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LOCAL GENERALIZED SOBOLEV SPACES

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I. Introduction

We introduced the generalized Sobolev spaces H^s_ω in [4]. In this paper, we introduce the space $H^s_{\omega c}(\Omega)$ of the generalized distributions in H^s_ω with compact supports in Ω and the local generalized Sobolev spaces $H^s_{\omega loc}(\Omega)$ of the generalized distributions on Ω which are locally in H^s_ω and study their properties.

For this purpose we briefly introduce the basic spaces which we need in this paper. The reader can find the details in [3]. Throughout this study, Ω denotes an open subset of R^n, and ω denotes an element of $\mathcal{M}_c$, the set of all continuous real valued functions ω on R^n which satisfy the following conditions :

$$(\alpha) \quad 0 = \omega(0) \le \omega(\xi + \eta) \le \omega(\xi) + \omega(\eta), \quad \xi, \eta \in R^n.$$

$$(\beta) \quad \int_{R^n} \frac{\omega(\xi)}{(1 + |\xi|)^{n+1}} d\xi < \infty.$$

(γ) $\omega(\xi) \ge a + \log(1 + |\xi|)$ for some constant a.

(δ) $\omega(\xi)$ is radial and increasing.

With the weight function ω and open set Ω in R^n. Björck defines $\mathcal{D}_\omega(\Omega)$ as the set of all ϕ in $L^1(R^n)$ such that ϕ has compact support in Ω and

$$\|\phi\|_\lambda = \int_{R^n} |\hat{\phi}(\xi)| e^{\lambda\omega(\xi)} d\xi < \infty$$

for all $\lambda > 0$. The space $\mathcal{D}_\omega(\Omega)$ equipped with the strict inductive limit topology is a strict (LF)-space, which is a complete (DF)-space.

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And we call $\mathcal{D}'_\omega(\Omega)$, the dual of $\mathcal{D}_\omega(\Omega)$, the Beurling's generalized distribution space. They denote by $\mathcal{E}_\omega(\Omega)$ the set of all complex-valued functions ψ in Ω such that $\psi\phi \in \mathcal{D}_\omega(\Omega)$ for all $\phi \in \mathcal{D}_\omega(\Omega)$ and the topology is given by the semi-norms $\|\phi\psi\|_\lambda$ for every $\lambda > 0$ and every ϕ in $\mathcal{D}_\omega(\Omega)$. The dual space $\mathcal{E}'_\omega(\Omega)$ of the space $\mathcal{E}_\omega(\Omega)$ can be identified with the set of all elements of $\mathcal{D}'_\omega(\Omega)$ which have compact supports contained in Ω . And $\mathcal{E}'_\omega(\Omega)$ can be considered as a subspace of $\mathcal{E}'_\omega(U)$ for any open subset U such that $\Omega \subseteq U \subseteq R^n$. They also introduced the generalized Schwartz space, denoted by $\mathcal{S}_\omega$, the space of all C^∞-function ϕ in $L^1(R^n)$ with the property that for each multi-index α and each non-negative number λ we have

$$P_{\alpha,\lambda}(\phi) = \sup_{x\in R^n} e^{\lambda\omega(x)}|D^\alpha\varphi(x)| < \infty$$

and

$$\Pi_{\alpha,\lambda}(\hat{\phi}) = \sup_{\xi\in R^n} e^{\lambda\omega(\xi)}|D^\alpha\hat{\phi}(\xi)| < \infty$$

and the dual space $\mathcal{S}'_\omega$ of the space $\mathcal{S}_\omega$.

II. Sobolev spaces with Compact Supports

Recall that $H^s_\omega = \{u \in \mathcal{S}'_\omega | \|u\|^\omega_s = [\int e^{2s\omega(\xi)}|\hat{u}(\xi)|^2 d\xi]^{\frac{1}{2}} < \infty\}$. Let Ω be an open subset of R^n and K any compact subset of R^n. Set $H^s_\omega(K) = \{u \in H^s_\omega | \operatorname{supp} u \subseteq K\}$ and $H^s_{\omega c}(\Omega) = H^s_\omega \cap \mathcal{E}'_\omega(\Omega)$. Then $H^s_\omega(K)$ is a Hilbert space with inner product given by

$$(u,v)^\omega_s = \int e^{2s\omega(\xi)}\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi$$

by Theorem 2.2 in [4]. We provide $H^s_{\omega c}(\Omega)$ with the strongest locally convex topology such that the inclusion map $H^s_\omega(K) \to H^s_{\omega c}(\Omega)$ is continuous for each compact subset K of Ω. A seminorm $\|\cdot\|$ on $H^s_{\omega c}(\Omega)$ is continuous if and only if for each compact subset K of Ω there is a constant C_K such that $\|u\| \le C_K\|u\|^\omega_s$ for each $u \in H^s_\omega(K)$. Since the topology of $H^s_{\omega c}(\Omega)$ may be defined by considering a sequence of compact sets increasing to Ω we see that $H^s_{\omega c}(\Omega)$ is an LB-space, a strict inductive limit of the Banach spaces $H^s_\omega(K)$. In particular, it is a complete Hausdorff non-metrizable locally convex space. Each $H^s_\omega(K)$ is a closed subspace of $H^s_{\omega c}(\Omega)$ and a subset B of $H^s_{\omega c}(\Omega)$ is bounded if and only if B is a bounded subset of $H^s_\omega(K)$ for some compact subset K of Ω.

PROPOSITION 1. *Let (ψ_m) be a locally finite partition of unity in $\mathcal{D}_\omega(\Omega)$. If a $= (a_m)$ is any sequence of non-negative integers, define $\|u\|^\omega_{a,s} = \sum a_m \|\psi_m u\|^\omega_s$ for all u in $H^s_{\omega c}(\Omega)$. Then the (uncountable) family of seminorms $\|u\|^\omega_{a,s}$ defines the topology $\mathcal{T}$ of $H^s_{\omega c}(\Omega)$.*

Proof. Since the sum is in fact a finite summation, they clearly define the seminorms on $H^s_{\omega c}(\Omega)$. By the proof of Lemma 2.8 in [4], we have $\|\psi_m u\|^\omega_s \le \|\psi_m\|_{|s|}\|u\|^\omega_s$ for all u in H^s_ω. Hence $\|u\|^\omega_{a,s} \le (2\pi)^{-\frac{n}{2}}(\sum a_m \|\psi_m\|_{|s|})\|u\|^\omega_s$. Now if $u \in H^s_\omega(K)$ then $\operatorname{supp} u \subseteq$ K, a compact set. Hence the above sum is a finite summation of non-negative real numbers. Hence the above inequality shows that the inclusion map $H^s_\omega(K) \to H^s_{\omega c}(\Omega)$ is continuous for each compact subset K of Ω with respect to the topology $\mathcal{T}'$ on $H^s_{\omega c}(\Omega)$ induced by the seminorms. Hence $\mathcal{T}$ is finer than $\mathcal{T}'$. In order to prove that $\mathcal{T}'$ is finer than $\mathcal{T}$, let G be any balanced and convex $\mathcal{T}$-neighborhood of 0. Since the inclusion map $I_K : H^s_\omega(K) \to H^s_{\omega c}(\Omega)$ is continuous for each compact subset K of Ω, $I_K^{-1}(G)$ is an open neighborhood of 0 in $H^s_\omega(K)$ for each K. If $K_j = \cup_{k=1}^j \operatorname{supp} \psi_k$, (K_j) is a sequence of compact subsets which increase to Ω. Then, for each j, there is an $\epsilon_j > 0$ such that $B(K_j, \epsilon_j) \equiv \{v \in H^s_\omega(K_j) | \|v\|^\omega_s < \epsilon_j\} \subseteq I_{K_j}^{-1}(G)$. Hence $B(K_j, \epsilon_j) = I_{K_j}(B(K_j, \epsilon_j)) \subseteq G$. Thus, $\cup_j B(K_j, \epsilon_j) \subseteq G$. Now, let $a = (a_j) = (2^j(1 + [\frac{1}{\epsilon_j}]))$ and consider $V = \{v \in H^s_{\omega c}(\Omega) | \|v\|^\omega_{a,s} < 1\}$. For each v in V, $v = \sum \frac{1}{2^j}(2^j \psi_j v)$. For each j, we have $\|2^j \psi_j v\|^\omega_s = \frac{2^j}{a_j} a_j \|\psi_j v\|^\omega_s \le \frac{2^j}{a_j}(\sum_k a_k \|\psi_k v\|^\omega_s) < \frac{2^j}{a_j} < \epsilon_j$. Hence, $2^j \psi_j v \in B(K_j, \epsilon_j) \subseteq G$. Since G is convex and $v = \sum \frac{1}{2^j}(2^j \psi_j v)$ is in fact a finite summation, we have $v \in G$. Hence G is a $\mathcal{T}'$-neighborhood of 0.

LEMMA 2. *The inclusion map $\mathcal{D}_\omega(\Omega)$ in $H^s_{\omega c}(\Omega)$ is continuous and has dense image. And the inclusion map $H^s_{\omega c}(\Omega)$ in H^s_ω is continuous.*

Proof. Let K be a compact subset of Ω. Then $\mathcal{D}_\omega(K) \to H^s_\omega(K)$ and $H^s_\omega(K) \to H^s_{\omega c}(\Omega)$ are continuous. But the continuity of $\mathcal{D}_\omega(K) \to H^s_{\omega c}(\Omega)$ for each compact subset K of Ω implies the continuity of

$\mathcal{D}_\omega(\Omega) \to H^s_{\omega c}(\Omega)$. If $u \in H^s_{\omega c}(\Omega)$ choose $\psi \in \mathcal{D}_\omega(\Omega)$ such that $\psi u = u$. By Theorem 2.2 in [4] we can choose $u_k \in \mathcal{S}_\omega$ so that $u_k \to u$ in H^s_ω. Then $\psi u_k \to \psi u = u$ in H^s_ω by the proof of Lemma 2.8 in [4]. If $K = \operatorname{supp}\psi$ then $\psi u_k \to u$ in $H^s_\omega(K)$ and in $H^s_\omega(\Omega)$. The last inclusion is also continuous since $\|u\|^\omega_s \le \sum \|\psi_m u\|^\omega_s = \|u\|^\omega_{a,s}$ for all $u \in H^s_{\omega c}(\Omega)$ and a $= (1, 1, \ldots)$.

In [4], we defined, for each non-negative integer k, the space $\mathcal{E}^k_\omega(\Omega)$ as the vector space of all locally integrable functions u on Ω such that

$$\|\phi u\|_k = \int e^{k\omega(\xi)}|\widehat{\phi u}(\xi)|d\xi < \infty$$

for all $\phi \in \mathcal{D}_\omega(\Omega)$. And we also defined the space $\mathcal{D}^k_\omega(\Omega)$ as the set of all u in $\mathcal{E}^k_\omega(\Omega)$ such that $\operatorname{supp} u$ is a compact subset of Ω with the inductive limit topology induced by the topologies on the spaces $\mathcal{D}^k_\omega(K)$ of the functions u of $\mathcal{E}^k_\omega(\Omega)$ with supports in compact subsets K of Ω. We have

PROPOSITION 3. *If k is a non-negative integer we have a continuous inclusion $\mathcal{D}^k_\omega(\Omega) \to H^k_{\omega c}(\Omega)$. If k is a non-negative integer and $s > k + \frac{n}{2}$ we have a continuous inclusion $H^s_{\omega c}(\Omega) \to \mathcal{D}^k_\omega(\Omega)$.*

Proof. Let (ψ_m) be the partition of unity in $D_\omega(\Omega)$ and let $a = (a_m)$ be any sequence of non-negative integers. For any compact subset K and each $u \in D^k_\omega(K)$, we have

$$\|u\|^\omega_{a,k} = \sum a_m \|\psi_m u\|^\omega_k \le \sum a_m \|\psi_m\|_k \|u\|^\omega_k \le C\|u\|^\omega_k.$$

Let ϕ be a local unit for K. Then, by Minkowski's inequality,

$$\begin{aligned}
\|u\|^\omega_k = \|\phi u\|^\omega_k &= \Big(\int e^{2k\omega(\xi)}|\widehat{\phi u}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \\
&= \Big(\int e^{2k\omega(\xi)}\Big|(2\pi)^{-n}\int \hat{u}(\eta)\hat{\phi}(\xi-\eta)d\eta\Big|^2 d\xi\Big)^{\frac{1}{2}} \\
&\le (2\pi)^{-\frac{n}{2}} \int |\hat{u}(\eta)| \Big(\int |\hat{\phi}(\xi-\eta)|^2 e^{2k\omega(\xi)} d\xi\Big)^{\frac{1}{2}} d\eta \\
&\le (2\pi)^{-\frac{n}{2}} \Big(\int |\hat{u}(\eta)| e^{k\omega(\eta)} d\eta\Big)\Big(\int |\hat{\phi}(\xi-\eta)|^2 e^{2k\omega(\xi-\eta)} d\xi\Big)^{\frac{1}{2}} \\
&\le C\|u\|_k \text{ for each } u \in \mathcal{D}^k_\omega(K).
\end{aligned}$$

The last inequality follows from Paley-Wiener Theorem in [3]. Hence $\mathcal{D}^k_\omega(\Omega) \to H^k_{\omega c}(\Omega)$ is continuous. Now suppose that $s > k + \frac{n}{2}$ and $u \in H^s_{\omega c}(\Omega)$. Then

$$\begin{aligned}\|u\|_k &= \int e^{k\omega(\xi)}|\hat{u}(\xi)|d\xi \\ &\le [\int e^{2s\omega(\xi)}|\hat{u}(\xi)|^2 d\xi]^{\frac{1}{2}}[\int e^{2(k-s)\omega(\xi)}d\xi]^{\frac{1}{2}} \\ &\le C\|u\|^\omega_s.\end{aligned}$$

But

$$\|u\|^\omega_s = \|\sum \psi_m u\|^\omega_s \le \sum \|\psi_m u\|^\omega_s = \|u\|^\omega_{a,s},$$

where a = (1, 1, ...). Therefore, $H^s_{\omega c}(\Omega) \to \mathcal{D}^k_\omega(\Omega)$ is continuous.

III. Local Sobolev Spaces

We set $H^s_{\omega loc}(\Omega) = \{u \in \mathcal{D}'_\omega(\Omega) | \phi u \in H^s_\omega$ for each $\phi \in \mathcal{D}_\omega(\Omega)\}$. We give $H^s_{\omega loc}(\Omega)$ the weakest topology so that the mapping $H^s_{\omega loc}(\Omega) \to H^s_\omega : u \mapsto \phi u$ is continuous for each $\phi \in \mathcal{D}_\omega(\Omega)$. Clearly there is a sequence $\phi_k \in \mathcal{D}_\omega(\Omega)$ such that whenever $\psi \in \mathcal{D}_\omega(\Omega)$ then there is k_0 such that $\phi_k\psi = \psi$ for $k \ge k_0$. Then for $k \ge k_0$, $\|\psi u\|^\omega_s = \|\phi_k\psi u\|^\omega_s \le C_\psi\|\phi_k u\|^\omega_s$. Hence the seminorms $u \mapsto \|\phi_k u\|^\omega_s, k = 0, 1, 2, \ldots$, determine the topology of $H^s_{\omega loc}(\Omega)$. In particular, it is metrizable. Moreover, we have

LEMMA 4. *$H^s_{\omega loc}(\Omega)$ is a Fréchet space.*

Proof. It suffices to show the completeness. Let (u_k) be a Cauchy sequence in $H^s_{\omega loc}(\Omega)$. If $\phi \in \mathcal{D}_\omega(\Omega)$ then $\phi u_k \to v_\phi$ in H^s_ω for some $v_\phi \in H^s_\omega$ since H^s_ω is complete. If $\phi, \psi \in \mathcal{D}_\omega(\Omega)$ then $\phi v_\psi = \psi v_\phi$, which is the limit of $(\phi\psi u_k)$. Hence there exists $v \in \mathcal{D}'_\omega(\Omega)$ such that $v_\phi = \phi v$ for each $\phi \in \mathcal{D}_\omega(\Omega)$. Then $\phi v \in H^s_\omega$ and $\phi u_k \to \phi v$ in H^s_ω. Therefore $v \in H^s_{\omega loc}(\Omega)$ and $u_k \to v$ in $H^s_{\omega loc}(\Omega)$.

We also have

LEMMA 5. *The inclusion map of $\mathcal{E}_\omega(\Omega)$ in $H^s_{\omega loc}(\Omega)$ is continuous. Moreover, $\mathcal{D}_\omega(\Omega)$ is dense in $H^s_{\omega loc}(\Omega)$.*

Proof. If $\phi \in \mathcal{D}_\omega(\Omega)$ then multiplication by ϕ maps $\mathcal{E}_\omega(\Omega)$ continuously into $\mathcal{D}_\omega(\Omega)$ which in turn is continuously included in H^s_ω since $\|\phi\|^\omega_s \le C_{\lambda,K}\|\phi\|_\lambda$ for $\lambda > \frac{n}{2} + s$ and $\phi \in \mathcal{D}_\omega(K)$ for each compact subset K of Ω. Thus $\phi \mapsto \|\phi u\|^\omega_s$ is a continuous seminorm on $\mathcal{E}_\omega(\Omega)$ which is equivalent to $\phi \mapsto \|\phi u\|_s$. Hence the inclusion map is continuous. In order to prove the density, let $\phi_k \in \mathcal{D}_\omega(\Omega)$ be such that if $\psi \in \mathcal{D}_\omega(\Omega)$ then there is k_0 such that $k \ge k_0$ implies $\psi = \phi_k\psi$. Let $u \in H^s_{\omega loc}(\Omega)$. Since $\phi_k u \in H^s_\omega$ and $\mathcal{D}_\omega(\Omega)$ is dense in H^s_ω by Theorem 2.2 in [4], there exists $v_k \in \mathcal{D}_\omega(\Omega)$ such that $\|v_k - \phi_k u\|^\omega_s \le \frac{1}{k}$. If $\psi \in \mathcal{D}_\omega(\Omega)$, choose k_0 as above. Then for $k \ge k_0$ we have $\|\psi(v_k - u)\|^\omega_s = \|\psi(v_k - \phi_k u)\|^\omega_s \le \frac{1}{k}C_\psi$ by the proof of Lemma 2.8 in [4]. Thus v_k converges to u in $H^s_{\omega loc}(\Omega)$.

Now we have

PROPOSITION 6. *If $m \ge 0$ is an integer, we have a continuous inclusion $\mathcal{E}^m_\omega(\Omega) \to H^m_{\omega loc}(\Omega)$. If $k \ge 0$ is an integer and $s > k + \frac{n}{2}$, we have a continuous inclusion $H^s_{\omega loc}(\Omega) \to \mathcal{E}^k_\omega(\Omega) \to C^k(\Omega)$.*

Proof. Let $u \in \mathcal{E}^m_\omega(\Omega)$ and $\phi_k \in \mathcal{D}_\omega(\Omega)$ be such that $\phi_k \equiv 1$ on Ω_k and $\phi_k \equiv 0$ on $\Omega - \Omega_{k+1}$, where $\Omega_k = \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > \frac{1}{k}, \|x\| < k\}$. For any $\phi \in \mathcal{D}_\omega(\Omega)$, choose k so that $\phi_k\phi = \phi$. Then $\|\phi u\|^\omega_m = \|\phi(\phi_k u)\|^\omega_m \le C_\phi\|\phi_k u\|_m$ as in the proof of Proposition 3. Hence the first inclusion is continuous. Now suppose $s > k + \frac{n}{2}$. Let $u \in H^s_{\omega loc}(\Omega)$ and $\phi \in \mathcal{D}_\omega(\Omega)$ and choose k so that $\phi_k\phi = \phi$. Then we have

$$
\begin{aligned}
\|\phi u\|_k = \|\phi(\phi_k u)\|_k &= \int e^{k\omega(\xi)}|\widehat{\phi\phi_k u}(\xi)|d\xi \\
&= \int e^{s\omega(\xi)}|\widehat{\phi\phi_k u}(\xi)|e^{(k-s)\omega(\xi)}d\xi \\
&\le \Big(\int e^{2s\omega(\xi)}|\widehat{\phi\phi_k u}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}\Big(\int e^{2(k-s)\omega(\xi)}d\xi\Big)^{\frac{1}{2}} \\
&\le C\|\phi\|_{|s|}\|\phi_k u\|^\omega_s.
\end{aligned}
$$

Hence the second inclusion is continuous. The continuity of the third one follows from Proposition 3.1 in [4].

PROPOSITION 7. *The Strong (anti)dual of $H^s_{\omega c}(\Omega)$ is $H^{-s}_{\omega loc}(\Omega)$. And the strong (anti)dual of $H^s_{\omega loc}(\Omega)$ is $H^{-s}_{\omega c}(\Omega)$.*

Proof. Let T be a continuous (conjugate) linear functional on $H^s_{\omega c}(\Omega)$ and $\phi \in \mathcal{D}_\omega(\Omega)$. Then ϕT is a continuous (conjugate) linear functional on $H^s_\omega(\operatorname{supp}\phi)$. Hence ϕT is a continuous (conjugate) linear functional on H^s_ω. By Theorem 2.6 in [4], H^{-s}_ω can be identified isometrically with the (anti)dual of H^s_ω by means of the pairing $(\phi T)(\psi) = (2\pi)^{-n}\int \widehat{\phi T}(\xi)\overline{\hat{\psi}}(\xi)d\xi$. Hence $\phi T \in H^{-s}_\omega$, $\operatorname{supp}\phi T \subset \Omega$, and $\|\phi T\| = \|\phi T\|^\omega_{-s}$. Since ϕ was arbitrary, this implies that $T \in H^{-s}_{\omega loc}(\Omega)$. Conversely, suppose $T \in H^{-s}_{\omega loc}(\Omega)$. Let (ψ_m) be the locally finite partition of unity in $\mathcal{D}_\omega(\Omega)$. For each u in $H^s_{\omega c}(\Omega)$, we define $\tilde{T}(u) = (2\pi)^{-n}\sum B_m \int \widehat{\psi_m T}(\xi)\hat{u}(\xi)d\xi$ where the summation runs over only all the integers m such that $\operatorname{supp}\psi_m \cap \operatorname{supp} u \neq \Phi$ and
$B_m = \dfrac{1}{2^m(1+\|\psi_m T\|^\omega_{-s})}$. By Hölder's inequality, we have

$$\begin{aligned}
|\tilde{T}(u)| &\leq \sum B_m \int |\widehat{\psi_m T}(\xi)|e^{-s\omega(\xi)}|\hat{u}(\xi)|e^{s\omega(\xi)}d\xi \\
&\leq \sum B_m\Big(\int |\widehat{\psi_m T}(\xi)|^2 e^{-2s\omega(\xi)}d\xi\Big)^{\frac{1}{2}}\Big(\int |\hat{u}(\xi)|^2 e^{2s\omega(\xi)}d\xi\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum B_m\|\psi_m T\|^\omega_{-s}\Big)\|u\|^\omega_s \leq \|u\|^\omega_{a,s}
\end{aligned}$$

for a= (1, 1, ...). Hence $\tilde{T}$ is a well-defined continuous (conjugate) linear functional on $H^s_{\omega c}(\Omega)$ which can be identified with T with norm $\leq \sum B_m\|T\psi_m\|^\omega_{-s}$. Since if u lies in a bounded subset of $H^s_{\omega c}(\Omega)$ then $\operatorname{supp} u$ is contained in a unique fixed compact subset of Ω, this implies that the strong (anti)dual of $H^s_{\omega c}(\Omega)$ can be identified with $H^{-s}_{\omega loc}(\Omega)$. On the other hand, if T is a continuous (conjugate) linear functional on $H^s_{\omega loc}(\Omega)$ then there is a constant C and a function $\phi_k \in \mathcal{D}_\omega(\Omega)$ such that $|T(u)| \leq C\|\phi_k u\|^\omega_s$ for all $u \in H^s_{\omega loc}(\Omega)$. Here (ϕ_k) is the sequence of test functions which defines the seminorms generating the topology on $H^s_{\omega loc}(\Omega)$. Then $\operatorname{supp} T \subseteq \operatorname{supp}\phi_k$ is a compact subset of Ω. Hence $T \in \mathcal{E}'_\omega(\Omega)$ and $|T(u)| \leq C\|\phi_k u\|^\omega_s \leq C'\|u\|^\omega_s$ for all u in $\mathcal{D}_\omega$. Hence, by Theorem 2.6 in [4], T $\in H^{-s}_\omega$. Hence $T \in H^{-s}_{\omega c}(\Omega)$. Conversely, suppose that $T \in H^{-s}_{\omega c}(\Omega)$. Then $T \in \mathcal{E}'_\omega(\Omega)$. If $u \in H^s_{\omega loc}(\Omega)$ and (ψ_m) is a locally finite partition of unity in $\mathcal{D}_\omega(\Omega)$, we define $\tilde{T}(u) =$

$(2\pi)^{-n}\sum\int\hat{T}(\xi)\overline{\widehat{u\psi_m}(\xi)}d\xi$ where the summation runs over only all the integers m such that $\operatorname{supp}\psi_m\cap\operatorname{supp}T\neq\Phi$. By means of Hölder's inequality, $\tilde{T}(u)$ is finite. Moreover, if $\phi\in\mathcal{D}_\omega(\Omega)$ is a local unit for the compact set $K=\cup\{\operatorname{supp}\psi_m:\operatorname{supp}\psi_m\cap\operatorname{supp}T\neq\Phi\}$ then

$$|\tilde{T}(u)|\leq\sum\|T\|_{-s}^{\omega}\|u\psi_m\|_s^{\omega}\leq(\sum\|\psi_m\|_{|s|}\|T\|_{-s}^{\omega})\|u\phi\|_s^{\omega}$$

for all u in $H_{\omega loc}^{s}(\Omega)$. Here the last sum is in fact a finite summation for those m such that $\operatorname{supp}\psi_m\cap\operatorname{supp}T\neq\Phi$, which is independent of u. Hence T can be identified with a continuous (conjugate) linear functional on $H_{\omega loc}^{s}(\Omega)$ with norm $\leq\sum\|\psi_m\|_{|s|}\|T\|_{-s}^{\omega}$. Since every convergent sequence in $H_{\omega c}^{-s}(\Omega)$ have supports contained in a unique compact subset, this implies that the strong (anti)dual of $H_{\omega loc}^{s}(\Omega)$ is $H_{\omega c}^{-s}(\Omega)$.

We immediately have, with the aid of Lemma 5,

COROLLARY 8. *The inclusion map $H_{\omega c}^{s}(\Omega)\to\mathcal{E}'_\omega(\Omega)$ and $H_{\omega loc}^{s}(\Omega)\to\mathcal{D}'_\omega(\Omega)$ are continuous even with the strong topologies on the distribution spaces.*

PROPOSITION 9. *If $s<t$ then the inclusion map $H_{\omega loc}^{t}(\Omega)\to H_{\omega loc}^{s}(\Omega)$ is continuous and the inclusion map $H_{\omega c}^{t}(\Omega)\to H_{\omega c}^{s}(\Omega)$ is compact.*

Proof. If $u\in H_{\omega loc}^{t}(\Omega)$ and $\phi\in\mathcal{D}_\omega(\Omega)$ then

$$\begin{aligned}(\|u\phi\|_s^{\omega})^2&=\int e^{2s\omega(\xi)}|\widehat{u\phi}(\xi)|^2d\xi\\&\leq\int e^{2t\omega(\xi)}|\widehat{u\phi}(\xi)|^2d\xi\\&=(\|u\varphi\|_t^{\omega})^2.\end{aligned}$$

Hence the first inclusion map is continuous. On the other hand, if (u_k) is a bounded sequence in $H_{\omega c}^{t}(\Omega)$, then, by the definition of the topology on this space, (u_k) is a bounded sequence in $H_\omega^t(K)$ for some compact subset K of Ω. But, $H_\omega^t(K)$ is continuously imbedded in $\overset{o}{H}{}_\omega^t(B(0,M))$, the closure of $\mathcal{D}_\omega(B(0,M))$ in the H_ω^t-norm, where $M=\sup\{\|x\|+1|x\in K\}$ and $B(0,M)=\{x\in R^n|\|x\|<M\}$. Hence,

by Theorem 3.6(Rellich's Compactness Theorem) in [4], (u_k) has a convergent subsequence in H^s_ω. Thus it has a convergent subsequence in $H^s_\omega(K)$ and therefore in $H^s_{\omega c}(\Omega)$ since $\|u\|^\omega_{a,s} \leq \sum a_m \|\psi_m\|_{|s|}\|u\|^\omega_s$. Consequently, the last inclusion map is compact for any open subset Ω of R^n.

PROPOSITION 10. *If $H^\infty_{\omega c}(\Omega) = \cap_s H^s_{\omega c}(\Omega)$ is given the weakest topology such that the inclusion map $H^\infty_{\omega c}(\Omega) \to H^s_{\omega c}(\Omega)$ is continuous for each s, then the inclusion map $\mathcal{D}_\omega(\Omega) \to H^\infty_{\omega c}(\Omega)$ is an algebraic isomorphism.*

Proof. The inclusion is obvious. If $u \in H^\infty_{\omega c}(\Omega)$ then $u \in H^s_{\omega c}(\Omega)$ for all s. Hence $\operatorname{supp} u$ is compact and $(\|u\|^\omega_s)^2 = \int e^{2s\omega(\xi)}|\hat{u}(\xi)|^2 d\xi < \infty$ for all s. By applying the Hölder's inequality, we have for any $\lambda \in R$

$$\begin{aligned} \|u\|_\lambda &= \int e^{\lambda\omega(\xi)}|\hat{u}(\xi)|d\xi \\ &= \int e^{(\lambda - s)\omega(\xi)} e^{s\omega(\xi)}|\hat{u}(\xi)|d\xi \\ &\leq \left(\int e^{2(\lambda - s)\omega(\xi)}\right)^{\frac{1}{2}} \|u\|^\omega_s \end{aligned}$$

for all sufficiently large s. Hence u is in $\mathcal{D}_\omega(\Omega)$. Thus the inclusion map is an algebraic isomorphism.

PROPOSITION 11. *If $H^\infty_{\omega loc}(\Omega) = \cap_s H^s_{\omega loc}(\Omega)$ is given the weakest topology such that the inclusion map $H^\infty_{\omega loc}(\Omega) \to H^s_{\omega loc}(\Omega)$ is continuous for each s, then $H^\infty_{\omega loc}(\Omega) = \mathcal{E}_\omega(\Omega)$ topologically.*

Proof. Clearly $\mathcal{E}_\omega(\Omega) \subseteq H^\infty_{\omega loc}(\Omega)$. If $u \in H^\infty_{\omega loc}(\Omega)$ then $u \in H^s_{\omega loc}(\Omega)$ for all s in R. Let $\phi \in \mathcal{D}_\omega(\Omega)$ be any test function. Then $\phi u \in H^s_\omega$ and $\operatorname{supp}(u\phi)$ is compact. Hence $\phi u \in H^s_{\omega c}(\Omega)$ for all s. Thus by Proposition 10 $\phi u \in \mathcal{D}_\omega(\Omega)$. Therefore $u \in \mathcal{E}_\omega(\Omega)$ which shows that $H^\infty_{\omega loc}(\Omega) = \mathcal{E}_\omega(\Omega)$. Since the inclusion map $\mathcal{E}_\omega(\Omega) \to H^s_{\omega loc}(\Omega)$ is continuous for all s in R, $\mathcal{E}_\omega$-topology on $H^\infty_{\omega loc}(\Omega)$ is finer than the given topology on $H^\infty_{\omega loc}(\Omega)$. Conversely, let G be any $\mathcal{E}_\omega$-open subset of $H^\infty_{\omega loc}(\Omega)$ and $u \in G$. Then there are constants $\epsilon > 0$ and $\lambda \in R$ and a function $\phi \in \mathcal{D}_\omega(\Omega)$ such that $\{v \in \mathcal{E}_\omega(\Omega) | \|\phi(v-u)\|_\lambda < \epsilon\} \subseteq G$. But if $v \in \mathcal{D}_\omega(\Omega) \cap H^s_\omega$ then, by Hölder's inequality,

$$\begin{aligned}\|v\|_{s-n} &= \int e^{(s-n)\omega(\xi)}|\hat{v}(\xi)|d\xi \\ &= \int e^{(-n)\omega(\xi)}e^{s\omega(\xi)}|\hat{v}(\xi)|d\xi \\ &\le \left(\int e^{(-2n)\omega(\xi)}d\xi\right)^{\frac{1}{2}}\|v\|^\omega_s \\ &= C\|v\|^\omega_s.\end{aligned}$$

Hence, $\{v \in \mathcal{E}_\omega(\Omega) | \|\phi(v-u)\|_\lambda < \epsilon\}$ contains $\{v \in H^{\lambda+n}_{\omega loc}(\Omega) | \|\phi(v-u)\|^\omega_{\lambda+n} < \frac{\epsilon}{C}\} \cap H^\infty_{\omega loc}(\Omega)$ which is an open subset of $H^\infty_{\omega loc}(\Omega)$ containing u. Therefore, G is $H^\infty_{\omega loc}(\Omega)$-open.

Recall that the space $\mathcal{D}'_{\omega,F}(\Omega)$ of generalized distributions of finite order is defined as the set of all $u \in \mathcal{D}'_\omega(\Omega)$ such that for each compact subset K of Ω there exist constants $C(K) > 0$ and $\lambda > 0$, independent of K, such that $|u(\phi)| \le C\|\phi\|_\lambda$ for all $\phi \in \mathcal{D}_\omega(K)$.

PROPOSITION 12. *We have* $\mathcal{E}'_\omega(\Omega) = \cup_s H^s_{\omega c}(\Omega)$ *and* $\mathcal{D}'_{\omega,F}(\Omega) = \cup_s H^s_{\omega loc}(\Omega)$. *Moreover, if* $u \in \mathcal{D}'_{\omega,F}(\Omega)$, *u has order* $\le k$ *and* $s > k + \frac{n}{2}$ *then* $u \in H^{-s}_{\omega loc}(\Omega)$.

Proof. By definition, $H^s_{\omega c}(\Omega) \subseteq \mathcal{E}'_\omega(\Omega)$ for all s in R. If $u \in \mathcal{E}'_\omega(\Omega)$ then, by Paley-Wiener Theorem, there is a constant $\lambda > 0$ such that $\int e^{-\lambda\omega(\xi)}|\hat{u}(\xi)|d\xi < \infty$. But, for any sequence $a = (a_m)$ of non-negative integers, we have $\|u\|^\omega_{a,-\lambda} = \sum a_m\|\psi_m u\|^\omega_{-\lambda} \le (\sum a_m\|\psi_m\|^\omega_\lambda)\|u\|_{-\lambda}$. Hence $u \in H^{-\lambda}_{\omega c}(\Omega)$. Therefore $\mathcal{E}'_\omega(\Omega) = \cup_s H^s_{\omega c}(\Omega)$. On the other hand, if $u \in H^{-s}_{\omega loc}(\Omega)$ then, by Proposition 7, u is a continuous (conjugate) linear functional on $H^s_{\omega c}(\Omega)$. Hence, there are constant C and sequence $a = (a_m)$ of non-negative integers such that $|u(\phi)| \le C\sum a_m\|\psi_m\phi\|^\omega_s$ for all $\phi \in H^s_{\omega c}(\Omega)$. But, for all $\phi \in \mathcal{D}_\omega(K)$, $\|\psi_m\phi\|^\omega_s \le \|\psi_m\|_{|s|}\|\phi\|^\omega_s \le C_K\|\psi_m\|_{|s|}\|\phi\|_s$ as in the proof of Proposition 3. Since (ψ_m) is a locally finite partition of unity, this implies that for any compact subset K of Ω there is a constant C such that $|u(\phi)| \le C\|\phi\|_s$ for all $\phi \in \mathcal{D}_\omega(K)$. Hence $u \in \mathcal{D}'_{\omega,F}(\Omega)$. Conversely, if $u \in \mathcal{D}'_{\omega,F}(\Omega)$ then for each compact subset K of Ω there are constants C_K and λ, independent of K, such that $|u(\phi)| \le C_K\|\phi\|_\lambda$ for all $\phi \in \mathcal{D}_\omega(K)$. As in the proof of Proposition

11, we have $|u(\phi)| \leq C_K \|\phi\|^{\omega}_{\lambda+n}$ for all $\phi \in \mathcal{D}_\omega(K)$. Since $\mathcal{D}_\omega(K)$ is dense in $H^{\lambda+n}_\omega(K)$, the above inequality holds on $H^{\lambda+n}_\omega(K)$. Hence u can be extended to a continuous (conjugate) linear functional on $H^{\lambda+n}_{\omega c}(\Omega)$. Thus by Proposition 7 $u \in H^{-(\lambda+n)}_{\omega loc}(\Omega)$. Consequently, $\mathcal{D}'_{\omega,F}(\Omega) = \cup_s H^s_{\omega loc}(\Omega)$. Moreover, if $u \in \mathcal{D}'_{\omega,F}(\Omega)$ has order $\leq k$ then for each compact subset K of Ω there is a constant C_K such that $|u(\phi)| \leq C_K \|\phi\|_k$ for all $\phi \in \mathcal{D}_\omega(K)$. By Hölder's inequality, we have

$$\begin{aligned}\|\phi\|_k &= \int e^{k\omega(\xi)}|\hat{\phi}(\xi)|d\xi = \int e^{(k-s)\omega(\xi)}e^{s\omega(\xi)}|\hat{\phi}(\xi)|d\xi \\ &\leq \left(\int e^{2(k-s)\omega(\xi)}d\xi\right)^{\frac{1}{2}}\|\phi\|^{\omega}_s = C'\|\phi\|^{\omega}_s\end{aligned}$$

since $2(k-s) < -n$. Hence we have $|u(\phi)| \leq C_K\|\phi\|^{\omega}_s$ for all $\phi \in \mathcal{D}_\omega(K)$. Since $\mathcal{D}_\omega(K)$ is dense in $H^s_\omega(K)$, the above inequality holds on $H^s_\omega(K)$. Therefore $u \in (H^s_{\omega c}(\Omega))' = H^{-s}_{\omega loc}(\Omega)$.

PROPOSITION 13. *If $u \in \mathcal{D}'_\omega(\Omega)$ and K is a compact subset of Ω then for some t in R we have $\phi u \in H^t_\omega$ for all $\phi \in \mathcal{D}_\omega(K)$.*

Proof. Since $u \in \mathcal{D}'_\omega(\Omega)$, there are constants C_K and $\lambda_K > 0$ such that $|u(\psi)| \leq C_K\|\psi\|_{\lambda_K}$ for all $\psi \in \mathcal{D}_\omega(K)$. Then for each $\phi \in \mathcal{D}_\omega(K)$, $|(\phi u)(\psi)| \leq C_K\|\phi\psi\|_{\lambda_K}$ for all $\psi \in \mathcal{D}_\omega(K)$. Let ψ_1 be a local unit for K in $\mathcal{D}_\omega(\Omega)$. Then $\psi = \psi_1 e^{-i<x,\xi>}$ is an element of $\mathcal{D}_\omega(\Omega)$ for each ξ. Therefore, by Paley-Wiener theorem and the conditions on ω,

$$\begin{aligned}|\widehat{\phi u}(\xi)| &= |(\phi u)(e^{-i<x,\xi>})| = |(\phi u)(\psi_1 e^{-i<x,\xi>})| \\ &\leq C_K\|\phi\psi_1 e^{-i<x,\xi>}\|_{\lambda_K} \\ &= (2\pi)^{-n}C_K\int e^{\lambda_K\omega(\eta)}\left|\int\hat{\phi}(\zeta)\hat{\psi}_1(\eta-\zeta+\xi)d\zeta\right|d\eta \\ &\leq C\|\phi\|_\lambda\|\psi_1\|_\mu\left(\int e^{(\lambda_K-\mu)\omega(\eta)}d\eta\right)\left(\int e^{(\mu-\lambda)\omega(\zeta)}d\zeta\right)e^{\mu\omega(\xi)}.\end{aligned}$$

Hence if we choose μ, λ so large that $\lambda_K - \mu < -n$ and $\mu - \lambda < -n$ then

$$\int |\widehat{\phi u}(\xi)|^2 e^{-2\lambda\omega(\xi)}d\xi \leq C\int e^{2(\mu-\lambda)\omega(\xi)}d\xi < \infty.$$

Therefore $\phi u \in H^{-\lambda}_\omega$ for all $\phi \in \mathcal{D}_\omega(K)$.

PROPOSITION 14. *If* $P(x,D)=\sum_{|\alpha|\le m} a_\alpha(x)\, D^\alpha$ *with* $a_\alpha\in\mathcal{E}_\omega(\Omega)$, *then* $P(x,D)$ *is a continuous linear map from* $H^s_{\omega loc}(\Omega)$ *into* $H^{s-m}_{\omega loc}(\Omega)$ *and a continuous linear map from* $H^s_{\omega c}(\Omega)$ *into* $H^{s-m}_{\omega loc}(\Omega)$ *and a continuous linear map from* $H^s_{\omega c}(\Omega)$ *into* $H^{s-m}_{\omega c}(\Omega)$.

Proof. If $u\in H^s_{\omega loc}(\Omega)$ and $\phi\in\mathcal{D}_\omega(\Omega)$ and ψ is a local unit for supp ϕ in $\mathcal{D}_\omega(\Omega)$ then we have

$$\begin{aligned}\|\phi P(x,D)u\|^\omega_{s-m} &= \|\phi P(x,D)(\psi u)\|^\omega_{s-m}\\ &\le \sum_{|\alpha|\le m}\|\phi a_\alpha\|_{|s-m|}\|D^\alpha(\psi u)\|^\omega_{s-m}\\ &\le \sum_{|\alpha|\le m}\|\phi a_\alpha\|_{|s-m|}e^{am}\|\psi u\|^\omega_s\end{aligned}$$

for all $u\in H^s_{\omega loc}(\Omega)$. On the other hand, if $\phi\in\mathcal{D}_\omega(\Omega)$ and $u\in H^s_{\omega c}(\Omega)$ then

$$\begin{aligned}\|\phi P(x,D)u\|^\omega_{s-m} &= \|\phi P(x,D)\sum_j(\psi_j u)\|^\omega_{s-m}\\ &\le \sum_{j\le M}\|\phi P(x,D)(\psi_j u)\|^\omega_{s-m}\\ &\le \sum_{j\le M}\sum_{|\alpha|\le m}\|\phi a_\alpha\|_{|s-m|}\|D^\alpha(\psi_j u)\|^\omega_{s-m}\\ &\le \sum_{j\le M}\sum_{|\alpha|\le m}\|\phi a_\alpha\|_{|s-m|}e^{am}\|(\psi_j u)\|^\omega_s\end{aligned}$$

for all $u\in H^s_{\omega c}(\Omega)$. Where (ψ_j) is the locally finite partition of unity and M is the maximum number of j such that supp $\phi\cap$ supp $\psi_j\ne\Phi$ and a is the constant on the condition (γ) on ω. Hence $\|\phi P(x,D)u\|^\omega_{s-m}\le C\sum_j\|\psi_j u\|^\omega_s = C\|u\|^\omega_{a,s}$ for a = (1, 1, ...). Therefore P(x, D) is a continuous linear map from $H^s_{\omega c}(\Omega)$ into $H^{s-m}_{\omega loc}(\Omega)$. Finally, for each compact subset K of Ω and every sequence a of non-negative integers, we have, for each $u \in H^s_\omega(K)$,

$$\begin{aligned}\|P(x,D)u\|^\omega_{a,s-m} &= \sum a_j \|\psi_j P(x,D)u\|^\omega_{s-m} \\ &\le \sum_{j\le M}\sum_{|\alpha|\le m} a_j \|\psi_j a_\alpha\|_{|s-m|}\|D^\alpha u\|^\omega_{s-m} \\ &\le \sum_{j\le M}\sum_{|\alpha|\le m} a_j \|\psi_j a_\alpha\|_{|s-m|} e^{am}\|u\|^\omega_s \\ &= C_K\|u\|^\omega_s.\end{aligned}$$

Here M is the maximum number of j such that supp $\psi_j \cap$ K is non-empty. Thus, P(x, D) is a continuous linear map from $H^s_\omega(K)$ into $H^{s-m}_{\omega c}(\Omega)$ for each compact subset K of Ω. Therefore, P(x, D) is a continuous linear map from $H^s_{\omega c}(\Omega)$ into $H^{s-m}_{\omega c}(\Omega)$.

PROPOSITION 15. *Let P(z) be a polynomial of degree m. Assume that for some s and some open subset Ω of R^n the set $N = \{u \in H^s_{\omega loc}(\Omega) | P(D)u = 0\}$ is a Montel space relative to the topology induced by $H^s_{\omega loc}(\Omega)$. Then, if $z = \xi + i\eta \in Z_P = \{z \in C^n | P(z) = 0\}$ and $|z| \to \infty$ then $|\eta| \to \infty$.*

Proof. Assume that the conclusion is false. Then there is a sequence $z_k \in Z_P$ such that $|z_k|$ is unbounded and $|\eta_k| \le C$. Passing to a subsequence if necessary we may assume that $\eta_k \to \eta_0$. Let $u_k(x) = e^{-s\omega(\xi_k)} e^{i<z_k,x>}$. Then $u_k \in N$. If $\phi \in \mathcal{D}_\omega(\Omega)$ we have $(\|\phi u_k\|^\omega_s)^2 = \int |\hat\phi(\xi - z_k)|^2 e^{-2s\omega(\xi_k)} e^{-2s\omega(\xi)} d\xi$. By the condition (α) on ω, $e^{-2|s|\omega(\xi-\xi_k)} \le e^{2s\omega(\xi)} e^{-2s\omega(\xi_k)} \le e^{2|s|\omega(\xi-\xi_k)}$. Thus

(*)

$$\int |\hat\phi(\xi - i\eta_k)|^2 e^{-2|s|\omega(\xi)} d\xi \le (\|\phi u_k\|^\omega_s)^2 \le \int |\hat\phi(\xi - i\eta_k)|^2 e^{2|s|\omega(\xi)} d\xi.$$

By Paley-Wiener theorem, if H is the support function of supp ϕ then for all constants M and $\epsilon > 0$ there is a constant $C_{M,\epsilon}$ such that

$$|\hat\phi(\xi - i\eta_k)| \le C_{M,\epsilon} \exp(H(-\eta_k) + \epsilon|\eta_k| - M\omega(\xi))$$

for all ξ and k. Since (η_k) is bounded we have

$$(**) \qquad |\hat\phi(\xi - i\eta_k)| \le C_M e^{-M\omega(\xi)}.$$

By taking M large enough we see that (**) implies that (ϕu_k) is a bounded sequence in H^s_ω. Thus (u_k) is a bounded sequence in N. Since N is Montel, there is a subsequence u'_k such that $u'_k \to u$ in N. If $\phi \in \mathcal{D}_\omega(\Omega)$ then $u_k(\phi) = \int u_k(x)\phi(x)dx = e^{-s\omega(\xi_k)}\hat{\phi}(-z_k)$. Thus by (**), $|u_k(\phi)| \le C'_M e^{(-M-s)\omega(\xi_k)}$ since ω is radial. Taking M large enough and noting that $|z_k| \to \infty$, $|\eta_k| \le C$ implies $|\xi_k| \to \infty$, we see that $u'_k \to 0$ weak* in $\mathcal{D}'_\omega(\Omega)$. Thus $u = 0$, that is, $\phi u'_k \to 0$ in H^s_ω for any $\phi \in \mathcal{D}_\omega(\Omega)$. By (*), (**) and the dominated convergence theorem it follows that $\int |\hat{\phi}(\xi - i\eta_0)|^2 e^{-2|s|\omega(\xi)}d\xi = 0$. Thus $\hat{\phi}(\xi - i\eta_0) = 0$ for all ξ, which implies that $\phi = 0$ for every $\phi \in \mathcal{D}_\omega(\Omega)$. This contradiction completes the proof.

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