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PEAK FUNCTION AND ITS APPLICATION

SANGHYUN CHO †

1. Introduction

Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $A(\Omega)$ denote the functions holomorphic on Ω and continuous on $\overline{\Omega}$. A point $p \in b\Omega$ is a peak point if there is a function $f \in A(\Omega)$ such that $f(p) = 1$, and $|f(z)| < 1$ for $z \in \overline{\Omega} - \{p\}$. The existence of peaking functions as well as the additional smoothness up to the boundary is one of the major topics in several complex variables. When Ω is strictly pseudoconvex, the situation with regard to peak functions is fairly well understood, but in the weakly pseudoconvex case we know very little. If $\Omega \subset\subset \mathbb{C}^2$ is pseudoconvex and $b\Omega$ is of finite type. Bedford and Fornaess [1], showed that there is a peak function in $A(\Omega)$. This method also works for finite type domains in $\mathbb{C}^n$ where the Levi-form of $b\Omega$ has (n-2)-positive eigenvalues. We also mention the work of Bloom [2], Hakim and Sibony [11], and Range [16] on the existence of peak functions with additional smoothness up to the boundary of Ω, i.e., in the various subclass of $A(\Omega)$.

Recently the author proposed a method [8] to construct a peak function for the domains in $\mathbb{C}^n$ where the optimal estimates of the Bergman kernel function are known. Namely, for each neighborhood V of $p \in b\Omega$ we construct a regular bumping family of pseudoconvex domains outside V, and use Bishop's $\frac{1}{4} - \frac{3}{4}$ method on bumped domains. This is a modification of Fornaess and McNeal's method [10] and can be applied to wide class of domains in $\mathbb{C}^n$. The optimal estimates of the Bergman kernel function and its derivatives are known, for example, for pseudoconvex domains of finite type in $\mathbb{C}^2$ [4,12], decoupled pseudoconvex

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domains of finite type domains in $\mathbb{C}^n$ [13], pseudoconvex domains of finite type in $\mathbb{C}^n$ where the Levi-form of $b\Omega$ has $(n-2)$-positive eigenvalues [6,7], and for the locally convex finite type domains in $\mathbb{C}^n$ [15]. In this paper, we want to prove the existence of the Hölder continuous peak functions on the domains we have just mentioned.

THEOREM 1.1. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $b\Omega$ be of finite type and $p \in b\Omega$. Assume that the optimal estimates of the Bergman kernel function and its derivatives are known on and off the diagonal near $p \in b\Omega$. Then for each small neighborhood V of p, there is a Hölder continuous peak function that peaks at p and extends holomorphically up to $b\Omega \setminus V$.*

REMARK 1.2. The author proved the existence of (continuous) peak functions for the locally convex domains in $\mathbb{C}^n$ [8]. Here we prove an additional smoothness (i.e., Hölder continuity) of the peak function up to the boundary for the domains, for example, we have mentioned before Theorem 1.1.

REMARK 1.3. Fornaess and McNeal also proved the existence of the Hölder continuous peak functions for the pseudoconvex domains of finite type in $\mathbb{C}^2$ and for decoupled pseudoconvex finite type domains in $\mathbb{C}^n$. Here we revise their proof and we show, in addition, that the peak function extends holomorphically up to $b\Omega \setminus V$.

The existence of peak functions for $A(\Omega)$ implies that Ω is complete in the Carathéodory metric. Since the Carathéodory metric is smaller than the Kobayashi metric and the Bergman metric, we obtain the following corollary as an immediate application of Theorem 1.1.

COROLLARY 1.4. *Let Ω be one of four kinds of domains we mentioned before Theorem 1.1. Then Ω is complete in the Kobayashi, Bergman and Carathéodory metrics.*

We will only prove the existence of Hölder continuous peak function on the pseudoconvex domains of finite type in $\mathbb{C}^n$ where the Levi-form of $b\Omega$ has $(n-2)$-positive eigenvalues. The same analysis will give us a Hölder continuous peak functions for the rest kinds of domains.

2. Smooth bumping families

Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r and let $0 \in b\Omega$. If $g : \mathbb{C} \longrightarrow \mathbb{C}$ is any smooth function with $g(0) = 0$, let $\nu(g)$ denote the order of vanishing of g at 0. For a vector valued $G = (g_1, \ldots g_n)$, let $\nu(G)$ denote the minimum order of vanishing of the g_i at 0.

DEFINITION 2.1. (D'Angelo). 0 is a point of finite 1-type if

$$\sup_G \frac{\nu(r \circ G)}{\nu(G)} = \Delta(0) < \infty,$$

where $G : \mathbb{C} \longrightarrow \mathbb{C}^n$ is a complex analytic map with $G(0) = 0$; $\Delta(0)$ is called the type of 0.

DEFINITION 2.2. Let $p \in b\Omega$ be an arbitrary point and let V be a neighborhood of p. By a smooth bumping family for Ω outside V we mean a family $\{\Omega_t\}_{0 \leq t \leq 1}$ of pseudoconvex domains with C^∞ defining functions $\{r_t\}$ with the following properties:

(a) $\Omega = \Omega_0$,
(b) $\Omega_{t_1} \subset \Omega_{t_2}$ if $t_1 < t_2$, and $r_t(z)$ is smooth in z and t,
(c) for any neighborhood U of $\partial\Omega \setminus V$ there is a $t_0 > 0$ such that $D_t \setminus U = D \setminus U$ for all $t \in [0, t_0]$.

The following theorem can be found in [5].

THEOREM 2.3. *Let p be a point of finite 1-type in the boundary of a pseudoconvex domain Ω in $\mathbb{C}^n$ with smooth defining function $r(z)$. Then for each neighborhood V of p, there exists a smooth 1-parameter family of pseudoconvex domains $\{\Omega_t\}_{0 \leq t < t_0}$, each defined by $\Omega_t = \{z; r(z,t) < 0\}$, where $r(z,t)$ has the following properties:*

(a) *$r(z,t)$ is smooth in z near $b\Omega$, and in t for $0 \leq t < t_0$,*
(b) *$r(z,t) = r(z)$ for $z \notin V$,*
(c) *$\frac{\partial r}{\partial t}(z,t) \leq 0$,*
(d) *$r(z,0) = r(z)$,*
(e) *for z in V, $\frac{\partial r}{\partial t} < 0$.*

DEFINITION 2.4. Suppose Ω, $p \in b\Omega$, V be as Theorem 2.3. Then we say $\{\Omega_t\}_{0 \leq t < t_0}$ a bumping family of Ω with front V.

THEOREM 2.5. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain and let $b\Omega$ be of finite 1-type. Assume $p \in b\Omega$ and V is a small neighborhood of p. Then there is a 1-parameter family of a smooth bumping family $\{\Omega_t\}$ outside V.*

Proof. Choose a neighborhood U of p such that $V \subset\subset U$. Since $b\Omega$ is compact, we can choose points $z_1, \ldots, z_N \in b\Omega$ and $\epsilon_1, \ldots, \epsilon_N > 0$ such that

(1) Ω is pseudoconvex and $b\Omega$ is of finite type,
(2) $\cup_{i=1}^N B(z_i, \epsilon_i/2) \supset b\Omega \setminus U$,
(3) $V \cap B(z_i, \epsilon_i) = \emptyset$, $i = 1, 2, \ldots, N$,
(4) $\overline{B(z_i, \epsilon_i)}$, is contained in a neighborhood V_i where Theorem 2.3 can be applicable, $i = 1, 2, \ldots, N$.

Set $V_i = B(z_i, \epsilon_i)$, $i = 1, 2, \ldots, N$, for the convenience. Consider a bumping family of Ω with front V_1. Since the type condition is stable under small C^∞-perturbations of $b\Omega$, we will get a family $\{\Omega_{t_1}\}_{0 \le t_1 < \alpha_1}$ of smooth pseudoconvex domains satisfying (1)-(4) for the domains Ω_{t_1} (instead of Ω) provided α_1 is sufficiently small. For each Ω_{t_1}, $0 \le t_1 < \alpha_1$, we consider a bumping family of Ω_{t_1} with front V_2 and call it $\{\Omega_{t_1 t_2}\}_{0 \le t_2 < \alpha_2}$. Again $\{\Omega_{t_1 t_2}\}_{0 \le t_2 < \alpha_2}$ will satisfy (1)-(4) provided α_2 is sufficiently small. Continuing in this manner, we will get a bumping family of pseudoconvex domains $\{\Omega_{t_1 t_2 \ldots t_N}\}$ outside V. Obviously we can regard this family as a 1-parameter family of pseudoconvex domains. $\square$

3. Estimates on the Bergman kernel

Let Ω be a bounded pseudoconvex domain of finite type in $\mathbb{C}^n$ with smooth defining function r and let $p \in b\Omega$. Suppose that p is a point of finite T in the sense of D'Angelo [9], and assume that the Levi-form of $b\Omega$ has $(n-2)$-positive eigenvalues at p. In this section we estimate the Bergman kernel function near p using the analysis of local geometry of $b\Omega$ in [6,7]. For the estimates of the kernel on the other domains, one can refer [4,12,13,15].

Let us take the coordinate functions $z_1, \ldots, z_n$ about $p \in b\Omega$. After a linear change of coordinates, we may assume that $|\frac{\partial r}{\partial z_1}(z)| \ge c > 0$ for all $z \in U$, for some neighborhood of p.

PROPOSITION 3.1. *([4, Proposition 2.2]) For each $z' \in U$ and positive even integer m, there is a biholomorphism $\Phi_{z'} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $\Phi_{z'}^{-1}(z') = 0$, $\Phi_{z'}^{-1}(z) = (\zeta_1, \ldots, \zeta_n)$ such that*

(3.1)

$$\begin{aligned} r(\Phi_{z'}(\zeta)) &= r(z') + Re\zeta_1 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\leq \frac{m}{2} \\ j,k>0}} Re\left(b_{j,k}^{\alpha}(z')\zeta_n^j\overline{\zeta}_n^k\zeta_\alpha\right) \\ &+ \sum_{\substack{j+k\leq m \\ j,k>0}} a_{j,k}(z')\zeta_n^j\overline{\zeta}_n^k + \sum_{\alpha=2}^{n-1} |\zeta_\alpha|^2 \\ &+ \mathcal{O}(|\zeta_1||\zeta| + |\zeta''|^2|\zeta| + |\zeta''||\zeta_n|^{\frac{m}{2}+1} + |\zeta_n|^{m+1}). \end{aligned}$$

Let us choose z' near p and fix it for a moment. We take m is equal to T in Proposition 3.1. Set $\rho(\zeta) = r \circ \Phi_{z'}(\zeta)$, and set

$$\tau(z', \delta) = min\{(\frac{\delta}{A_l(z')})^{\frac{1}{l}} : 2 \leq l \leq T\}.$$

Since $A_T(z_0) \geq c > 0$, it follows that $A_T(z') \geq c' > 0$ for all $z' \in U$ if U is sufficiently small. This gives the inequality,

$$\delta^{\frac{1}{2}} \lesssim \tau(z', \delta) \lesssim \delta^{\frac{1}{T}}, \quad z' \in U. \tag{3.2}$$

The definition of $\tau(z', \delta)$ easily implies that if $\delta' < \delta''$, then

$$(\delta'/\delta'')^{\frac{1}{2}}\tau(z', \delta'') \leq \tau(z', \delta') \leq (\delta'/\delta'')^{\frac{1}{T}}\tau(z', \delta''). \tag{3.3}$$

Now set $\tau_1 = \delta, \tau_2 = \ldots = \tau_{n-1} = \delta^{\frac{1}{2}}$, $\tau_n = \tau(z', \delta) = \tau$ and define

$$\begin{aligned} R_\delta(z') &= \{\zeta \in \mathbb{C}^n; |\zeta_k| < \tau_k,\ k = 1, 2, \ldots, n\}, \text{ and} \\ Q_\delta(z') &= \{\Phi_{z'}(\zeta); \zeta \in R_\delta(z')\}. \end{aligned} \tag{3.4}$$

In [1], the author proved that for $z \in Q_\delta(z')$ it follows that

$$\tau(z', \delta) \approx \tau(z, \delta). \tag{3.5}$$

We recall the estimates on the Bergman kernel function and its derivatives for the domain Ω considered in this section [6,7]. Because of the transformation formula for the Bergman kernel function, we state our estimates the special coordinates in Proposition 3.1.

THEOREM 3.2. *Let Ω and $p \in b\Omega$ be as in the begining of this section. For $z^1, z^2 \in U \cap \Omega$, set $\zeta^i = \Phi_{z^1}^{-1}(z^i)$, $i = 1, 2$. Then there exist a neighborhood U of p and constants $C_{\alpha,\beta}$, independent of $z^1, z^2 \in U \cap \Omega$, such that*

$$|D_{\zeta^1}^{\alpha} \overline{D}_{\zeta^2}^{\beta} K_{\Omega_{z^1}}(\zeta^1, \zeta^2)| \leq C_{\alpha,\beta} \delta^{-n-\alpha_1-\beta_1-\frac{1}{2}\gamma} \cdot \tau(z^1, \delta)^{-2-\alpha_n-\beta_n}$$

where $\delta = |\rho(\zeta^1)| + |\rho(\zeta^2)| + M(\zeta^1, \zeta^2)$, $\gamma = \alpha_2 + \beta_2 + \ldots + \alpha_{n-1} + \beta_{n-1}$, and $M(\zeta^1, \zeta^2) = |\zeta_1^1 - \zeta_1^2| + \sum_{j=2}^{n-1} |\zeta_j^1 - \zeta_j^2|^2 + \sum_{l=2}^{T} A_l(z')|\zeta_n^1 - \zeta_n^2|^l$, and where $\rho = r \circ \Phi_{z^1}$.

REMARK 3.3. Let Ω and $p \in b\Omega$ be as above. Then the author [6] estimated the Bergman kernel function $K(z, z)$ on the diagonal as follows

$$K(z, z) \approx \sum_{l=2}^{m} A_l(z')^{\frac{2}{l}} r(z)^{-n-\frac{2}{l}} \approx r(z)^{-n} \cdot \tau(z, r(z))^{-2}, \tag{3.6}$$

near p, and this is the case when $z^1 = z^2$, and $\alpha, \beta = 0$, in Theorem 3.2.

Since p is a point of finite type T, Catlin's theorem [3] says that there are $\epsilon > 0$ and a neighborhood U in which $\bar{\partial}$-Neumann problem satisfies a subelliptic estimate of order $\epsilon > 0$ on (0,1)-forms. For each $w \in U \cap \Omega$, define the function

$$h_w(z) = \frac{K_\Omega(z, w)}{K_\Omega(w, w)}.$$

Then the estimates of Theorem 3.2 and (3.6) give us

$$|D_z^{\alpha} h_w(z)| \lesssim \delta^{-\alpha_1 - \frac{1}{2}(\alpha_2 + \ldots, \alpha_{n-1})} \cdot \tau(z, \delta)^{-\alpha_n}, \tag{3.7}$$

where $\delta = |r(w)| + |r(z)| + M(w, z)$, and where

$$M(w, z) = |z_1 - w_1| + \sum_{j=2}^{n-1} |z_j - w_j|^2 + \sum_{l=2}^{T} A_l(w)|z_n - w_n|^l.$$

We will estimate $K_\Omega(z,w)$ for z outside a certain neighborhood of w and will show that $|h_w(z)|$ is quite small outside that neighborhood. For the convenience in notation, we denote by C the various constants that follows. Let π be a projection onto $b\Omega$ and set $p = \pi(w)$. For $\eta > 0$, let $B(p,\eta)$ denote the euclidean ball centered at p of radius η. Let $\zeta \in C^\infty(\mathbb{C}^n)$ be a function with the property that $\zeta \equiv 1$ on $\Omega \setminus B(p,\eta)$ and $\zeta \equiv 0$ on $B(p,\frac{\eta}{2})$, and let N denote the $\overline{\partial}$-Neumann operator on (0,1)-forms. The following theorem was proved in [14].

PROPOSITION 3.4. *Let Ω, U, and $\epsilon > 0$ be as above. Let $s,t \in \mathbb{R}^+$. If α is a smooth (0,1)-form in the domain of the Kohn Laplacian and $supp\alpha \subset B(p,\frac{\eta}{8})$ then there is a constant $C_{st} > 0$ so that*

$$\|\zeta N\alpha\|_s^2 \le C_{st}\eta^{-2(\frac{s+t}{\epsilon}+4)}\|\alpha\|_{-t}^2.$$

Let $\phi \in C_0^\infty(0,1)$ be a non-negative radial function with $\int \phi = 1$. For $w \in U$, set

$$\phi_w(z) = (\frac{\delta(w)}{2})^{-2n}\phi(\frac{z-w}{\delta(w)/2}).$$

Then from the Kohn's formula,

$$K_\Omega(z,w) = \phi_w(z) - \overline{\partial}^* N\overline{\partial}\phi_w(z).$$

Assume $supp\phi_w \subset B(p,\frac{\eta}{8})$. Then from the Proposition 4.1 with $s = r+1$, we have

$$\begin{aligned}\|\zeta(\cdot)K_\Omega(\cdot,w)\|_r^2 &\le C\|\zeta N\overline{\partial}\phi_w(\cdot)\|_{r+1}^2\\ &\le C\eta^{-2(\frac{r+1+t}{\epsilon}+4)}\|\overline{\partial}\phi_w(\cdot)\|_{-t}^2.\end{aligned}$$

If $t > n+1$, Sobolev's lemma gives

$$\begin{aligned}\|\phi_w\|_{-(t-1)}^2 &= sup\{|(\phi_w,f)|; f \in C_0^\infty,\ \ \|f\|_{t-1} \le 1\}\\ &\le C|\int \phi_w| \le C\end{aligned}$$

for some $C > 0$. If we choose $r > n+1$, another application of Sobolev's lemma shows

$$sup_z|\zeta(z)K_\Omega(z,w)| \le C\|\zeta(\cdot)K_\Omega(\cdot,w)\|_r.$$

Hence

$$sup_z|\zeta(z)K_\Omega(z,w)| \le C\eta^{-(\frac{2n+4}{\epsilon}+4)}.$$

Now set $\eta = \delta(w)^{\frac{\epsilon}{2(2n+4)}}$. If we combine Theorem 3.2 and the estimate (3.6), we have proved the following

PROPOSITION 3.5. *Let Ω and η be as above. There exists a constant $C > 0$, independent of $w \in U$, so that*

$$|h_w(z)| \leq C\delta(w)$$

for $w \in U$ and $z \in \overline{\Omega} \setminus B(\pi(w), \eta)$.

REMARK 3.6. In [1], the author showed that the sharp subelliptic estimate holds near p. So we may, in fact, take $\epsilon = \frac{1}{T}$.

4. A construction of Hölder continuous peak functions

In this section, we construct a Hölder continuous peak function at p. This construction can be done by careful observation of the estimates in Theorem 3.2 and (3.6), (3.7). Let Ω and $p \in b\Omega$ be as in the begining of section 3 and assume that the type of p is T. Then by Remark 3.6, the sharp subelliptic estimate of order $\frac{1}{T}$ holds near p. Set $\eta = \frac{1}{2(2n+4)T}$. Now we denote by N the interior normal to the boundary of Ω at p.

LEMMA 4.1. *For every q on N, let $|q - p| = d$. There exists a constant $C > 0$ such that for every point q on N sufficiently close to p, there exist a neighborhood U_p of p and a holomorphic function $h = h_q$ on Ω such that*

(1) $|h| < C$ *on* Ω,
(2) $h(q) = 1$,
(3) $|h(z)| < Cd$ *for* $z \in \Omega \setminus U_p$,
(4) $|Dh| < \frac{C}{d}$.

Proof. Define $h_q(z) = K(z, q)/K(q, q)$. Property (2) is clear. From Proposition 3.5, we have $|h_q(z)| \leq Cd$ for $z \in \overline{\Omega} \setminus B(\pi(q), d^\eta)$. This proves (3) with $U_p = B(p, d^\eta)$. The estimates of Theorem 3.2 and (3.6) give $|h| \leq C$ for $z \in U_p = B(\pi(q), d^\eta)$. This fact together (3) gives (1). Also from the estimates of Theorem 3.2 and (3.6), (3.7), we have $|Dh| < \frac{C}{d}$. □

REMARK 4.2. Actually (4) of Lemma 4.1 can be sharpened as in (3.7).

We now ready to costruct a Hölder continuous peak functions at p;

For simplicity we assume that $p = 0$. Let the type of p is equal to T and choose a neighborhood U of p such that the subelliptic estimates for $\overline{\partial}$-equation of order $\frac{1}{T}$ hold on U and Theorem 3.2 and the estimates of Remark 3.3 hold on U. We denote by N the interior normal to the boundary of Ω at p. For each neighborhood $V \subset\subset U$ of p, choose a neighborhood V_1, $V_1 \subset\subset V \subset\subset U$. Next we consider a 1-parameter family of a smooth bumping family $\{\Omega_t\}_{0\le t<t_0}$ outside V_1. We may assume that $V_1 \cap \Omega = V_1 \cap \Omega_t$ for all $t > 0$, after perhaps shrinking V_1, and $\Omega \setminus V \subset\subset \Omega_t \setminus V$ for all $0 < t < t_0$. Now fix $0 < t_1 < t_0$ and consider the pseudoconvex domain Ω_{t_1}. Since the type condition is stable, we may assume that $b\Omega_{t_1}$ is of finite type. Also we have $d = dist\ (\Omega \setminus V, b\Omega_{t_1} \setminus V) > 0$. Let $q_n = p + \frac{s}{2^n T} N$, for a small constant s to be determined. Hence $d_n = \frac{s}{2^n T}$. Define $h_n(z) = h_{q_n}(z + q_n - p)$, where h_{q_n} is the function defined on Ω_{t_1} (instead of Ω) as in Lemma 4.1 associated with q_n. Notice that $h_n(z)$ is defined on $\overline{\Omega} \setminus V$, and on U intersect a translate of Ω which contains $\overline{\Omega} \cap \overline{V}$, provided s is sufficiently small. Therefore h_n is well defined and holomorphic on $\overline{\Omega}$ for each $n \ge 0$. Let U_n denote the neighborhood corresponding to h_n as in Lemma 4.1. For a suitable constant $0 < c < 1$, to be determined later, let $r = 1 - c$ and define a peak function as $H = r\sum_{n=0}^{\infty} c^n h_n$. During the proof we will choose the constants more precisely. Let us estimate H on various sets. First outside U_0. By the property (3) of Lemma 4.1 we have $|h_n| < \frac{1}{2}$ for every n provided that s is sufficiently small. So we get that $|H| < r\sum_{n=0}^{\infty} \frac{c^n}{2} = \frac{1}{2}$.

Now let $\mathbb{P}(z) = |z_1| + \sum_{j=2}^{n-1} |z_j|^2 + \sum_{l=2}^{T} A_l(0)|z_n|^l$, with the notation of section 3. Next assume that z is in $U_n \setminus U_{n+1}$. Let m be the largest integer so that $\mathbb{P}(z) < d_m$. Then from the estimates of the first derivatives of h_k and by virtue of (3.2) and (3.5), we have for $k \le m$ that

$$
\begin{aligned}
|h_k| - 1 &\le C[d_k^{-1}|z_1| + \sum_{j=2}^{n-1} d_k^{-\frac{1}{2}}|z_j| + \tau(0, d_k)^{-1}|z_n|] \\
&\le C[d_k^{-1}|z_1| + \sum_{j=2}^{n-1} d_k^{-\frac{1}{2}}|z_j| + \tau(0, d_k)^{-1} \cdot \tau(0, \mathbb{P}(z))]
\end{aligned}
$$

because $\tau(0, \mathbb{P}(z))^{-1} \le \frac{C}{|z_n|}$. Thus by virtue of (3.3) and (3.4) we get that

$$|h_k| - 1 \leq C[d_m \cdot d_k^{-1} + d_m^{\frac{1}{2}} \cdot d_k^{-\frac{1}{2}} + \tau(0, d_k)^{-1} \cdot \tau(0, d_m)]$$
$$\leq C(2^{T(k-m)} + 2^{\frac{T}{2}(k-m)} + 2^{k-m}).$$

Therefore $|h_k| < 1 + C2^{k-m}$ for $k \leq m$. Similarly, if we combine the estimates in Theorem 3.2 and (3.6), we have that $|h_k| < C\frac{d_k^2}{\mathbb{P}^2(z)}$ for $m < k < n$. Also, $|h_n| < C$ and $|h_k| < C\frac{s}{2^{kT}}$ if $k > n$. Hence we estimate H as follows:

$$|H| < r[\sum_{k \leq m} c^k(1 + C2^{k-m}) + \sum_{m<k<n} c^k C\frac{d_k}{\mathbb{P}^2} + c^n C + \sum_{k>n} c^k C\frac{s}{2^k}]$$
$$< r\frac{(1 - c^{m+1})}{(1-c)} + rC\frac{((2c)^{m+1} - 1)}{(2c-1)2^m}$$
$$+ rCc^{m+1} \cdot \frac{4}{3} + rc^n C + rCs\frac{(\frac{1}{2})^{n+1}}{(1-c)}$$
$$= (1 - c^{m+1}) + \frac{2C(1-c)}{(2c-1)}(c^{m+1} - 2^{-m-1})$$
$$+ \frac{4}{3}rCc^{m+1} + rc^n C + Cs(\frac{1}{2})^{n+1}.$$

Now set $r = \frac{1}{10C}$, (i.e., $c = 1 - \frac{1}{10C}$) for instance. Then

$$|H| < (1 - c^{m+1}) + \frac{1}{5(2c-1)}(c^{m+1} - 2^{-m-1})$$
$$+ \frac{2}{15}c^{m+1} + \frac{1}{10}c^n + Cs(\frac{1}{2})^{n+1}$$
$$< 1 - \frac{1}{2}c^{m+1} + \frac{1}{4}(\frac{1}{2})^{n+1} < 1,$$

if s is sufficiently small. Here we assumed that $C >> 1$ and hence $1 > c > \frac{9}{10}$, for instance.

Next we consider Hölder estimates. Let $V_k = \{z : |z| \leq 2^{-kT}\}$ and choose any $z, w \in V_1$. Without loss of generality, we may assume that $0 < |w| < |z|$. We will estimate $|H(z) - H(w)| < r\sum c^n |h_n(z) - h_n(w)|$. First fix $m \leq n$ so that $w \in V_n \setminus V_{n+1}$ and $z \in V_m \setminus V_{m+1}$. From (4) of Lemma 4.1, we have $|h_j(z)-h_j(w)| \leq C|z-w|\cdot d_j^{-1} \leq C|z-w|2^{jT}$. Without loss of generality, we may assume that $mlog\frac{1}{c} \geq 2(10T+1)log2$. We thus get that

$$\begin{aligned} r\sum_{j<m+10} c^j|h_j(z)-h_j(w)| &\leq C|z-w|\cdot(2^Tc)^{m+10} \\ &\leq C|z-w|^{\nu} \end{aligned}$$

if $|z-w|^{1-\nu}\cdot(2^Tc)^{m+10} \leq 1$, which holds if $\nu < \frac{log\frac{1}{c}}{2Tlog2}$.

We consider next the tail end of the series.

Case I. $n > m+3$

Then we have

$$r\sum_{j>m+8} c^j|h_j(z)-h_j(w)| \leq Cc^m.$$

We need $c^m \leq |z-w|^{\nu}$ or $c^m \leq 2^{-mT\nu}$. So if we take $\nu \leq \frac{log\frac{1}{c}}{Tlog2}$, then

$$r\sum_{j>m+8} c^j|h_j(z)-h_j(w)| \leq C|z-w|^{\nu}.$$

Case II. $n \leq m+3$ and $|z-w| \leq C|w|^T$.

Note that $|h_j(z)-h_j(w)| \leq C|Dh_j(z')||z-w|$ for some $z' \in Q_{M(z,w)}(z)$. Since $|w| \approx |z'| \approx 2^{-mT}$, the estimate (3.7) gives us

$$\begin{aligned} |h_j(z)-h_j(w)| &\leq C2^{mT}|z-w| \\ &\leq C2^{mT}|z-w|^{\frac{1}{T}}\cdot|w|^{T-1} \\ &\leq C|z-w|^{\frac{1}{T}}. \end{aligned}$$

So

$$r\sum_{j>m+8} c^j|h_j(z)-h_j(w)| \leq C|z-w|^{\frac{1}{T}}.$$

Case III. $n \le m+3$ and $|z-w| >> |w|^T$.

Let $\tilde{z}$, $\tilde{w}$ be the projections of z and w onto $b\Omega$ respectively and let us denote $\nu_{\tilde{z}}$, $\nu_{\tilde{w}}$ the corresponding outward normal vectors at $\tilde{z}$ and $\tilde{w}$. Set $z' = z - |w|\nu_{\tilde{z}}$ and $w' = w - |w|\nu_{\tilde{w}}$. Note that

$$|h_j(z') - h_j(w')| \le C|Dh_j(z'')||z' - w'|,$$

where $|r(z'')| \gtrsim |w|$. Again from the estimate in (3.7), we get that

$$|h_j(z') - h_j(w')| \le C|w|^{-1}|z' - w'| \le C2^{mT}|z' - w'| \lesssim 1.$$

Therefore,

$$\begin{aligned} r\sum_{j>m+8} c^j|h_j(z) - h_j(w)| &\le r\sum_{j>m+8} c^j|h_j(z) - h_j(z')| \\ &+ r\sum_{j>m+8} c^j|h_j(z') - h_j(w')| \\ &+ r\sum_{j>m+8} c^j|h_j(w') - h(w)| \\ &\le C(c^m + c^m + c^m) \le Cc^m. \end{aligned}$$

We want $c^m \le |z-w|^\nu$, which will follow if $c^m \le |w|^{T\nu} \approx 2^{-mT^2\nu}$. This leads to the estimate $\nu < \frac{log\frac{1}{c}}{T^2 log2}$. So $H = r\sum_{i=0}^{\infty} c^n h_n$ is a Hölder continuous (of order $\nu \le \frac{log\frac{1}{c}}{T^2 log2}$) peak function at p.

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Department of Mathematics Edu.
Pusan University
Pusan 609-735, Korea
e-mail: cho@hyowon.cc.pusan.ac.kr