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FINITELY BASED LATTICE VARIETIES (I)

SANG GYOU YIM* AND YOUNG YUG KANG

1. Introduction

In R. McKenzie[12], it is shown that the cardinality of the lattice variety is $2^{\aleph_0}$ and K. Baker[1] contains the stronger result that $\mathbf{M}$, the variety of all modular lattices, has $2^{\aleph_0}$ subvarieties. It follows that there exists a variety of modular lattices that is not finitely based. In fact, K. Baker[2] gave an example of such a variety. And it was proved by K. Baker[2] and B. Jónsson[8] that join of two finitely based lattice varieties is not always finitely based. K. Baker[2] gave an explicit example of case of modular lattice variety and B. Jónsson[8] gave one of case of nonmodular lattice variety. Then it is proposed whether the join of two finitely based varieties is finitely based under certain conditions. The answer to the question is not affirmative.

B. Jónsson[8] gave a necessary and sufficient condition for the join of two finitely based varieties to be finitely based. In B. Jónsson[9], it was conjectured that the join of a finitely based lattice variety and a lattice variety generated by a finite lattice is always finitely based. Here the original question for modular lattice varieties is investigated under certain conditions. Let A_1 be the lattice pictured in figure 1. Actually we obtain the following result.

THEOREM 1. *Let* $\mathbf{V}$ *and* $\mathbf{V}'$ *be finitely based modular lattice varieties. If* $A_1 \notin \mathbf{V}$ *and* $A_5 \notin \mathbf{V}'$, *then* $\mathbf{V} + \mathbf{V}'$ *is finitely based.*

The rest of this paper is divided into two sections. In section 2, we will give some preliminary definitions and facts. And finally in last section we shall prove the theorem 1 and state their corollary. For

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standard concepts and facts from lattice theory, we refer the reader to G. Grätzer[5] and S. Burris and H.P. Sankappanavar[13]. However we use $+$ and $\cdot$ instead of $\vee$ and $\wedge$, respectively, for the lattice operations.

2. Preliminaries

We first review some well-known definitions for lattice. Now consider two quotients b/a and d/c in a lattice L. If $b+c=d$ and $a \leq c$, then we say that *b/a transposes weakly up onto d/c* (in symbols $b/a \nearrow_w d/c$) and dually, if $a \cdot d = c$ and $d \leq b$, then we say that *b/a transposes weakly down onto d/c* (in symbols $b/a \searrow_w d/c$). If there exists a sequence of quotients

$$b/a = b_0/a_0, b_1/a_1, \ldots, b_n/a_n = d/c$$

such that for $i = 0, 1, \ldots, n-1$ $b_i/a_i \nearrow_w b_{i+1}/a_{i+1}$ or $b_i/a_i \searrow_w b_{i+1}/a_{i+1}$, then b/a is said to *projectly weakly onto d/c* in n steps. If both $b/a \nearrow_w d/c$ and $d/c \searrow_w b/a$, that is, if $b+c=d$ and $b \cdot c = a$, then we say that *b/a transposes up onto d/c* and that *d/c transposes down onto b/a* (in symbols $b/a \nearrow d/c$ and $d/c \searrow b/a$). If there exists a equence of quotients

$$b/a = b_0/a_0, b_1/a_1, \ldots, b_n/a_n = d/c$$

such that for $i = 0, 1, \ldots, n-1, b_i/a_i \nearrow b_{i+1}/a_{i+1}$ or $b_i/a_i \searrow b_{i+1}/a_{i+1}$, then we say that *$b/a$ projects onto d/c* in n steps. By the *projective distance between two quotients b/a* and d/c, in symbols $P(b/a, d/c)$, we means that the smallest nonnegative integer n such that some nontrivial subquotients b'/a' of b/a and d'/c' of d/c are projective to each other in n steps. If no such n exists, then we write $P(b/a, d/c) = \infty$. By the *weak projective distance* from a quotient b/a to a quotient d/c in a lattice L (in symbols $P_w(b/a, d/c)$) we mean that the smallest nonnegative integer n such that a nontrivial subquotient b'/a' of b/a projects weakly onto a nontrivial subquotient d'/c' of d/c in n steps. Let L be a modular lattice. By a *sequence of transposes* in L we mean a (finite) sequence of quotients

$$b_0/a_0, b_1/a_1, \ldots, b_n/a_n \tag{1}$$

such that for each $k < n$, b_k/a_k transposes either up or down onto b_{k+1}/a_{k+1}. The sequence of transposes (1) is said to be *reduced* if, for $0 < k < n$, either

$$(2) \qquad b_{k-1}/a_{k-1} \nearrow b_k/a_k \searrow b_{k+1}/a_{k+1}$$

or else

$$(3) \qquad b_{k-1}/a_{k-1} \searrow b_k/a_k \nearrow b_{k+1}/a_{k+1}.$$

The sequence of transposes (1) is said to be *normal* if is reduced and, for $0 < k < n$,

$$(4) \qquad b_k = b_{k-1} + b_{k+1} \quad \text{if} \quad (2) \quad \text{holds}$$

and

$$(5) \qquad a_k = a_{k-1} \cdot a_{k+1} \quad \text{if} \quad (3) \quad \text{holds.}$$

Finally, sequence of transposes (1) is said to be a *strongly normal sequence of transposes* if is normal and for, $0 < k < n$,

$$(6) \qquad b_{k-1} \cdot b_{k+1} \leq a_k \quad \text{if} \quad (2) \quad \text{holds}$$

and

$$(4) \qquad a_{k-1} + a_{k+1} \geq b_k \quad \text{if} \quad (3) \quad \text{holds.}$$

Suppose (1) is strongly normal sequence of transposes in L. The sublattice of L generated by the endpoints of three successive quotients b_i/a_i, with $i = k-1, k, k+1$, is actually generated by three these elements, namely by b_{k-1}, a_k and b_{k+1} if (2) holds, but by a_{k-1}, b_k and a_{k+1} if (3) holds. This lattice is therefore finite. It is, in fact, a homomorphic image of the lattice pictured in figure 2 if (2) holds, but of its dual if (3) holds. We denote the five-element modular nondistributive lattice by $\mathbf{M}_3$ pictured in figure 3. By a *diamond* we mean that a five-termed sequence $[a \leq x, y, z \leq b]$ of elements of L whose terms are all equal (in which case the diamond is said to be *degenerate*) or else form $\mathbf{M}_3$. Any nonidentity permutation of x, y and z yield a diamond, which by definition is distinct from the original diamond, even though they represent the same sublattice of L. We see from figure 2 that if a quotient b_{k-1}/a_{k-1} is nontrivial, the figure contains a nontrivial diamond

$$D_k = [a_k < x_k, y_k, z_k < b_k] \tag{8}$$

where

$$D_k = [a_{k-1} + a_{k+1} < b_{k-1} + a_{k+1}, a_k, a_{k-1} + b_{k+1} < b_k] \tag{9}$$

if (2) holds, and

$$D_k = [a_k < a_{k-1} \cdot b_{k+1}, b_k, b_{k-1} \cdot a_{k+1} < b_{k-1} \cdot b_{k+1}] \tag{10}$$

if (3) holds.

By this argument, a strongly normal sequence of transposes (1) in L generates a sequence of $(n-1)$ diamonds $D_1, D_2, \dots, D_{n-1}$. This is said to be *associated sequence of diamonds*. In Jónsson [7], it was proved that if two quotients in a modular lattice L project onto each other in n steps, then there exists a nontrivial subquotient of them which project strongly normally onto each other in less than or equal to n steps. Therefore, with each sequence of projectivities there is always an associated sequence of diamonds. We must now investigate how these diamonds fit together. First, we define some notions. Given two diamonds $D_i = [a_i < x_i, y_i, z_i < b_i], i = 1, 2$, we say that D_1 *transposes down onto* D_2 (in symbols $D_1 \searrow_{(1)} D_2$) or D_2 *transposes up onto* D_1 (in symbols $D_2 \nearrow_{(1)} D_1$) if $b_1/a_1 \searrow b_2/a_2$ and under this transposition the vertices x_1, y_1 and z_1 mapped onto the corresponding vertices x_2, y_2 and z_2, or $b_2/a_2 \nearrow b_1/a_1$ and under this transposition the vertices x_1, y_1 and z_1 mapped onto the corresponding vertices x_2, y_2 and z_2, respectively (See figure 4). Also we say that D_1 *translates up onto* D_2 (in symbols $D_1 \nearrow_{(2)} D_2$) and that D_1 *translates down onto* D_2 (in symbols $D_1 \searrow_{(2)} D_2$) if $b_1/z_1 \nearrow x_2/a_2$ and if $z_1/a_1 \searrow b_2/x_2$, respectively (See figure 4).

If $D = [a < x, y, z < b]$ is a diamond, then D^* is defined to be the diamond $[a < z, x, y < b]$. So $D_1 \searrow_{(1)} D_2^*$ means that $b_1/a_1 \searrow$

$b_2/a_2, x_1 \cdot b_2 = z_2, y_1 \cdot b_2 = x_2$ and $z_1 \cdot b_2 = y_2$. The investigation of how these associated diamonds fit together was done by D.X. Hong[6]. Hong[6] contains the following very useful theorem. We call it ***Hong's Theorem*** in this paper.

THEOREM 2.1 (HONG'S THEOREM). *Let b/a and d/c be nontrivial quotients in a modular lattice such that $P(b/a, d/c) = n, 2 < n < \infty$. Then some nontrivial subquotients b'/a' and d'/c' of b/a and d/c, respectively, can be connected by a strongly normal sequence of transposes*

$$b'/a' = b_0/a_0, b_1/a_1, \ldots, b_n/a_n = d'/c'$$

such that the associated diamonds $D_1, D_2, \ldots, D_{n-1}$ satisfy

(i) $D_k \nearrow_{(1)} D^*_{k+1}$ or $D_k \nearrow_{(2)} D_{k+1}$ *if* $b_k/a_k \nearrow b_{k+1}/a_{k+1}$, *and*
$D_k \searrow_{(1)} D^*_{k+1}$ or $D_k \searrow_{(2)} D_{k+1}$ *if* $b_k/a_k \searrow b_{k+1}/a_{k+1}$,
where $k = 1, 2, \ldots, n-2$.

(ii) *If* $D_k \nearrow_{(1)} D^*_{k+1}$ or $D_k \searrow_{(1)} D^*_{k+1}$, *then* $D_k = D^*_{k+1}$, *where* $k = 2, 3, \ldots, n-2$.

(iii) *If* $D_k \nearrow_{(1)} D^*_{k+1}$ or $D_k \searrow_{(1)} D^*_{k+1}$, *then it cannot happen that* $D_{k+1} \searrow_{(1)} D^*_{k+2}$ or $D_{k+1} \nearrow_{(1)} D^*_{k+2}$, *respectively.*

If the condition (i), (ii) and (iii) are satisfied, then we refer to the strongly normal sequence of transposes in Hong's theorem as a ***Hong sequence***.

A nonempty class $\mathbf{V}$ of algebras of same type is called *variety* if it is closed under subalgebras, homomorphic images and direct products. A class $\mathbf{K}$ of algebras of same type is *finitely based* if $\mathbf{K}$ is the class of all models of some finite set of identities. As an example, the class Λ of all lattices is a finitely based variety. Now we introduce Jónsson's criterion and its corollary.

THEOREM 2.2. (JÓNSSON [8]). *Suppose $\mathbf{U}$ is a lattice variety and let $\mathbf{V}$ and $\mathbf{V}'$ be subvarieties of $\mathbf{U}$ defined, relative to $\mathbf{U}$, by the identities $\alpha = \beta$ and $\gamma = \delta$, respectively, where the inclusions $\beta \leq \alpha$ and $\delta \leq \gamma$ hold in $\mathbf{U}$. In order for $\mathbf{V} + \mathbf{V}'$ to be finitely based relative to $\mathbf{U}$, it is necessary and sufficient that there exists a positive integer n with the following property*

$P(n)$: For any lattice $L \in \mathbf{U}$, if there exist $\mu, \nu \in {}^{\omega}L$ and $c, d \in L$ with $c < d$ such that both $\alpha(\mu)/\beta(\mu)$ and $\gamma(\nu)/\delta(\nu)$ project weakly onto d/c, then there exist $\mu', \nu' \in {}^{\omega}L$ and $c', d' \in L$ with $c' < d'$ such that both $\alpha(\mu')/\beta(\mu')$ and $\gamma(\nu')/\delta(\nu')$ project weakly onto d'/c' in n steps.

COROLLARY 2.3. *Let $\mathbf{V}$ and $\mathbf{V}'$ be subvarieties of $\mathbf{M}$ defined, relative to $\mathbf{M}$, by the identities $\alpha = \beta$ and $\gamma = \delta$, respectively, where the inclusions $\beta \leq \alpha$ and $\delta \leq \gamma$ hold in $\mathbf{M}$. In order for $\mathbf{V} + \mathbf{V}'$ to be finitely based relative to $\mathbf{M}$, it is necessary and sufficient that there exists a positive integer n with the following property*

$P(n)$: For any lattice $L \in \mathbf{M}$, if there exist $\mu, \nu \in {}^{\omega}L$ such that a nontrivial subquotient of $\alpha(\mu)/\beta(\mu)$ projects onto a nontrivial subquotient of $\gamma(\nu)/\delta(\nu)$, then there exist $\mu', \nu' \in {}^{\omega}L$ such that a nontrivial subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto a nontrivial subquotient of $\gamma(\nu')/\delta(\nu')$ in n steps.

For any lattice L, if there exists a nonnegative integer n such that for all $a, b, c, d \in L$ with $a < b$ and $c < d$, whenever b/a projects weakly onto d/c, then b/a projects weakly onto a nontrivial subquotient of d/c in n steps, then the smallest such n is denoted by $R(L)$. If no such n exists, then we write $R(L) = \infty$. For a class $\mathbf{K}$ of lattices, $R(\mathbf{K})$ denotes the supremum of $R(L)$ for $L \in \mathbf{K}$. We denote $\mathbf{F}_{\mathbf{M}}(a, b, c)$ by free modular lattice generated by a, b and c(See [5]). For a lattice L and arbitrary two elements a, b in L, $con_L(a, b)$ or $con(a, b)$ denotes the smallest congruence relation on L that identifies a and b. Generally, it is well-known fact that the cardinality of $\mathbf{F}_{\mathbf{M}}(a, b, c, d)$, where any two of them are incomparable, is infinite and that $\mathbf{F}_{\mathbf{M}}(3)$ has 28 elements. And in [14], Takeuchi showed that $\mathbf{F}_{\mathbf{M}}(a, b < c, d)$ has 138 elements. Its Hasses diagram is pictured in figure 5.

3. Proof of main theorem

By a *critical* quotient of a lattice L we mean a quotient that is collapsed by every nontrivial congruence relation on L.

LEMMA 3.1. *Let L be a modular lattice generated by five diamonds D_1, D_2, D_3, D_4 and D_5 with the property that $D_1 \nearrow_{(2)} D_2 \searrow_{(2)} D_3 \nearrow_{(2)} D_4 \searrow_{(2)} D_5$. Let $a = a_1, b_1$ or $x_1, b = a_3, c = b_3$ and $d = a_5, b_5$ or z_5 in L. Then, the modular lattice G gengrated by a, b, c and d is a homomorphic image of $L_M(a, b < c, d)$ pictured in figure 6. i.e., G contains at most 25 elements.*

Proof. In the lattice L, for arbitrary four elements a, b, c and d with $b < c$, they have always the following three properties:

$$(a+b)\cdot(b+d) = b, \tag{1}$$

$$(a+c)\cdot(c+d) = c, \quad \text{and} \tag{2}$$

$$a+c+d = a+b+d. \tag{3}$$

To prove the lemma, we use the $\mathbf{F}_{\mathbf{M}}(a, b < c, d)$. In order to use the simplified notation, we use their numbers in figure 5. Let $a = 62, b = 29, c = 110$ and $d = 77$. Hereafter in the proof, for the natural numbers m, n, p and q, "$m = n$" implies that m and n are identified, and "$m = n$" $\longmapsto$ "$p = q$" implies that if m and n are identified, then so are p and q. Then

(1) is equal that "86 = 29"
(2) is equal that "132 = 110", and
(3) is equal that "138 = 136".

Thus then by the property of modular lattice,

(1) is equivalent to that (a_1) : "86 = 71", (a_2) : "71 = 56", (a_3) : "71 = 54", (a_4) : "56 = 42", (a_5) : "54 = 42" and (a_6) : "42 = 29".

(2) is equivalent to that (b_1) : "132 = 128", (b_2) : "128 = 121" and (b_3) : "121 = 110".

In fact,

by (3), "138 = 136" $\longmapsto$ "137 = 134" and "110 = 97".

By (a_1), "86 = 71" $\longmapsto$ "128 = 121", "67 = 57", "108 = 96", "59 = 44", "49 = 37", "68 = 53", "64 = 52" and "67 = 57".

By (a_2), "71 = 56" $\longmapsto$ "41 = 31", "27 = 20". "103 = 91", "57 = 46", "41 = 27", "22 = 13" and "92 = 78".

By (a_3), "71 = 54" $\longmapsto$ "121 = 110", "27 = 17", "56 = 42", "96 = 83", "35 = 23" and "65 = 50".

By (a_6), "42 = 29" $\longmapsto$ "5 = 2".

By (b_1), "132 = 128" $\longmapsto$ "97 = 85", "61 = 47", "106 = 95", "109 = 99", "134 = 130", "63 = 49", "102 = 87" and "119 = 108".

Hence the set of all representative elements of G is at most $\{1, 2, 4, 7, 15, 23, 26, 29, 33, 53, 62, 66, 73, 77, 78, 91, 97, 114, 116, 124, 126, 131, 133, 137, 138\}$. The Hasses diagram is pictured in figure 6. □

LEMMA 3.2. *In a modular lattice L generated by five diamonds D_1, D_2, D_3, D_4 and D_5 with the property that $D_1 \nearrow_{(2)} D_2 \searrow_{(2)} D_3 \nearrow_{(2)} D_4 \searrow_{(2)} D_5$, If $a_1 + a_5 = a_2 + a_4$, then $b_2 + b_4 = x_1 + z_5$.*

Proof. Let $a_1 + a_5 = a_2 + a_4$. Then

$$\begin{aligned} b_2 + b_4 =& (x_1 + a_2 + b_3) + (b_3 + a_4 + z_5) \\ =& x_1 + a_2 + b_3 + a_4 + z_5 \\ =& x_1 + a_2 + (x_3 + z_3) + a_4 + z_5 \\ =& x_1 + a_2 + a_4 + z_5 \\ =& x_1 + a_1 + a_5 + z_5 \text{ by hypothesis} \\ =& x_1 + z_5. \quad \square \end{aligned}$$

Let $\bar{x}$ denote the image of each $x \in L$ in the homomorphic image $\bar{L}$ of L, and we shall use this notation for any homomorphic image of a given lattice.

LEMMA 3.3. *The modular lattice L generated by five diamonds D_1, D_2, D_3, D_4 and D_5 with the property that $D_1 \nearrow_{(2)} D_2 \searrow_{(2)} D_3 \nearrow_{(2)} D_4 \searrow_{(2)} D_5$ has a homomorphic image of $\overline{G}_5$ as a homomorphic image. Furthermore, L has the finite simple lattice A_5 as a homomorphic image.(See figure 7 and figure 8).*

Proof. We will show that $L/con(a_1 + a_5, a_2 + a_4)$ is a homomorphic image of $\overline{G}_5$. For this, by lemma 3.1 and $F_M(3)$, it is enough to show that the following two identities hold:

(a) $$\bar{a}_1 \cdot \bar{a}_4 + \bar{a}_2 \cdot \bar{a}_5 = \bar{a}_3, \quad \text{and}$$

(b) $$\bar{a}_1 \cdot \bar{b}_5 + \bar{b}_1 \cdot \bar{a}_5 = \bar{b}_1 \cdot \bar{b}_5.$$

In fact, by $con/(a_1 + a_5, a_2 + a_4)$, we have

(*1) $$\bar{a}_1 + \bar{a}_5 = \bar{a}_2 + \bar{a}_4 = \bar{a}_1 + \bar{a}_3 + \bar{a}_5 \geq \bar{a}_3, \quad \text{and}$$

(*2)
$$\bar{a}_1 + \bar{a}_5 = \bar{a}_2 + \bar{a}_4 = \bar{a}_2 + \bar{x}_3 + \bar{z}_3 + \bar{a}_4 = \bar{a}_2 + \bar{b}_3 + \bar{a}_4 \geq \bar{b}_3.$$

On the other hand, since $\bar{b}_1 \cdot \bar{b}_5 < \bar{x}_2 \cdot \bar{z}_4 = \bar{a}_3 \leq \bar{b}_3$, and by (*1) and (*2), we get

(*3) $$\bar{b}_1 \cdot \bar{b}_5 \leq (\bar{a}_1 + \bar{a}_5).$$

Therefore

$$\begin{aligned}
\text{(a); } \bar{a}_1 \cdot \bar{a}_4 + \bar{a}_2 \cdot \bar{a}_5 &= \bar{a}_1 \cdot \bar{a}_3 + \bar{a}_3 \cdot \bar{a}_5 \\
&= \bar{a}_3 \cdot (\bar{a}_1 + \bar{a}_3 \cdot \bar{a}_5) \\
&= \bar{a}_3 \cdot (\bar{a}_1 + \bar{a}_2 \cdot \bar{a}_5) \\
&= \bar{a}_3 \cdot \bar{a}_2 \cdot (\bar{a}_1 + \bar{a}_5) \\
&= \bar{a}_3 \quad [\text{by } (*1)],
\end{aligned}$$

and

$$\begin{aligned}
\text{(b); } \bar{a}_1 \cdot \bar{b}_5 + \bar{b}_1 \cdot \bar{a}_5 &= \bar{b}_1 \cdot (\bar{a}_1 \cdot \bar{b}_5 + \bar{a}_5) \\
&= \bar{b}_1 \cdot \bar{b}_5 \cdot (\bar{a}_1 + \bar{a}_5) \\
&= \bar{b}_1 \cdot \bar{b}_5 \quad [\text{by } (*3)].
\end{aligned}$$

This completes the proof. □

LEMMA 3.4(BAKER[3]). *For arbitrary two lattices L and M, let $f : L \to M$ be a surjective lattice homomorphism, and let b/a and d/c be nontrivial quotients in L. Suppose that $f(b)/f(a)$ projects weakly onto $f(d)/f(c)$ with $f(c) < f(d)$ in k steps in M for some $k \geq 0$. Then there exist $c', d' \in L$, with $c \leqslant c' < d' \leqslant d$, such that $f(c) = f(c')$ and $f(d) = f(d')$, and such that b/a projects weakly onto d'/c' in $(k+1)$ steps if $k > 0$ and in 2 steps when $k = 0$.*

From above lemma 3.4, we get easily the following as its corollary.

COROLLARY 3.5. *For arbitrary two modular lattices M_1 and M_2, let $f : M_1 \to M_2$ be a surjective lattice homomorphism, and let b/a and d/c be nontrivial quotients in M_1. Suppose that $f(b)/f(a)$ projects weakly onto $f(d)/f(c)$ with $f(c) < f(d)$ in k steps in M_2 for some $k \geq 0$. Then there exist $c', d' \in M_1$, with $c \leqslant c' < d' \leqslant d$, such that $f(c) = f(c')$ and $f(d) = f(d')$, and such that a subquotient of b/a projects onto d'/c' in $(k+1)$ steps if $k > 0$ and in 2 steps when $k = 0$.*

Proof of Theorem 1. Let $\mathbf{V}$ and $\mathbf{V}'$ be defined by the identities $\alpha = \beta$ and $\gamma = \delta$, respectively, relative to $\mathbf{M}$. We may assume that the inclusions $\beta \leq \alpha$ and $\delta \leq \gamma$ hold in every modular lattice. Letting $\mathbf{U} = \mathbf{M}$, we are going to show that the condition $P(n)$ in corollary 2.3 holds for $n = 25$. Since $\mathbf{M}$ is finitely based, it follows that $\mathbf{V} + \mathbf{V}'$ is finitely based. Consider a lattice $L \in \mathbf{M}$, and suppose there exist $\mu, \nu \in^\omega L$ such that a nontrivial subquotient of $\alpha(\mu)/\beta(\mu)$ projects onto a nontrivial subquotient of $\gamma(\mu)/\delta(\nu)$ in m steps.

Assuming that the two quotients have been so chosen that m is as small as possible, we shall show that assumption $m > n$ leads to a contradiction. There exists, by Hong's Theorem, a Hong sequence

$$b/a = b_0/a_0, b_1/a_1, \dots, b_m/a_m = d/c$$

for some nontrivial subquotients b/a and d/c of $\alpha(\mu)/\beta(\mu)$ and $\gamma(\mu)/\delta(\nu)$, respectively. Let $D_1, D_2, \dots, D_{m-1}$ be the associated sequence of diamonds. Then we have the following two cases:

(1) There exists a subsequence $D_k, D_{k+1}, D_{k+2}, D_{k+3}$ and D_{k+4} for $0 < k < m - 10$ with $D_k \nearrow_{(2)} D_{k+1} \searrow_{(2)} D_{k+2} \nearrow_{(2)} D_{k+3} \searrow_{(2)} D_{k+4}$.

(2) there does not exist any such subsequence.

Case (1): By lemma 3.3, the lattice A_5 is a homomorphic image of the sublattice L_0 of L generated by $D_k, D_{k+1}, D_{k+2}, D_{k+3}$ and D_{k+4}. Furthermore, b_k/a_k is a nontrivial quotient in L_0 and $\bar{b}_k/\bar{a}_k$ is a critical quotient in A_5. Since $A_5 \notin \mathbf{V}', \delta(\bar{\nu'}) < \gamma(\bar{\nu'})$ for some $\nu' \in {}^\omega L_0$. Observe that $\gamma(\bar{\nu'}) = \overline{\gamma(\nu')}$ and $\delta(\bar{\nu'}) = \overline{\delta(\nu')}$. Also $R(A_5) = 8$. Hence $\overline{\gamma(\nu')}/\overline{\delta(\nu')}$ projects weakly onto $\bar{b}_k/\bar{a}_k$ in 8 steps. Since b_k/a_k and $\gamma(\nu')/\delta(\nu')$ are nontrivial quotients in L_0, by corollary 3.5, there exists a nontrivial subquotient y/x, with $a_k \leq x < y \leq b_k$, such that $\bar{y} = \bar{b}_k$ and $\bar{x} = \bar{a}_k$, and such that a subquotient of $\gamma(\nu')/\delta(\nu')$ projects onto y/x in 9 steps. Since L_0 is a modular lattice, therefore a nontrivial quotient of b/a projects onto a nontrivial quotient of $\gamma(\nu')/\delta(\nu')$ in $(k+9)$ steps. Since $0 < k < (m-10)$, it contains a contradiction.

Case (2): We can choose some integer t, with $7 < t < m - 14$ such that

$$D_t \nearrow_{(2)} D_{t+1} = D^*_{t+2}, D_{t+2} \nearrow_{(2)} D_{k+3}$$

or

$$D_t \searrow_{(2)} D_{t+1} = D^*_{t+2}, D_{t+2} \searrow_{(2)} D_{k+3}.$$

Then $D_t \cup D_{t+1} \cup D_{t+3}$ forms a sublattice L_1 of L which contains A_1 as a homomorphic image, and b_{t+3}/a_{t+3} is a prime quotient in L_1. Since $A_1 \notin \mathbf{V}$, $\alpha(\mu') > \beta(\mu')$ for some $\mu' \in {}^{\nu}L_1$. Since $R(A_1) \leq 7, \overline{\alpha(\mu')}/\overline{\beta(\mu')}$ projects weakly onto b_{t+3}/a_{t+3} in 7 steps. Therefore by corollary 3.5, a nontrivial subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto b_{t+3}/a_{t+3} in 8 steps. Since L_1 is a modular lattice, a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto b_{t+3}/a_{t+3} in 8 steps. So a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto a nontrivial subquotient of d/c in $(m-t+5)$ steps. This too leads to a contradiction. This completes the proof. $\square$

COROLLARY 3.6. *Let $\mathbf{V}$ and $\mathbf{V}'$ be finitely based modular lattice varieties. If $\mathbf{A}_1 \notin \mathbf{V}$ and A_5^d, the dual of A_5, is not contained in $\mathbf{V}'$, then $\mathbf{V} + \mathbf{V}'$ is finitely based.*

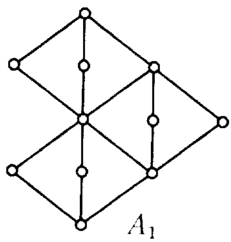


Figure 1

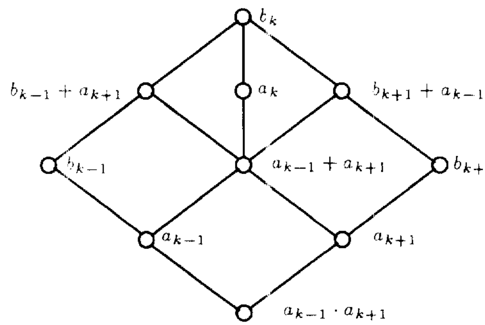


Figure 2

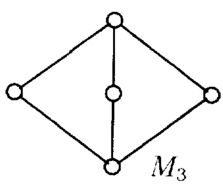


Figure 3

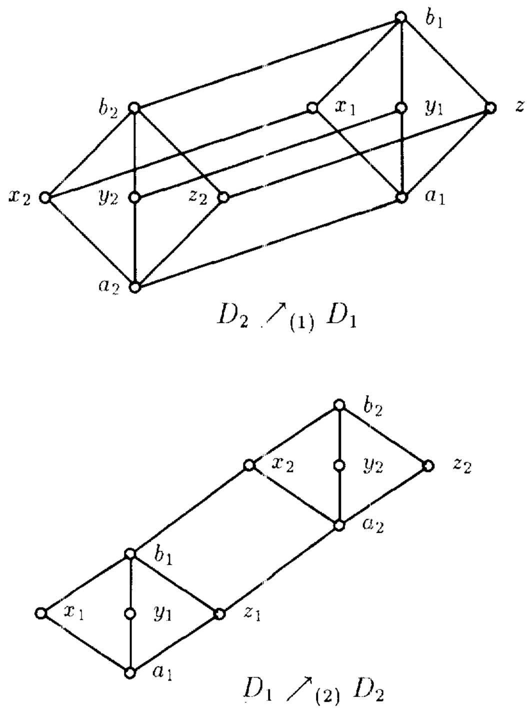
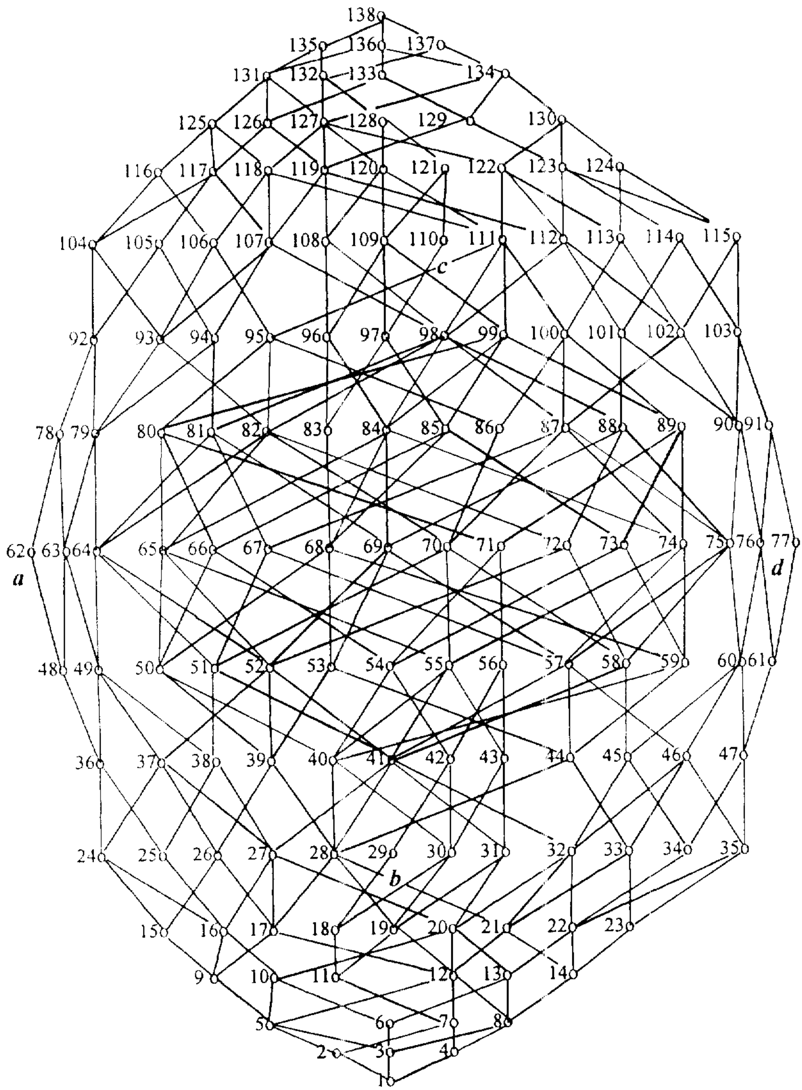
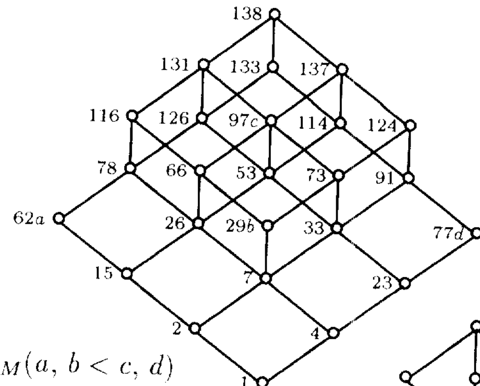


Figure 4



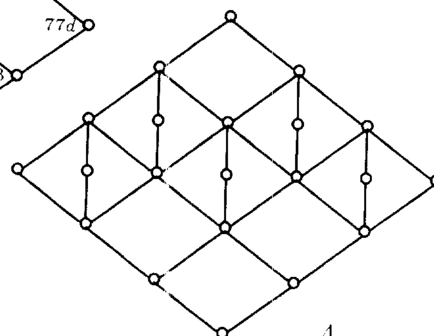
$\mathrm{FM}(a, b < c, d)$

Figure 5



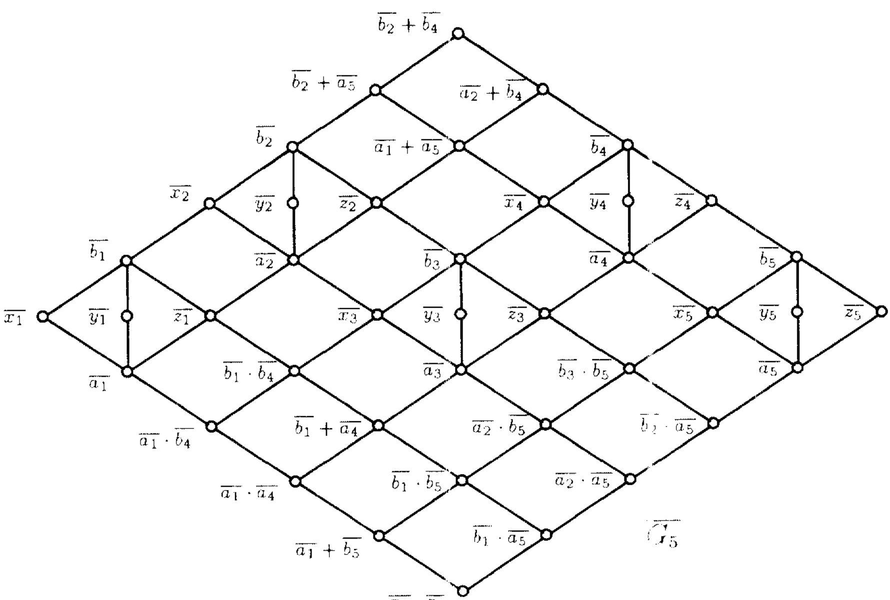
$L_M(a,\, b < c,\, d)$

Figure 6



A_5

Figure 8



$\overline{G_5}$

Figure 7

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Sang Gyou Yim
Department of Mathematics
Yeungnam University
Kyongsan 712-749, Korea

Young Yug Kang
Department of Mathematics
Keimyung University
Taegu 704-701, Korea