x) = (\frac{1}{\sqrt{2\pi t}})^k \int_{\mathbb{R}^k} f(z) e^{\frac{-|z-x|^2}{2t}} dz.$$

In the infinite dimensional case, Gross [6] has studied the analogue of equation (1) on an abstract Wiener spaces (H, B):

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\Delta_G u(t,x), \quad t > 0, \ x \in B$$
$$u(0,x) = f(x), \tag{5}$$

where Δ_G is Gross Laplacian operator defined in section 1.

Let P_t denote the Wiener measure on B with variance t. For a bounded Lip–1 function f defined on B, the function $u(t,x) = \int_B f(x+y)P_t(dy)$ satisfies equation (5).

THEOREM 2.1. *Let f be a bounded Lip-1 function in B. Then*

(1) $D^2 P_t f(x) \in L_{(1)}(H)$, *for all $x \in B$, where $L_{(1)}$ is a trace class operator.*
(2) *For each $c > 0$, the map $(t,x) \mapsto D^2 P_t f(x)$ is uniformly continuous from $[c,\infty) \times B$ into $L_{(1)}(H)$.*
(3) *$u(t,x) = P_t f(x)$ is uniformly continous on $[c,\infty) \times B$.*
(4) *For each $t > 0$, $\frac{\partial u}{\partial t}$ exists uniformly on x and*

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2} \mathrm{trace} D^2 u(t,x).$$

For the proof, see [6], [9].

REMARK. $\lim_{t\to 0} u(t,x) = f(x)$ uniformly in x. Hence $P_t f(x)$ solves the heat equation $\frac{\partial u}{\partial t}(t,x) = \frac{1}{2} trace D^2 u(t,x)$ with initial condition f.

3. The Heat Equation in the White Noise Space

Consider the heat equation in the white noise space

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta_G u(t,x), \quad t > 0, \quad x \in \mathcal{S}'(\mathbb{R}), \tag{6}$$

where $\Delta_G = \int \partial_t^2 dt$ is the Gross Laplacian operator, as described in section 1.

Let P_t denote the Wiener measure on $\mathcal{S}'(\mathbb{R})$ with $t > 0$. Then the following holds.

LEMMA 3.1. *Let* $\varphi_n(x) =<: x^{\otimes n} :, f_n >$, $f_n \in \widehat{\mathcal{S}(\mathbb{R}^n)}$, $x \in \mathcal{S}'(\mathbb{R})$. *Define*

$$P_t\varphi_n(x) = \int \varphi(x+y)P_t(dy).$$

Then $P_t\varphi_n(x)$ *is the solution of the equation (6).*

Proof.

$$\begin{aligned} P_t\varphi_n(x) &= \int \varphi_n(x+y)P_t(dy) \\ &= \int <: (x+y)^{\otimes n} :, f_n > P_t(dy) \\ &= \int <: (x+\sqrt{t}y)^{\otimes n} :, f_n > \mu(dy) \\ &= \int \sum_{k=0}^{\infty} \binom{n}{k} <: (\sqrt{t}y)^{\otimes k} :, g_k > \mu(dy), \end{aligned}$$

where $g_k =< x^{\otimes n-k}, f_n >$. Now compute as follows.

$$\begin{aligned} &<: (\sqrt{t}y)^{\otimes k} :, g_k > \\ &= \sum_{j=0}^{[k/2]} \binom{k}{2j}(2j-1)!!(-1)^j(\sqrt{t})^{k-2j}(1-t)^j <: y^{\otimes k-2j} : \hat{\otimes}\mathrm{Tr}^{\otimes j}, g_k > \\ &= \sum_{j=0}^{[k/2]} \binom{k}{2j}(2j-1)!!(-1)^j(1-t)^j <: y^{\otimes k-2j} :, \mathrm{Tr}^{\otimes j} g_k >, \end{aligned}$$

where we used a formula for $:(\lambda y)^{\otimes n}:$ from [13]. Note that

$$\int <: y^{\otimes k-2j} :, \mathrm{Tr}^{\otimes j} g_k >= 0$$

unless $k = 2j$. Hence we have

$$\int <: (\sqrt{t}y)^{\otimes k} :, g_k > \mu(dy) = (2j-1)!!(-1)^j(1-t)^j \mathrm{Tr}^j g_{2j}.$$

Therefore,

$$\begin{aligned}
P_t\varphi_n(x) &= \sum_{k=0}^{n} \binom{n}{k} \int <:(\sqrt{t}y)^{\otimes k}:, g_k> \mu(dy) \\
&= \sum_{\substack{k=0 \\ k=2j}}^{n} \binom{n}{2j} (2j-1)!!(-1)^j(1-t)^j \mathrm{Tr}^j g_{2j} \\
&= \sum_{\substack{k=0 \\ k=2j}}^{n} \binom{n}{2j} (2j-1)!!(-1)^j(1-t)^j \mathrm{Tr}^j < x^{\otimes n-k}, f_n > \\
&= \sum_{j=0}^{[n/2]} \binom{n}{2j} (2j-1)!!(-1)^j(1-t)^j < x^{\otimes(n-2j)} \hat{\otimes} \mathrm{Tr}^j, f_n > . \qquad (7)
\end{aligned}$$

Now differentiate with respect to t. Then we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}[P_t\varphi_n(x)] &= \sum_{j=0}^{[n/2]} \binom{n}{2j} (2j-1)!!j(-1)^{j-1}(1-t)^{j-1} \\
&\qquad \cdot < x^{\otimes(n-2j)} \hat{\otimes} \mathrm{Tr}^j, f_n > \\
&= \sum_{k=0}^{[n/2]-1} \binom{n}{2k+2} (2k+1)!!(k+1)(-1)^k(1-t)^k \cdot \\
&\qquad \cdot < x^{\otimes(n-2k-2)} \hat{\otimes} \mathrm{Tr}^{k+1}, f_n > . \qquad (8)
\end{aligned}$$

On the other hand, we can compute the Gross' Laplacian of $P_t\varphi$ as follows.

$$\begin{aligned}
&\Delta_G[P_t\varphi(x)] \\
&= \Delta_G \sum_{j=0}^{[n/2]} \binom{n}{2j} (2j-1)!!(-1)^j(1-t)^j < x^{\otimes(n-2j)} \hat{\otimes} \mathrm{Tr}^j, f_n > \\
&= \sum_{j=0}^{[n/2]} \binom{n}{2j} (2j-1)!!(-1)^j(1-t)^j (n-2j)(n-2j-1) \cdot \\
&\qquad \cdot < x^{\otimes(n-2j-2)} \hat{\otimes} \mathrm{Tr}^j, \mathrm{tr} f_n >
\end{aligned}$$

$$= \sum_{j=0}^{[n/2]-1} \binom{n}{2j} (2j-1)!!(-1)^j(1-t)^j(n-2j)(n-2j-1) \cdot$$
$$\cdot < x^{\otimes(n-2j-2)} \hat{\otimes} \mathrm{Tr}^{j+1}, f_n >$$
$$= \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-1)^k(1-t)^k(n-2k)(n-2k-1) \cdot$$
$$\cdot < x^{\otimes(n-2k-2)} \hat{\otimes} \mathrm{Tr}^{k+1}, f_n > .$$

It is easy to see that

$$\frac{1}{2}\binom{n}{2k}(2k-1)!!(n-2k)(n-2k-1) = \binom{n}{2k+2}(k+1)(2k+1)!!.$$

Hence we have

$$\frac{\partial}{\partial t}[P_t\varphi_n] = \frac{1}{2}\Delta_G P_t \varphi_n.$$

□

Lemma 3.1 shows that for φ in n-th homogenuous chaos, $P_t\varphi$ is a solution of the equation (6).

Also, one can consider the algebra of exponential function $\mathcal{A}$. We have the following result.

LEMMA 3.2. *Let* $\varphi(x) = e^{<x,\xi>}$, $\xi \in \mathcal{S}(\mathbb{R})$. *Then* $P_t\varphi(x)$ *is the solution of the equation (6).*

Proof.

$$P_t\varphi(x) = \int_{S'(R)} e^{<x+\sqrt{t}y,\xi>}\mu(dy)$$
$$= e^{<x,\xi>}\int_{S'(R)} e^{<\sqrt{t}y,\xi>}\mu(dy)$$
$$= e^{<x,\xi>}e^{\frac{t}{2}|\xi|_2^2}.$$

Then

$$(9) \qquad \frac{\partial}{\partial t}(P_t\varphi)(x) = \frac{1}{2}e^{<x,\xi>+\frac{t}{2}|\xi|_2^2}\cdot|\xi|_2^2$$

And

$$\Delta_G(P_t\varphi)(x) = \int \partial_u{}^2 e^{<x,\xi>+\frac{t}{2}|\xi|_2^2} du$$

$$= e^{<x,\xi>+\frac{t}{2}|\xi|_2^2} \cdot |\xi|_2^2. \tag{10}$$

Hence $\frac{\partial}{\partial t}(P_t\varphi)(x) = \frac{1}{2}\Delta_G(P_t\varphi)(x)$. Therefore, by linearity we can conclude that for every $\varphi \in \mathcal{A}$, $P_t\varphi(x)$ is a solution of equation (6). □

Now we extend the function φ to $(\mathcal{S})$. Consider $\varphi \in (\mathcal{S})$. Then by Theorem 4.53 of [7], there exist $p_0 > \frac{1}{2}$ and constant $C > 0$ such that for all $w > p_0$,

$$\left|\varphi(x+\sqrt{t}y)\right| \le C\|\varphi\|_{2,w} e^{|x+\sqrt{t}y|_{2,-w}^2}$$

$$\le C\|\varphi\|_{2,w} e^{2|x|_{2,-w}^2 + 2t|y|_{2,-w}^2}. \tag{11}$$

Make w large enough (depending on t) so that

$$2t\,|y|_{2,-w}^2 \le \alpha\,|y|_{2,-v}^2$$

where $v > \frac{1}{2}$, and α is such that Fernique's theorem (cf. [9]) applies. Then

$$\int e^{\alpha|y|_{2,-v}^2} d\mu(y) = C_0,$$

where C_0 is some constant. Note that $(L^2(\mathbb{R}), \mathcal{S}_{-v}(\mathbb{R}), \mu)$ is an abstract Wiener space if $v > \frac{1}{2}$.

Therefore, we have

$$\int |\varphi(x+\sqrt{t}y)| d\mu(y) \le Const.\|\varphi\|_{2,w} e^{2|x|_{2,-w}^2}, \tag{12}$$

for all w large enough, and the constant depends on x and t. Thus, for all $x \in \mathcal{S}'(\mathbb{R})$, $t \ge 0$, $\varphi \in (\mathcal{S})$, $P_t\varphi(x)$ is well defined. Since P_t is linear, the above estimation shows that $\varphi \to P_t\varphi(x)$ is continuous from $(\mathcal{S})$ into $\mathbb{C}$. Therefore, if φ_n converges to φ in $(\mathcal{S})$, then for every $x \in \mathcal{S}'(\mathbb{R})$, $t \in \mathbb{R}_+$, $P_t\varphi_n(x)$ converges to $P_t\varphi(x)$ in $\mathbb{C}$. In particular this convergence holds if φ_n approximates φ in $(\mathcal{S})$ by a polynomial or by a sum of exponential functions. Now consider the function $x \mapsto P_t\varphi(x)$ for $t > 0$ fixed, $\varphi \in (\mathcal{S})$ fixed. From the Lemma 5.2 of [11], the map is C^∞ in every direction of $\mathcal{S}'(\mathbb{R})$ in Gâteaux sense. Now we want to prove the following fact.

THEOREM 3.3. *For any* $\varphi \in (\mathcal{S})$, $P_t\varphi \in (\mathcal{S})$.

To prove the theorem, one can consider the $\mathcal{A}_p$ norm introduced in [5], see also [7]:

$\mathcal{A}_p$, for $p \in \mathbb{N}$, is the space of all (strongly) continuous function f on $\mathcal{S}'(\mathbb{R})$, such that the restriction of f to $\mathcal{S}_{-p}(\mathbb{R})$ is entire analytic and f is finite in the following norm:

$$\|f\|_{\mathcal{A}_p} := \sup_{z \in \mathcal{S}_{-p}(\mathbb{R})} |f(z)|\, e^{-\frac{1}{2}|z|_{2,-p}^2}.$$

To see that $P_t\varphi$ belongs to $\mathcal{A}_p$, one needs to check the following:

(1) $P_t\varphi$ is analytic on $\mathcal{S}_{-p}(\mathbb{R})$.

(2) $\sup_{z \in \mathcal{S}_{-p}(\mathbb{R})} |P_t\varphi(z)|\, e^{-\frac{1}{2}|z|_{2,-p}^2} < \infty$.

Proof of 1. Set $P_t\varphi(x+zy) := \int \varphi(x+zy+\sqrt{t}u)\mu(du)$. This is well defined as can be proved by an estimate as above.
Now we show that $P_t\varphi$ is entire analytic in the sense of Hille and Phillips [12], i.e.,

$$z \mapsto P_t\varphi(x+zy)$$

is entire analytic. First we give an argument that this mapping is continuous. By Lee's result [5] (cf. also [7]), φ has an entire analytic extention to the complexified Schwartz space of tempered distributions $\mathcal{S}'_{\mathbb{C}}(\mathbb{R})$. Therefore, in particular the mapping $z \mapsto \varphi(x+zy)$ is continuous from $\mathbb{C}$ into itself for all $\varphi \in (\mathcal{S}), x, y \in \mathcal{S}'(\mathbb{R})$. On the other hand, the estimation (11), (12) shows that $\varphi(x+zy+\sqrt{t}u)$ is integrable with respect to $\mu(du)$, uniformly in z on bounded subsets of $\mathbb{C}$. Thus we may apply the uniform convergence theorem to conclude that

$$z \mapsto \int \varphi(x+zy+\sqrt{t}u)\mu(du) = P_t\varphi(x+zy)$$

is continuous. Therefore, we may integrate $z \mapsto P_t\varphi(x+zy)$ over any closed contour in $\mathbb{C}$:

$$\begin{aligned}\oint P_t\varphi(x+zy)dz &= \oint\Big(\int \varphi(x+zy+\sqrt{t}u)\mu(du)\Big)dz \\ &= \int \oint \varphi(x+zy+\sqrt{t}u)dz\mu(du) \\ &= 0,\end{aligned}$$

where we used Fubini's theorem in the first step, and Cauchy's theorem in the last. (Recall that $z \mapsto \varphi(x + zy + \sqrt{t}u)$ is analytic.) Hence by Morera's theorem, $P_t\varphi$ is analytic.

Proof of 2. By using a theorem of [5] (cf. also [7]) and (12),

$$\begin{aligned}
\|P_t\varphi\|_{\mathcal{A}_p} &= \sup_{x\in\mathcal{S}_{-p}(\mathbb{R})} |P_t\varphi(x)|\, e^{-\frac{1}{2}|x|^2_{2,-p}} \\
&\leq \sup_{x\in\mathcal{S}_{-p}(\mathbb{R})} C\|\varphi\|_{2,w} e^{2|x|^2_{2,-w}} e^{-\frac{1}{2}|x|^2_{2,-p}} \\
&\leq \sup_{x\in\mathcal{S}_{-p}(\mathbb{R})} C_q\|\varphi\|_{2,q} e^{\frac{1}{2}(|x|^2_{2,-p}-|x|^2_{2,-p})}
\end{aligned}$$

for large enough $q > p + 2$. Thus we have for all $p \in \mathbb{N}_0$,

$$\|P_t\varphi\|_{\mathcal{A}_p} \leq C_q\|\varphi\|_{2,q}.$$

Hence $P_t\varphi$ belongs to $\mathcal{A}_p$ for every $p \in \mathbb{N}_0$. Since by the result in [7], [13], $\mathcal{A} = \cap\mathcal{A}_p$ is homeomorphic to $(\mathcal{S})$, $P_t\varphi$ belongs to $(\mathcal{S})$. Therefore, the theorem is proved. □

From the proof of the Theorem 3.3, we have the following fact.

COROLLARY 3.4. *P_t is continuous on $(\mathcal{S})$.*

Moreover, from [11], $(P_t\varphi)(x)$ has the following representation:

$$(P_t\varphi)(x) = \langle\langle \frac{d\mu_{t,x}}{d\mu}, \varphi\rangle\rangle,$$

where $\frac{d\mu_{t,x}}{d\mu} \in (\mathcal{S})^*$, $t \geq 0$, $x \in \mathcal{S}'(\mathbb{R})$. So if we can prove that $(P_t\varphi)(x)$ solve the heat equation, then $\frac{d\mu_{t,x}}{d\mu}$ is the fundamental solution. To show that for $\varphi \in (\mathcal{S})$, $P_t\varphi(x)$ is the solution of the heat equation, we need the following lemma.

LEMMA 3.5. *Let $\varphi \in (\mathcal{S})$ have chaos decomposition given by*

$$\varphi = \sum_{n=0}^{\infty} \varphi_n.$$

Then we have

$$\frac{\partial}{\partial t}P_t\varphi = \sum_{n=0}^{\infty}\frac{\partial}{\partial t}P_t\varphi_n$$

Proof. By Corollary 3.4, we know that $P_t\sum_{m=0}^{n}\varphi_m$ converges to $P_t\varphi$, since $\sum_{m=0}^{n}\varphi_m$ converges to φ in $(\mathcal{S})$. Now we have the following facts;

(1) From [11], $P_t\varphi_n(x)$ is differentiable on $I\!R_+$, with respect to t,

(2) By the result in [7], Δ_G is continuous on $(\mathcal{S})$,

(3) By the Corollary 2.3.4, P_t are continuous on $(\mathcal{S})$,

(4) By the Lemma 2.3.1, $\frac{\partial}{\partial t}P_t\varphi_n = \frac{1}{2}\Delta_G P_t\varphi_n \in (\mathcal{S})$.

By the facts above, $\sum_{n=1}^{\infty}\frac{\partial}{\partial t}P_t\varphi_n = \frac{1}{2}\Delta_G P_t\sum_{n=1}^{\infty}\varphi_n$.

Next, we need to see the uniform convergence of $\sum_{n=1}^{\infty}\frac{\partial}{\partial t}P_t\varphi_n(x)$ on $t \in T$, where T is some interval in $I\!R_+$. Since $\sum_{n=1}^{\infty}\frac{\partial}{\partial t}P_t\varphi_n = \frac{1}{2}\Delta_G P_t\sum_{n=1}^{\infty}\varphi_n$, it is enough to show that $\sum_{n=1}^{\infty}\Delta_G P_t\varphi_n$ converges uniformly in t on some interval T. Using the continutity of Δ_G, we find the following estimate similarly as above: For some $p \in I\!N$.

$$\begin{aligned}|\Delta_G P_t\varphi_n(x)| &\le C_p\|\Delta_G P_t\varphi_n\|_{2,p}e^{2|x|_{2,-p}^2}\\ &\le C\|P_t\varphi_n\|_{2,p+q}e^{2|x|_{2,-p}^2},\end{aligned}$$

where C is the some constant and where q is large enough. Also

$$\begin{aligned}\|P_t\varphi_n\|_{2,p+q} &= \|\int\varphi_n(\cdot+\sqrt{t}y)\mu(dy)\|_{2,p+q}\\ &\le \int\|\varphi_n(\cdot+\sqrt{t}y)\|_{2,p+q}\mu(dy).\end{aligned}$$

Since $\varphi_n(x+\sqrt{t}y) = (\tau_{\sqrt{t}y}\varphi_n)(x)$, where $(\tau_y\varphi)(x) = \varphi(x+y)$,

$$\|\varphi_n(\cdot+\sqrt{t}y)\|_{2,p+q} = \|\tau_{\sqrt{t}y}\varphi_n\|_{2,p+q}.$$

Also since $t \mapsto \|\tau_{\sqrt{t}y}\varphi_n\|_{2,p+q}$ is continuous from $I\!R_+$ into $I\!R_+$, one can conclude that for every bounded interval $T \subset [0,\infty)$, $\|\tau_{\sqrt{t}y}\varphi_n\|_{2,p+q}$ is bounded.

Therefore, for any $t \in T$,

$$\begin{aligned}\|\tau_{\sqrt{t}y}\varphi_n\|_{2,p+q} &\leq K_{y,T}\|\varphi_n\|_{2,p+q} \\ &\leq K_{y,T}\sqrt{n!}\|f_n\|_{2,p+q}.\end{aligned}$$

Hence

$$|\Delta_G P_t \varphi_n(x)| \leq K_{y,T}\sqrt{n!}e^{2|x|^2_{2,-p}}\|f_n\|_{2,p+q},$$

where $K_{y,T}$ is some constant depending on T and y.
Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}|\Delta_G P_t\varphi_n(x)| &\leq K_{y,T}e^{2|x|^2_{2,-p}}\sum_{n=0}^{\infty}\sqrt{n!}\|f_n\|_{2,p+q} \\ &= K_{y,T}e^{2|x|^2_{2,-p}}\sum_{n=0}^{\infty}\sqrt{n!}\|f_n\|_{2,p+q+1}\cdot 2^{-n} \\ &\leq K_{y,T}e^{2|x|^2_{2,-p}}\|\varphi\|_{2,p+q+1}(\sum_{n=0}^{\infty}2^{-n})^{\frac{1}{2}}.\end{aligned}$$

Therefore, $\sum \frac{\partial}{\partial t}P_t\varphi_n(x)$ is uniformly convergent with respect to t on some bounded interval T. Hence the statement of the Lemma follows. □

By Lemmas 3.1, and 3.4, we have the following result.

THEOREM 3.6. *For any $\varphi \in (\mathcal{S})$, $\varphi(t,\cdot) = P_t\varphi \in (\mathcal{S})$ solves the initial value problem*

$$\frac{\partial}{\partial t}\varphi(t,x) = \frac{1}{2}\Delta_G\varphi(t,x), \quad x \in \mathcal{S}'(\mathbb{R}),$$

$$\varphi(0,x) = \varphi(x).$$

Moreover, $\frac{d\mu_{t,x}}{d\mu} \in (\mathcal{S})^*$ is the fundamental solution of the heat equation.

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