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TOTALLY UMBILIC LORENTZIAN SUBMANIFOLDS

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1. Introduction

A totally umbilic submanifold of a pseudo-Riemannian manifold is a submanifold whose first fundamental form and second fundamental form are proportional. An ordinary hypersphere $S^n(r)$ of an affine $(n+1)$-space of the Euclidean space E^m is the best known example of totally umbilic submanifolds of E^m. From the point of views in differential geometry, the totally umbilic submanifolds are the simplest submanifolds next to totally geodesic submanifolds. The totally umbilic submanifolds of a Riemannian space form $\bar{M}^m(c)$ with constant sectional curvature c are well known ([2], p.129).

In this paper we classify the totally umbilic submanifolds of the pseudo-Euclidean space E_t^m and prove that a submanifold M_s^n of E_t^m with indefinite metric (i.e., $1 \leq s \leq n-1$) is totally umbilic if and only if null geodesics of M_s^n are all straight lines.

2. Notations and Terminologies

Let E_t^m be the m-dimensional pseudo-Euclidean space with the standard flat metric given by

$$\bar{g} = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{m} dx_j^2,$$

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where $(x_1, \cdots, x_m)$ is a rectangular coordinate system of E_t^m. For a positive number r and a point $c \in E_t^m$, we denote by $S_t^{m-1}(c, r)$ and $H_{t-1}^{m-1}(c, -r)$, the pseudo-Riemannian sphere and the pseudo-hyperbolic space defined respectively by

$$S_t^{m-1}(c, r) = \{x \in E_t^m | < x - c, x - c >= r^2\},$$

$$H_{t-1}^{m-1}(c, -r) = \{x \in E_t^m | < x - c, x - c >= -r^2\},$$

where $<,>$ denotes the indefinite inner product on the pseudo-Euclidean space. The point c is called the center of $S_t^{m-1}(c, r)$ and of $H_{t-1}^{m-1}(c, -r)$, respectively. We simply denote $S_t^{m-1}(0, 1)$ and $H_{t-1}^{m-1}(0, -1)$ by S_t^{m-1} and H_{t-1}^{m-1}, respectively. In physics, S_1^{m-1}, H_1^{m-1} and E_1^m are known as de Sitter space-time, anti-de Sitter space-time and the Minkowski space-time, respectively. A vector X in E_t^m is said to be space-like (respectively, time-like or light-like) if $< X, X >> 0$ or $X = 0$ (respectively, $< X, X >< 0$ or $< X, X >= 0$ with $X \neq 0$).

Let M_s^n be a submanifold of a pseudo-Euclidean space E_t^m. $\nabla, \tilde{\nabla}, D$, h and A_ξ denote the Levi-Civita connection on M_s^n, the flat connection on E_t^m, the normal connection on the normal bundle of M_s^n, the second fundamental form and the Weingarten map with respect to ξ in the normal bundle, respectively. Note that $H = \frac{1}{n} trace\ h$ is called the mean curvature vector of the submanifold M_s^n of E_t^m. If $\{e_1, \cdots, e_n\}$ is a local orthonormal frame of the tangent bundle of M_s^n, then $trace\ h = \Sigma_{i=1}^n \epsilon_i h(e_i, e_i)$, where $\epsilon_i = g(e_i, e_i) = \pm 1$ for $i = 1, \cdots, n$. If $H \equiv 0$, then M_s^n is called a minimal submanifold of E_t^m. We may find the basic notations and formulae in [2, 4].

3. Main Theorems

Recall that totally umbilic submanifolds are the submanifolds satisfying

$$h(X, Y) =< X, Y > H, \quad X, Y \in TM. \tag{3.1}$$

For a fixed null vector $x_0 \in E_{s+1}^{n+1}$, we let $U_s^n(x_0)$ denote the pseudo-Riemannian submanifold given by ([3])

$$U_s^n(x_0) = \{x \in E_{s+1}^{n+2} | \ < x, x >= 0, \ < x, x_0 >= -1\}. \tag{3.2}$$

Then, since $\{x_0, x\}$ is a basis of the normal bundle, we have $h(X,Y) = < X, Y > x_0$. Hence $U^n_s(x_0)$ is a flat totally umbilic submanifold of E^{n+2}_{s+1} with constant mean curvature vector field $H \equiv x_0$. In fact, $U^n_s(x_0)$ is isometric to E^n_s and for any null vector x_0, $U^n_s(x_0)$ are all congruent in E^{n+2}_{s+1}.

PROPOSITION 3.1. *Let M^n_s, $n \geq 2$, be a submanifold of E^m_t. Then M^n_s is totally umbilic if and only if M^n_s is, up to congruences of E^m_t, an open part of the following :*

E^n_s, $S^n_s(0,r)$, $H^n_s(0,r)$, $U^n_s(x_0)$.

Proof. Suppose that M^n_s is totally umbilic. Then, as in the proof of Riemannian case ([2]), (3.1) and Codazzi equation implies that

$$DH = 0, \quad A_H = < H, H > I \tag{3.3}$$

and $< H, H >$ is constant.

If $H \equiv 0$, then (3.1) shows that M^n_s is totally geodesic.

Now assume that $H \neq 0$.

(1) If $< H, H >= \epsilon\alpha^2$ with $\epsilon = \pm 1$ and $\alpha > 0$, then $H = \alpha e_{n+1}$. Then as in the proof of Riemannian case, M^n_s lies in the fixed $(n+1)$-dimensional linear subspace of E^m_t generated by $\{e_1, \cdots, e_n, e_{n+1}\}$. Thus M^n_s is contained in E^{n+1}_s (if $\epsilon = 1$) or E^{n+1}_{s+1} (if $\epsilon = -1$). And it can be shown that M^n_s is, up to congruences, an open part of $S^n_s(0,r)$ (if $\epsilon = 1$) or $H^n_s(0,r)$ (if $\epsilon = -1$).

(2) If $< H, H >= 0$, then (3.3) implies that H is a constant null vector x_0. Note that (3.1) implies

$$\tilde{\nabla}_X e_i = \nabla_X e_i + < X, e_i > x_0, \quad i = 1, \cdots, n, \quad X \in TM. \tag{3.4}$$

Hence we have from (3.4)

$$\tilde{\nabla}_X (e_1 \wedge \cdots \wedge e_n \wedge x_0) = 0, \quad X \in TM. \tag{3.5}$$

(3.5) shows that M^n_s lies in a fixed degenerate $(n+1)$-dimensional affine space in E^{n+2}_{s+1}. Choose a null vector field y of the normal bundle of M^n_s in E^{n+2}_{s+1} such that $< y, x_0 >= -1$. Then from (3.1) we obtain

$$A_y = -I, \quad Dy = 0. \tag{3.6}$$

Hence $x - y$ is a constant vector y_0. Thus we have

$$< x - y_0, x - y_0 >= 0, \quad < x - y_0, x_0 >= -1.$$

Therefore M_s^n is, up to congruences, an open part of $U_s^n(x_0)$.

The converse is obvious. □

Let M_s^n $(1 \le s \le n-1)$ denote the totally umbilic submanifolds in Proposition 3.1, then it can be easily shown that every null geodesic of M_s^n is a straight line.

Conversely we prove the following :

THEOREM 3.2. *Let M_s^n $(1 \le s \le n-1)$ be a submanifold of E_t^m with an indefinite metric. Then M_s^n is totally umbilic if every null geodesic of M_s^n is a straight line.*

We first prove the following lemma :

LEMMA 3.3. *Let V_s^n be an n-dimensional scalar product space with index $s = 1, \cdots, n-1$ and $h : V_s^n \times V_s^n \to W$ be a symmetric bilinear map. Then the following are equivalent.*

(1) $h(v, v) = 0$ for all null vector $v \in V_s^n$.

(2) $h(X, Y) =< X, Y > H$ for all $X, Y \in V_s^n$, where $H = \frac{1}{n} tr h$.

Proof. Choose an orthonormal basis $\{e_1, \cdots, e_n\}$ so that $< e_i, e_j > = \epsilon_i \delta_{ij}$ where $\epsilon_i = -1$ for $i \in \{1, \cdots, s\}$ and $\epsilon_j = 1$ for $j \in \{s+1, \cdots, n\}$. For $i \in \{1, \cdots, s\}$ and $j \in \{s+1, \cdots, n\}$, since $e_i \pm e_j$ is null, we have

(3.7)
$$h(e_i, e_j) = 0 \text{ and } -h(e_i, e_i) = h(e_j, e_j), \quad 1 \le i \le s, \ s+1 \le j \le n.$$

For $i, j \in \{1, \cdots, s\}$ with $i \ne j$, since $e_i + e_j + \sqrt{2} e_n$ is null, (3.7) implies that

$$h(e_i, e_j) = 0, \quad 1 \le i, j \le s, \ i \ne j \tag{3.8}$$

Now let $i, j \in \{s+1, \cdots, n\}$ with $i \ne j$. Then $\sqrt{2} e_1 + e_i + e_j$ is null. Hence (3.7) shows that

$$h(e_i, e_j) = 0, \quad s+1 \le i, j \le n, \ i \ne j. \tag{3.9}$$

Thus from (3.7), (3.8) and (3.9) we know that $h(X, Y) = \langle X, Y \rangle H$.
The converse is obvious. □

Proof of Theorem 3.2. For any fixed point $p \in M_s^n$ and a null vector $v \in T_pM$, consider the null geodesic $\gamma(t)$ of M_s^n with $\dot{\gamma}(0) = v$. Since $\gamma(t)$ is a straight line, we have $\bar{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$. Hence we obtain $h(v, v) = 0$ for all null vector $v \in T_pM$. Thus Lemma **3.3** shows that M_s^n is totally umbilic. □

Similarly, we may prove the following :

THEOREM 3.4. *Let M_s^n be a submanifold of $\bar{M}_t^m$ with index $1 \leq s \leq n-1$. Then the following are equivalent.*

(1) *M_s^n is totally umbilic.*

(2) *Every null geodesic of M_s^n is a geodesic of the ambient space $\bar{M}_t^m$.*

References

1. B.-Y. Chen, *Geometry of submanifolds and its application*, Science University of Tokyo,Toyko, 1981.
2. B.-Y. Chen, *Total mean curvature and submanifolds of finite type*, World scientific, New Jersey and Singapore, 1984.
3. M. A. Magid, *Isometric immersions of Lorentz space with parallel second fundamental forms*, Tsukuba Jour.Math. **8** (1984), 31-54.
4. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

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